1. Provide a proof if the statement is true and a counterexample if it is false.

(a) Suppose that \( A \) is an \( n \times n \) invertible matrix and \( B \) is any \( m \times n \) matrix. Then
\[
\text{rank}(B) = \text{rank}(BA).
\]

(b) Suppose that \( A \) is an \( n \times n \) invertible matrix and \( B \) is any \( m \times n \) matrix. Then \( B \) and \( BA \) have exactly the same nullspace.

2. Let \( V \) and \( W \) be two vector spaces, and let \( T : V \rightarrow W \) be a linear function. Define the nullspace of \( T \) by
\[
\text{nullspace}(T) = \{ X \in V : T(X) = 0 \}.
\]

(a) Prove that \( \text{nullspace}(T) \) is a subspace of \( V \).

(b) Prove that the following statements are equivalent.
   (i) \( \text{nullspace}(T) = \{ 0 \} \).
   (ii) \( T \) is injective (i.e., one-to-one.)
   (iii) If \( \mathcal{B} \) is a linearly independent subset of \( V \), then its image \( T(\mathcal{B}) \) is a linearly independent subset of \( W \).

(c) Suppose that \( T \) is a bijection, and let \( \mathcal{B} \) be any basis of \( V \). Prove that the image \( T(\mathcal{B}) \) is a basis of \( W \).

Note: a linear transformation that is a bijection is called an isomorphism.