Math 1520 – Sample Final Exam

You may use a graphing calculator (TI-83, 84, for example) on this exam, but not one that can perform symbolic integration (TI-89, for example).

There are 15 questions, worth a total of 150 points.

(10) 1. Find the solution to the separable differential equation \( \frac{dy}{dx} = x\sqrt{y} \) with initial condition \( y = 4 \) when \( x = 0 \).

**Solution:** \( y = \left( \frac{x^2}{4} + 2 \right)^2 \)

(10) 2. The slope field for the differential equation \( \frac{dy}{dx} = x - y \) is shown below.

(a) Sketch the two solutions which have initial conditions \((-3, 3)\) and \((0, -3)\).

(b) Guess one linear solution to the differential equation and check that it works.

**Solution:** \( y = x - 1 \).

(10) 3. The Pioneer 10 spacecraft is powered by plutonium radio-thermal generators. The power produced depends directly on the amount of plutonium remaining. The amount of plutonium, \( P \), decays according to the differential equation \( \frac{dP}{dt} = -rP \).

(a) Find \( r \), the decay coefficient, given that the half-life of plutonium is 87.72 years.

**Solution:** \( r = \frac{\log(2)}{87.72} \approx 0.0079 \)
(b) At launch, the power generated was 2580 Watts. How much power was being generated when it sent its last signal in 2003, 31 years after launch?

**Solution:** Power after 31 years is $2580e^{-r \cdot 31} \approx 2019$ Watts.

(10) 4. The (infinite) region bounded by the curve $y = e^{-x}$, the positive $y$-axis and the positive $x$-axis is revolved around the $x$-axis. Find the volume of this solid of revolution.

**Solution:**

$$
\int_0^\infty \pi (e^{-x})^2 \, dx = -\frac{1}{2} \pi e^{-2x} \bigg|_0^\infty = \frac{\pi}{2}
$$

(10) 5. Integrate $\int \frac{dx}{x^2 + 2x}$

**Solution:** $\frac{1}{2} \log |\frac{x}{x+2}| + C$

(10) 6. Integrate $\int \frac{dx}{x^2 + 2x + 1}$

**Solution:** $\frac{-1}{x+1} + C$

(10) 7. Integrate $\int \frac{dx}{x^2 + 2x + 2}$

**Solution:** $\arctan(x + 1) + C$

(10) 8. $2e$ or not $2e$? For each, decide if the value is $2e$ or not $2e$.

(a) $\sum_{n=0}^{\infty} \frac{2}{n!}$

(b) $\lim_{n \to \infty} 2 \left(1 + \frac{1}{n}\right)^n$

(c) $\int_{2}^{\infty} \frac{dx}{x}$

(d) $\sum_{n=0}^{\infty} \frac{e}{2^n}$

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<th>(a)</th>
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<td>$2e$</td>
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Solution: For all of these, a good approach is to try computing the values for a few
n, say n = 2, 5, 10, 100. For e, f, and maybe b, that’s the only approach you would
be expected to do.

(a) \(\sum_{n=0}^{\infty} \frac{2}{n!} = 2 \sum_{n=0}^{\infty} \frac{1}{n!} = 2e^1 = 2e\)

(b) \(\lim_{n \to \infty} \log \left(1 + \frac{1}{n}\right)^n = \lim_{n \to \infty} n \log \left(1 + \frac{1}{n}\right) = \lim_{x \to 0} \log(1+x) = 1\). Here, we set

\(x = \frac{1}{n}\) and used L’Hopital’s rule in the final step. From this, \(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e\)
and \(\lim_{n \to \infty} 2 \left(1 + \frac{1}{n}\right)^n = 2e\).

(c) \(\int_2^{\infty} \frac{dx}{x} = \log(x)|_2^\infty = \infty\).

(d) \(\sum_{n=0}^{\infty} \frac{e}{2^n} = e \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = e \frac{1}{1 - \frac{1}{2}} = 2e\), from the geometric series.

(e) This requires the theory of continued fractions. See, for example http://
people.math.binghamton.edu/dikran/478/Ch7.pdf, formula (7.14).

(f) Proving this formula requires Stirling’s formula, \(\log(n!) = n \log(n) - n + O(\log(n))\)
where “big O” means that part grows no faster than \(\log(n)\). Stirling’s formula
is proven by relating \(\log(n!)=\sum_{x=1}^{n} \log(x)\) to the trapezoid rule approximation
to the integral \(\int_1^n \log(x)dx\). See https://en.wikipedia.org/wiki/Stirling’s
approximation for details. With Stirling’s approximation:

\[\log \left(\frac{2n}{\sqrt{n!}}\right) = \log(2) + \log(n) - \frac{\log(n!)}{n} = \log(2) + \log(n) - \log(n) + 1 - O(\log(n))/n\]
so \( \lim_{n \to \infty} \log \frac{2n}{\sqrt{n!}} = \log(2) + 1 \). Taking exp of both sides gives \( \lim_{n \to \infty} \frac{2n}{\sqrt{n!}} = e^{\log(2)+1} = 2e \.

\[ (g) \sum_{n=0}^{\infty} \frac{n+1}{n!} = \sum_{n=0}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} = \left( \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \right) + e = \left( \sum_{n=0}^{\infty} \frac{1}{n!} \right) + e = 2e \.]

9. Find the sum of the geometric series \( \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \frac{5}{243} + \cdots \).

**Solution:**

\[
\frac{5}{3} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{5}{2}
\]

10. Define a sequence by \( a_0 = 1, a_n = \frac{1+a_{n-1}}{2+a_{n-1}} \). Write out the first five terms of this sequence. Bonus: What can you say about the limit of this sequence?

**Solution:** \( 1, \frac{3}{5}, \frac{5}{8}, \frac{13}{21}, \frac{34}{55}, \ldots \) The sequence is decreasing and bounded below (by 0), so it has a limit. The limit satisfies \( L = \frac{1+L}{2+L} \), so \( L = \frac{\sqrt{5}-1}{2} = 0.618\ldots \)

11. Match the description to the series:

- (a) \( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \)
- (b) \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \)
- (c) \( 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots \)
- (d) \( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \)
- (e) Giants beat Rangers in 5 games.

**Solution:** (a) is geometric, (b) is harmonic, (c) is alternating, (d) is a power series, (e) was the 2010 World Series.

12. The Fourier series for \( f(x) = x \) on the interval \([ -\pi, \pi ]\) is given by

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).
\]
where

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \, dx. \]

Compute the coefficients \( b_n \) and write the first five terms of the Fourier series for \( f \).

**Solution:** \[ b_n = \frac{-2 \cos(n\pi)}{n}, \] so

\[ f(x) = 2 \sin(x) - \sin(2x) + \frac{2}{3} \sin(3x) - \frac{1}{2} \sin(4x) + \frac{2}{5} \sin(5x) + \cdots \]

(10) 13. Find the Taylor series for \( f(x) = e^x \) at the point \( x = 1 \). Write using summation notation or show at least five terms.

**Solution:**

\[ f(x) = \sum_{n=0}^{\infty} \frac{e}{n!} (x - 1)^n = e + e(x - 1) + \frac{e}{2} (x - 1)^2 + \frac{e}{6} (x - 1)^3 + \frac{e}{24} (x - 1)^4 + \cdots \]

(10) 14. Find the fifth derivative of \( f(x) = \frac{x}{1 - x^2} \) at \( x = 0 \).

**Solution:**

\[ f(x) = \frac{x}{1 - x^2} = x(1 + x^2 + x^4 + x^6 + \cdots) = x + x^3 + x^5 + x^7 + \cdots \]

so \( f^{(5)}(0) = 5! = 120. \)

(10) 15. Give an example of a power series centered at 3 with radius of convergence equal to 5. Does your series converge at \( x = 0 \)? Does it converge at \( x = 8 \)?

**Solution:**

\[ \sum_{n=0}^{\infty} \frac{(x - 3)^n}{5^n} \]

This series does converge at \( x = 0 \) but not at \( x = 8 \).