MT 315-01 091 TEST 3

Name: ____________________________

(1) (15 pts) Define the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by the following rule: \( T(X) \) is the result of first rotating \( X \) counterclockwise by an angle of \( \pi/4 \) radians, and then multiplying by

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}
\]

Find a matrix \( B \) such that \( T(X) = BX \).

(2) (10 pts) Let \( S, T : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) \) be the functions defined by

\[
S(f) = \int_0^1 f(x)dx, \quad \text{and} \quad T(f) = f + 1.
\]

Which one of these functions is linear and which one is not? Justify your answer.
(3) (15 pts) For the given matrix $A$, find a non-zero $3 \times 2$ matrix $B$ such that $AB = 0$. Prove that any such matrix $B$ must have rank 1. (Hint: the columns of $B$ belong to the nullspace of $A$.)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}. $$

(4) (10 pts) Let $A$ be a $2 \times 2$ non-singular matrix and let $B = \{X_1, X_2, X_3\}$ be a subset of $\mathbb{R}^2$ that spans $\mathbb{R}^2$. Prove that $\{AX_1, AX_2, AX_3\}$ also spans $\mathbb{R}^2$. 
(5) (15 pts) Let $A$ be any $m \times m$ invertable matrix and $B$ any $m \times n$ matrix. Prove that $\text{rank}(AB) = \text{rank}(B)$. Explain why this shows that $B$ is invertable if and only if $AB$ is invertable.

(6) (10 pts) Given the matrix $A$ and an eigenbasis $\mathcal{B}$, find the matrix for $T_A$ relative to the basis $\mathcal{B}$.

$$A = \begin{bmatrix} -7 & 3 \\ -18 & 8 \end{bmatrix} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$
(7) (15 pts) Let $V$ and $W$ be vector spaces and let $T : V \rightarrow W$ be a linear function. Suppose that $\mathcal{B}$ is a linearly independent subset of $V$. Prove that if $\text{nullspace}(T) = \{0\}$, then $T(\mathcal{B})$ is linearly independent. (Recall that $\text{nullspace}(T) = \{X \in V : T(X) = 0\}$.)

(8) (10 pts) Let $\mathcal{P}_2$ be the vector space of polynomials of degree no more than 2. Find a basis for the subspace of $\mathcal{P}_2$ spanned by the set

$$\{1 - 2x, 3x + 2x^2, 3 + 4x^2, 2 - x + 2x^2\}.$$ 

Express the other polynomials in this set as linear combinations of the basis elements.