(1) (15 pts) Define the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by the following rule: $T(X)$ is the result of first rotating $X$ counterclockwise by an angle of $\pi/2$ radians, and then multiplying by

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Find a matrix $B$ such that $T(X) = BX$.

(2) (10 pts) Let $A$ and $B$ be $n \times n$ matrices and suppose that $AB$ is invertable. Prove that $A$ and $B$ are invertable. (Suggestion: consider the ranks of $A$, $B$, and $AB$.)
(3) (15 pts) Find a matrix $A$ such that multiplication by $A$ transforms the parallelogram with vertices $(0,0), (2,0), (1,1), (3,1)$ onto the unit square.

(4) (10 pts) Let $A$ be a $2 \times 2$ non-singular matrix and let $\mathcal{B} = \{X_1, X_2\}$ be any basis of $\mathbb{R}^2$. Using any results from class or assignments, explain how you know that $\{AX_1, AX_2\}$ is also a basis of $\mathbb{R}^2$. 
(5) (20 pts) Let $\mathcal{B}$ be the ordered basis of $\mathbb{R}^3$ defined by

$$
\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

(a) Find $P_\mathcal{B}$ and $C_\mathcal{B}$.

(b) Find the $\mathcal{B}$-coordinate vector for $X = [1, 2, 3]^t$.

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(X) = AX$ where

$$
A = \begin{bmatrix}
0 & -1 & 2 \\
1 & 2 & 2 \\
0 & 0 & 3
\end{bmatrix}.
$$

Find the matrix of $T$ relative to the basis $\mathcal{B}$. 
(6) (20 pts) Let $L : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ be defined by

$$L(f) = x^2 f'' + f'.$$

(a) Prove that $L$ is a linear function.

(b) Find the matrix of $L$ relative to the standard basis of $\mathcal{P}_2$.

(7) (10 pts) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and let $T : \mathcal{V} \rightarrow \mathcal{W}$ be a linear function. Define

$$\text{null}(T) = \{X \in \mathcal{V} : T(X) = 0\}.$$

Prove that $\text{null}(T)$ is a subspace of $\mathcal{V}$. 