(1) (10 pts) Explain why the vectors shown are independent.

\[
\begin{bmatrix}
1 \\
0 \\
1 \\
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
\end{bmatrix}
\]

(2) (15 pts) Consider the following statements.

(I) “A subset of a linearly dependent set is linearly dependent.”

(II) “A set that contains a linearly dependent set is linearly dependent.”

(a) Which of these two statements are true and which are false?

(b) Provide a counterexample for the statement that is false.
(3) (30 pts) Given the system of equations
\[
\begin{align*}
x - y &= 2 \\
2y + 2z &= -4 \\
x &= 0
\end{align*}
\]
(a) write the augmented matrix for this system and find an echelon form for this matrix. Express the solution set in vector form, using a translation vector and spanning vector(s).

(b) Write down the coefficient matrix \( A \). Determine whether the following statement is true or false, and carefully justify your answer: “The set of all \( B \in \mathbb{R}^3 \) such that \( A \cdot X = B \) is solvable is a line in \( \mathbb{R}^3 \).”
(4) (30 pts) Given the $2 \times 4$ matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix}$$

(a) show that the column vector $[-2 \ 5 \ 1 \ 0]^t$ belongs to the nullspace of $A$. Show your work explicitly.

(b) How many spanning vectors are needed to span the nullspace of $A$? Find the nullspace of $A$. (There is a shortcut here.)

(c) Let $B = [b_1 \ b_2]^t$ be any vector in $\mathbb{R}^2$ and let $T$ be a vector in $\mathbb{R}^4$ such that $A \cdot T = B$. What can you say about the complete solution to the system $A \cdot X = B$? Explain.
(5) (15 pts) Let \(X\) and \(Y\) be elements of a vector space \(V\), and let \(W = aX + bY\) for some real numbers \(a\) and \(b\). Suppose that \(X = sY\) for some real number \(s\). (a) Use substitution and two vector space (scalar multiplication) properties to prove that \(W = (as + b)Y\).

(b) In the above situation, what does part (a) prove about the relationship between the two sets: \(\text{span}\{X, Y\}\) and \(\text{span}\{Y\}\)? Explain.