1. Let \( T : C^\infty(\mathbb{R}) \to \mathbb{R}^2 \) be defined by
\[
T(f) = \left[ \int_0^{2\pi} f(x) \cos x \, dx, \int_0^{2\pi} f(x) \sin x \, dx \right], \quad f \in C^\infty(\mathbb{R}).
\]
Prove that \( T \) is a linear function.

2. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the translation function defined by
\[
T(X) = X + [1, 2]^t.
\]
Prove that \( T \) is not a linear function.

3. Find two square matrices \( A \) and \( B \) of the same size such that \( AB \neq BA \). Justify your answer.

4. Let \( P \) be the parallelogram with vertices \([0, 0]^t, [-2, 3]^t, [2, 0]^t \) and \([0, 3]^t\). Find a linear transformation that transforms \( P \) into the unit square.

5. Referring to Proposition 2 on page 178,
   (a) find two non-zero matrices \( A \) and \( B \) such that \( \text{rank}(AB) = \text{rank}(A) = \text{rank}(B) \).
   (b) Find two matrices \( A \) and \( B \) such that \( \text{rank}(AB) < \text{rank}(A) \) and \( \text{rank}(AB) < \text{rank}(B) \).

6. Let \( A \) be a \( 3 \times 2 \) matrix and \( B \) a \( 2 \times 3 \) matrix.
   (a) What is the size of the matrix \( AB \) ?
   (b) Prove that \( AB \) is not invertible.

7. Let us say that a linear function \( T \) dilates a vector \( X \) if there is some non-zero number \( \lambda \) such that \( T(X) = \lambda X \) (\( \lambda \) is also called an eigenvalue for \( T \)). Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear function given by \( T(X) = AX \) where
\[
A = \begin{bmatrix} 4 & 2 \\
2 & 4 \end{bmatrix}
\]
   (a) Show that \( T \) dilates the vectors \( P_1 = [1, 1]^t \) and \( P_2 = [1, -1]^t \) and find the corresponding eigenvalues.
   (b) A non-singular matrix is “diagonal” if it dilates the standard basis vectors. Now let \( P \) be the matrix with columns \( P_1 \) and \( P_2 \). Show that
\[
P^{-1}AP
\]
is a diagonal matrix. How is this explained in terms of the answer to part (a) ?

(c) Now let \( T : \mathbb{R}^n \to \mathbb{R}^n \) and \( A \) an \( n \times n \) matrix such that \( T(X) = AX \). Suppose that there is a basis \( \{P_1, P_2, \ldots, P_n\} \) for \( \mathbb{R}^n \) such that each vector \( P_j \) is dilated by \( T \), that is, \( T(P_j) = \lambda_j P_j, \ 1 \leq j \leq n \). Prove that \( P^{-1}AP \) is a diagonal matrix.

(Extra credit) Prove the statement if it is true, give a counterexample if it is false: “It is impossible for a linear function \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) to transform the unit square into a triangle.”