Section 19 – The Product Topology

We will consider these products of topological spaces:

- **The finite Cartesian product** \( X_1 \times X_2 \times \ldots \times X_n \)
- **The infinite Cartesian Product** \( X_1 \times X_2 \times X_3 \times \ldots \)

There are two topologies: the box topology and the product topology.

**Basis for the box topology:** sets of the form \( U_1 \times U_2 \times \ldots \times U_n \) (resp.)

**Sub-Basis for the product topology:** sets of the form \( \pi_i^{-1}(U_i) \)

These topologies are the same for the finite products, but they do not agree on the infinite products.

**Convention:** When considering a product space, we shall assume it is given the product topology unless stated otherwise.

Some Definitions

Let \( J \) be an index set. Given a set \( X \), we define an **\( J \)-tuple of elements of \( X \)** to be a function \( x : J \to X \). We will often write \( x(\alpha) = x_\alpha \); we call it the \( \alpha \) th coordinate of \( x \). The function itself can also be denoted by \( (x_\alpha)_{\alpha \in J} \).

**The set of all \( J \)-tuples of elements of \( X \)** is denoted by \( X^J \).

Let \( (A_\alpha)_{\alpha \in J} \) be an indexed family of sets; Let \( X = \bigcup_{\alpha \in J} A_\alpha \). **The Cartesian product** of this indexed family, denoted by \( \prod_{\alpha \in J} A_\alpha \) is defined to the set of all \( J \)-tuples \( (x_\alpha)_{\alpha \in J} \) of elements of \( X \) such that \( x_\alpha \in A_\alpha \) for each \( \alpha \in J \).

If the sets \( A_\alpha \) are all equal to \( X \), then the product is just the set \( X^J \) of all \( J \)-tuples of \( X \).

**The Box Topology:** For the indexed family of topological spaces \( X = \bigcup_{\alpha \in J} A_\alpha \) The basis consisting of all sets of the form \( \prod_{\alpha \in J} U_\alpha \), where \( U_\alpha \) is open in \( X_\alpha \) generates the box topology.
We define the projection map $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ to be the function assigning to each element of the product space its $\beta$th coordinate $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$.

**Theorem 19.1 – Comparison of the box and product topologies.**
The box topology on $\prod X_\alpha$ has as a basis all sets of the form $\prod U_\alpha$ where $U_\alpha$ is open in $X_\alpha$ for each $\alpha$.
The product topology ....