We refer here to Ross, Section 13, Example 2. Let
\[ \mathbb{R}^k = \{ x = (x_1, x_2, \ldots, x_k) : x_j \in \mathbb{R} \}. \]
For \( x, y \in \mathbb{R}^k \) define the scalar product on \( \mathbb{R}^k \) by
\[ \langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_k y_k \]
and the norm on \( \mathbb{R}^k \) by
\[ \| x \| = \sqrt{\langle x, x \rangle} = \left( \sum_{j=1}^{k} x_j^2 \right)^{1/2}. \]
Observe that the scalar product is bilinear: \( \langle x + ty, z \rangle = \langle x, z \rangle + t \langle y, z \rangle \) and \( \langle x, y + tz \rangle = \langle x, y \rangle + t \langle x, z \rangle \).

1. Prove the Cauchy-Schwartz inequality by following the indicated steps.

**Lemma 1** (Cauchy-Schwartz inequality). For all \( x, y \in \mathbb{R}^k \), one has
\[ |\langle x, y \rangle| \leq \| x \| \| y \|. \]
(a) Show that the function \( p(t) = \langle x + ty, x + ty \rangle \) is a quadratic function of \( t \) that is non-negative for all \( t \).
(b) Explain why the lemma is true if \( y = 0 \). Suppose \( y \neq 0 \). Substitute \( t_0 = -\langle x, y \rangle / \langle y, y \rangle \) into \( p(t) \), simplify, and combine with (a) to get the result.

2. Use (1) to prove that the norm satisfies the triangle inequality \( \| x + y \| \leq \| x \| + \| y \| \).

3. Prove that \( D(x, y) = \| x - y \| \) defines a metric function on \( \mathbb{R}^k \).

4. Do Exercises 1(a) and 2 at the end of Section 4.2 in Kaplansky.