ADMISSIBILITY FOR A CLASS OF QUASIREGULAR REPRESENTATIONS

BRADLEY N. CURREY
Saint Louis University

Abstract. Given a semidirect product $G = N \rtimes H$ where $N$ is nilpotent, connected, simply connected and normal in $G$ and where $H$ is a vector group for which $\text{ad}(h)$ is completely reducible and $\mathbb{R}$-split, let $\tau$ denote the quasiregular representation of $G$ in $L^2(N)$. An element $\psi \in L^2(N)$ is said to be admissible if the wavelet transform $f \mapsto \langle f, \tau(\cdot)\psi \rangle$ defines an isometry from $L^2(N)$ into $L^2(G)$. In this paper we give an explicit construction of admissible vectors in the case where $G$ is not unimodular and the stabilizers in $H$ of its action on $\hat{N}$ are almost-everywhere trivial. In this situation we prove orthogonality relations and we construct an explicit decomposition of $L^2(G)$ into $G$-invariant, multiplicity-free subspaces each of which is the image of a wavelet transform. We also show that, with the assumption of (almost-everywhere) trivial stabilizers, non-unimodularity is necessary for the existence of admissible vectors.

0. Introduction

For the most general notion of continuous wavelet transform, we start with a separable, locally compact topological group $G$, and a unitary representation $\tau$ of $G$ acting in the Hilbert space $\mathcal{H}_\tau$. Given a vector $\psi \in \mathcal{H}_\tau$, we have a linear mapping $W_\psi$ from $\mathcal{H}_\tau$ into the space of bounded continuous functions on $G$ defined by $W_\psi(f) = \langle f, \tau(\cdot)\psi \rangle$. In the event that $W_\psi$ actually defines an isometry of $\mathcal{H}_\tau$ into $L^2(G)$, then we say that $W_\psi$ is a continuous wavelet transform, and that $\psi$ is admissible for $\tau$. When $G$ has Type I reduced dual, the two extreme cases - where $\tau$ is irreducible or where $\tau$ is the regular representation - are well understood [8, 11]. Most closely related to discrete wavelets is the case where $G$ is a semidirect product $G = N \rtimes H$ with $N$ normal, and where $\tau$ is the quasiregular representation of $G$ in $L^2(N)$. The simplest example of this case is the “$ax + b$” group $G = \mathbb{R} \rtimes \mathbb{R}^*_+$, where the quasiregular representation of $G$ in $L^2(\mathbb{R})$ certainly does have admissible vectors, since it is the direct sum of two (square-integrable) irreducible representations. General semidirect products of the form $G = \mathbb{R}^n \rtimes H$, where $H$ is a closed subgroup of $GL(n, \mathbb{R})$, are studied in [13, 22]. There $H$ is said to be admissible if the corresponding quasiregular representation has an admissible vector, and an (almost) characterization of all admissible $H$ is proved.

It is natural then to consider the continuous wavelet transform for the quasiregular representation of $G = N \rtimes H$ when $\mathbb{R}^n$ is replaced by a locally compact, connected, unimodular group $N$. The paper [12] lays out the general theory under that assumption that both of the following conditions hold: (1) for a.e. $\lambda$ belonging to the dual $\hat{N}$, the stabilizer $H_\lambda$ in $H$ is compact and (2) $\hat{N}$ has a co-null subset consisting of finitely many open orbits. There are a number of important situations in which these assumptions hold (see for example [10]). Assumption (1) is certainly a natural one: in the case where $N = \mathbb{R}^n$ it is shown relatively easily in [13] that (1) is in fact a necessary condition for admissibility. The necessity of (1) in the case where $N$ is not abelian remains an open question however, and seems to be quite difficult even in simple examples. On the other hand, easy examples and the general results of [13] show that (2) is not necessary.

In this paper we consider the class of $G = N \rtimes H$ satisfying the following conditions:

\begin{enumerate}
  \item[(i)] $N$ is any connected, simply connected nilpotent Lie group, and
\end{enumerate}

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(ii) $H$ is a vector group acting on $N$ in such a way that the Lie algebra $\text{ad}(\mathfrak{h})$ is completely reducible and $\mathbb{R}$-split.

The group $G$ is exponential, meaning that the exponential map defined on its Lie algebra $\mathfrak{g}$ is a bijection onto $G$. The orbit method applies both to $N$ and $G$, and the relationship between coadjoint orbits in the linear dual $\mathfrak{n}^*$ of $\mathfrak{n}$, and coadjoint orbits of $G$ in $\mathfrak{g}^*$ is well-understood. A great deal is also known about the spectral decomposition of the quasiregular representation in this context [14, 15]. In this paper we clarify the relationship between explicit orbital parametrizations in $\mathfrak{n}^*$ and $\mathfrak{g}^*$ as well. In Section 1 we recall the method of stratification by which the collective orbit structure can be described, applying this method both to $\mathfrak{n}^*$ and to $\mathfrak{g}^*$. With carefully chosen bases for $\mathfrak{n}$ and $\mathfrak{g}$, this procedure yields subsets $\Lambda_\circ$ of $\mathfrak{n}^*$ and $\Lambda$ of $\mathfrak{g}^*$, which parametrize a.e. the duals $\hat{N}$ and $\hat{G}$ respectively, and such that if $p : \mathfrak{g}^* \to \mathfrak{n}^*$ is the restriction map, then $p(\Lambda)$ is explicitly described as a subset of $\Lambda_\circ$. The action of $H$ on $\hat{N}$ is realized a.e. as an action of $H$ on $\Lambda_\circ$, and the Fourier transform of a function in $L^2(\hat{N})$ has domain $\Lambda_\circ$ by means of Pukanzsky’s explicit version of the Plancherel formula. Thus the issues surrounding the conditions (1) and (2) above - the “size” of the stabilizers and the collective structure of the $H$-orbits in $\hat{N}$ - can be addressed in concrete terms.

In Section 1 we show that there is a Zariski open subset $\Lambda$ of $\Lambda_\circ$ and a single vector subgroup $H_0$ of $H$ such that $H_0 = H_\lambda$ holds for all $\lambda \in \Lambda$. Thus, in light of the preceding constructions, condition (1) is simplified: it is just says that $H_0 = (1)$. Nevertheless, it is still an open question as to whether this is necessary for the existence of $\tau$-admissible vectors. Therefore, for the purposes of this paper we make the assumption that condition (1) holds, and hence that $H_0 = (1)$. With this assumption in place, we describe the action of $H$ on $\Lambda$, and obtain an explicit cross-section $\Sigma \subset \Lambda$ for the $H$-orbits in $\Lambda$. It is shown that $p|_\Lambda$ is a bijection onto $\Sigma$. A decomposition of $\tau$ is described in terms of an explicit measure on $\Sigma$. The observation is made that, if $N$ is not abelian, then the irreducible decomposition of $\tau$ has infinite multiplicity. In fact we construct an explicit, direct-sum decomposition of $L^2(N)$ into $\tau$-invariant subspaces $L^2(N)_\beta$ that are pairwise isomorphic and multiplicity-free. In the case where $N = \mathbb{R}^n$, one has $L^2(N)_\beta = L^2(N)$.

By virtue of the results [13, Theorem 1.8] and [11, Theorem 0.2], we expect the existence of admissible vectors to be tied to the non-unimodularity of $G$, and this is shown to be precisely the case. Note that in this context, both $H$ and $N$ are unimodular, so $G$ is non-unimodular if and only if the $H$-action on $N$ is non-unimodular. First we prove a Caldéron condition for the admissibility with respect to the subrepresentations $\tau^\beta$ of $\tau$ acting in $L^2(N)_\beta$. The construction of $\tau^\beta$-admissible vectors is now relatively easy when $G$ is non-unimodular, and we use this construction, together with the relationship between $\Sigma$ and $\Lambda$ described above, to prove the following

**Theorem.** Let $G = N \times H$ where $N$ is a connected, simply connected nilpotent Lie group and $H$ is a vector group such that the Lie algebra $\text{ad}(\mathfrak{h})$ is $\mathbb{R}$-split and completely reducible. Assume furthermore that for a.e. $\lambda \in \hat{N}$, the stabilizer $H_\lambda$ is trivial. Let $\tau$ be the quasiregular representation of $G$ in $L^2(N)$. Then $\tau$ has an admissible vector if and only if $G$ is not unimodular.

Finally, in the case where admissible vectors exist, we generalize the methods of [18] to show that the wavelet transform yields an explicit direct-sum decomposition of the regular representation of $G$ into pairwise isomorphic, multiplicity-free subrepresentations, each of which is isomorphic with $\tau^\beta$.

### 1. Orbital Parameters in $\mathfrak{n}^*$ and in $\mathfrak{g}^*$

We begin by setting some notation. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ of the form $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where $\mathfrak{n}$ is nilpotent, $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$, and where $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ with $\text{ad}(\mathfrak{h})$ completely reducible and $\mathbb{R}$-split. Let $G = N \times H$ be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ (resp. $\mathfrak{n}^*$) be the linear dual of $\mathfrak{g}$ (resp. $\mathfrak{n}$), and let $p : \mathfrak{g}^* \to \mathfrak{n}^*$ be the restriction mapping. For a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ let $\mathfrak{s}^\perp = \{ \xi \in \mathfrak{g}^* \mid [\xi, \mathfrak{s}] = 0 \}$. We denote the coadjoint action of $G$ on $\mathfrak{g}^*$ multiplicatively, as well as the coadjoint action of $N$ on $\mathfrak{n}^*$ and the “restricted coadjoint action” of $G$ on $\mathfrak{n}^*$. For any subset $\mathfrak{t}$ of $\mathfrak{g}$, if $f$ is a linear functional defined on $[\mathfrak{g}, \mathfrak{t}]$, then set

$$\mathfrak{t}^f = \{ Z \in \mathfrak{g} \mid f[Z, T] = 0 \text{ holds for every } T \in \mathfrak{t} \}.$$
If \( t \) is an ideal in \( g \), then \( t^f \) is a subalgebra of \( g \). Recall that for any \( \ell \in g^* \), the Lie algebra \( g(\ell) \) of its stabilizer \( G(\ell) \) in \( G \) is \( g^f \), and similarly for \( f \in n^* \), the Lie algebra of its stabilizer \( N(f) \) in \( N \) is \( n(f) = n^f \cap n \).

Next we summarize some results concerning the classification and parametrization of coadjoint orbits [6, 7].

Let \( g \) be any completely solvable Lie algebra, and choose any Jordan-Holder sequence \((0) = g_0 \subset g_1 \subset \cdots \subset g_n = g \), with ordered basis \( \{Z_1, Z_2, \ldots, Z_n\} \) so that \( Z_j \in g_j - g_{j-1} \). Let \( \delta_j \) be the character of \( G \) such that

\[
\text{Ad}(s)Z_j = \delta_j(s)Z_j \mod g_{j-1}
\]

and let \( d\delta_j \) denote its differential.

(1) To each \( \ell \in g^* \) there is associated an index set \( e(\ell) \subset \{1, 2, \ldots, n\} \) defined by

\[
e(\ell) = \{1 \leq j \leq n \mid g_j \not\subset g_{j-1} + g(\ell)\}.
\]

For a subset \( e \) of \( \{1, 2, \ldots, n\} \), the set \( \Omega_e = \{ \ell \in g^* \mid e(\ell) = e \} \) is \( G \)-invariant. The \( \Omega_e \) are determined by polynomials as follows: to each index set \( e \) one associates the skew-symmetric matrix

\[
M_e(\ell) = [\ell[Z_i, Z_j]]_{i,j \in e}.
\]

Setting

\[
Q_e(\ell) = \det M_e(\ell),
\]

one finds that there is a total ordering \( \prec \) on the set \( E = \{ e \mid \Omega_e \neq \emptyset \} \) such that

\[
\Omega_e = \{ \ell \in g^* \mid Q_{e'}(\ell) = 0 \text{ for all } e' \prec e, \text{ and } Q_e(\ell) \neq 0 \}.
\]

We refer to the collection of non-empty \( \Omega_e \) as the coarse stratification of \( g^* \), and to its elements as coarse layers.

(2) Let \( e \in E \); then \( |e| \) is even, and we set \( d = |e|/2 \). To each \( \ell \in \Omega_e \) there is associated a “polarizing sequence” of subalgebras

\[
g = p_0(\ell) \supset p_1(\ell) \supset \cdots \supset p_d(\ell) = p(\ell),
\]

and an index sequence pair \( i(\ell) = \{i_1 < i_2 < \cdots < i_d\} \) and \( j(\ell) = \{j_1, j_2, \ldots, j_d\} \), having values in \( e(\ell) \), defined by the recursive equations:

\[
i_k = \min\{1 \leq j \leq n \mid g_j \cap p_{k-1}(\ell) \not\subset p_{k-1}(\ell)^f\},
\]

\[
p_k(\ell) = (p_{k-1}(\ell) \cap g_{i_k})^f \cap p_{k-1}(\ell),
\]

and

\[
j_k = \min\{1 \leq j \leq n \mid g_j \cap p_{k-1}(\ell) \not\subset p_k(\ell)\}.
\]

For each \( k \), \( i_k < j_k \), and \( e(\ell) \) is the disjoint union of the values of \( i(\ell) \) and \( j(\ell) \). Note that since \( i(\ell) \) must be increasing, it is determined by \( e(\ell) \) and \( j(\ell) \). For any splitting of \( e \) into such a sequence pair \( (i, j) \) we set \( \Omega_{e,i,j} = \{ \ell \in \Omega_e \mid j(\ell) = j \} \). These sets are also algebraic and \( G \)-invariant, and we refer to the collection of non-empty \( \Omega_{e,i,j} \) as the fine stratification of \( g^* \). For \( 1 \leq k \leq d \), if we set

\[
M_{e,k}(\ell) = [\ell[Z_i, Z_j]]_{i,j \in \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}}
\]

let \( \text{Pf}_{e,k}(\ell) \) denote the Pfaffian of \( M_{e,k}(\ell) \), and let \( \text{P}_{e,j}(\ell) = \text{Pf}_{e,1}(\ell)\text{Pf}_{e,2}(\ell) \cdots \text{Pf}_{e,d}(\ell) \), then there is a total ordering \( \prec \prec \) on the pairs \( e, j \) such that

\[
\Omega_{e,j} = \{ \ell \in g^* \mid \text{P}_{e',j'}(\ell) = 0 \text{ for all } (e', j') \prec \prec (e, j) \text{ and } \text{P}_{e,j}(\ell) \neq 0 \}.
\]
The following rational functions are naturally associated with the fine stratification. Fix $\ell \in \Omega$. Define $\rho_0(Z, \ell) = Z$; assume that $\rho_{k-1}(Z, \ell)$ is defined and set

$$
\rho_k(Z, \ell) = \rho_{k-1}(Z, \ell) - \frac{\ell[\rho_{k-1}(Z, \ell), \rho_{k-1}(Z_{i_k}, \ell)]}{\ell[\rho_{k-1}(Z_{j_k}, \ell), \rho_{k-1}(Z_{i_k}, \ell)]} \rho_{k-1}(Z_{j_k}, \ell)
$$

Set $Y_k(\ell) = \rho_{k-1}(Z_{i_k}, \ell)$, and $X_k(\ell) = \rho_{k-1}(Z_{j_k}, \ell)$, $1 \leq k \leq d$; then it can be shown [2, Lemma 1.5] that for each $1 \leq k \leq d$,

$$
\text{Pr}_{e,k}(\ell) = \ell[Y_1(\ell), X_1(\ell)] \ell[Y_2(\ell), X_2(\ell)] \cdots \ell[Y_k(\ell), X_k(\ell)].
$$

If we set

$$
m_k(\ell) = \text{span}\{Y_1(\ell), Y_2(\ell), \ldots, Y_k(\ell), X_1(\ell), X_2(\ell), \ldots, X_k(\ell)\},
$$

then for each $\ell \in \Omega$, $g = m_k(\ell) \oplus m_k(\ell)^{\ell}$ and $\rho_k(Z, \ell)$ is the projection of $Z$ into $m_k(\ell)^{\ell}$ parallel to $m_k(\ell)$. It follows that

$$
\ell[\rho_k(Z, \ell), \rho_k(T, \ell)] = \ell[\rho_k(Z, \ell), T], \ Z, T \in g, \ell \in g^*.
$$

The functions $\rho_k(\cdot, \ell)$ have the additional properties:

(i) $\rho_k(g_j, \ell) \subset g_j$, $1 \leq j \leq n, 0 \leq k \leq d$, and

(ii) $\rho_k(g, \ell) \cap g_{k+1} \subset g(\ell), 0 \leq k \leq d - 1$.

Finally, if $\alpha$ is an automorphism of $g$ such that $\alpha(g_j) = g_j$ holds for every $j$, then $\alpha^*$ leaves each fine layer invariant.

(3) Now fix a layer $\Omega_{e,j}$ in the fine stratification. For each $\ell \in \Omega_{e,j}$, define the “dilation set”

$$
\varphi(\ell) = \{j \in e | g_j^{\ell} \cap \ker(d\delta_j) = g_j^{\ell} \cap \ker(d\delta_j)\}.
$$

The index set $\varphi(\ell)$ identifies those directions in the orbit of $\ell$ where the coadjoint action of $G$ “dilates” by the character $\delta_j^{-1}$. The indices in $\varphi(\ell)$ are included in the values of the sequence $i$, and are defined by

$$
\varphi(\ell) = \{i_k | d\delta_{i_k}(X_k(\ell)) \neq 0\}.
$$

There are examples where $\varphi(\ell)$ is not constant on the fine layer. For each subset $\varphi$ of the values of $i$, the set $\Omega_{e,j,\varphi} = \{\ell \in \Omega_{e,j} | \varphi(\ell) = \varphi\}$ is an algebraic subset of $\Omega_{e,j}$, and we refer to this further refinement of the fine stratification as the ultra-fine stratification of $g^\varphi$. The ultra-fine stratification also has an ordering for which the minimal layer is a Zariski open subset of the minimal fine layer.

(4) Now fix an ultra-fine layer $\Omega = \Omega_{e,j,\varphi}$, and for $\ell \in \Omega$, $j = i_k \in \varphi$, set

$$
q_j(\ell) = \frac{d\delta_j(X_k(\ell))}{\ell[X_k(\ell), Z_j]}.
$$

Let

$$
V = V_{e,\varphi} = \{\ell \in g^\varphi | \text{if } j \in e - \varphi, \text{ then } \ell(Z_j) = 0\}
$$

Then the set

$$
\Lambda = \Lambda_{e,j,\varphi} = \{\ell \in V \cap \Omega | \text{ and for every } j \in \varphi, |q_j(\ell)| = 1\}
$$

is a topological cross-section for the orbits in $\Omega$. If $g$ is nilpotent, then the ultra-fine stratification coincides with the fine stratification and $\Lambda = V \cap \Omega$. 

4
We now return to the case where \( g = n \oplus \mathfrak{h} \) as described above, and we apply the stratification procedure first to the nilpotent Lie algebra \( \mathfrak{n} \). We fix once and for all an ordered basis \( \{ Z_1, Z_2, \ldots, Z_n \} \) of \( \mathfrak{n} \) for which both

(i) \( \mathfrak{n}_j = \text{span} \{ Z_1, Z_2, \ldots, Z_j \} \) is an ideal in \( \mathfrak{g} \) and

(ii) for each \( A \in \mathfrak{h} \), \( Z_j \) is an eigenvector for \( \text{ad} A \).

hold for all \( 1 \leq j \leq n \). Having chosen the basis \( Z_1, Z_2, \ldots, Z_n \) for \( \mathfrak{n} \), let \( \Omega^o \) be the minimal (and hence Zariski open) fine layer in \( \mathfrak{n}^* \), with \( \Lambda^o \) its cross-section. Denote the objects referred to in (1)-(3) above by \( \mathfrak{e}^o, \mathfrak{i}^o, \mathfrak{j}^o \), and \( \rho^o_k \). For each \( 1 \leq j \leq n \), set \( e_j = Z_j^* \in \mathfrak{n}^* \) and set \( \gamma_j = -d\delta_j \) so that

\[
\text{ad}^o A(e_j) = \gamma_j(A) e_j, \quad A \in \mathfrak{h}.
\]

For each \( h \in H \), since \( \text{Ad}^o(h)(\Omega^o) = \Omega^o \) and the \( e_j \) are eigenvectors of \( \text{Ad}^o(h) \), we have that \( \text{Ad}^o(h)(\Lambda^o) = \Lambda^o \).

With this in mind we choose a convenient basis for \( \mathfrak{h} \). Set \( c = n - 2d^o \), write \( \{ 1, \ldots, n \} - \mathfrak{e}^o = \{ u_1 < u_2 < \cdots < u_c \} \), and set \( \lambda_a = \ell(Z_{u_a}), 1 \leq a \leq c \). Then \( \ell \to \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_c) \) identifies \( \Lambda^o \) with a Zariski open subset of \( \mathbb{R}^c \). We select a subset \( \alpha_v, 1 \leq v \leq r \) of \( \gamma_{u_a}, 1 \leq a \leq c \) as follows: \( a_1 = \min \{ 1 \leq a \leq c \mid \gamma_{u_a} \neq 0 \} \); \( a_2 = \min \{ 1 \leq a \leq c \mid \gamma_{u_a} \text{ is not a multiple of } \gamma_{u_{a_1}} \} \); \( a_3 = \min \{ 1 \leq a \leq c \mid \gamma_{u_a} \text{ is not in the span of } \gamma_{u_{a_1}}, \gamma_{u_{a_2}} \} \); and so on, until for some \( r > 0 \), every \( \gamma_j \) belongs to the span of \( \gamma_{u_{a_1}}, \gamma_{u_{a_2}}, \ldots, \gamma_{u_{a_r}} \). Set

\[
\alpha_v = \gamma_{u_{a_v}}, \quad 1 \leq v \leq r.
\]

We shall refer to the set \( \{ \alpha_v \mid 1 \leq v \leq r \} \) as the minimal spanning set of roots with respect to the orbital cross-section \( \Lambda^o \). We shall use the notation \( \mathfrak{h}_v = \cap_{v-1}^v \ker \alpha_w, 1 \leq v \leq r \). We now make an important observation: let \( f \in \Lambda^o \); for each \( j \in \mathfrak{e}^o \), \( f(Z_j) = 0 \), and if \( j \notin \mathfrak{e}^o, 1 \leq j \leq n \), then \( \mathfrak{h}_v \subset \ker \gamma_j \). It follows that \( \mathfrak{h}_r \subset \mathfrak{n}^* \) holds for every \( f \in \Lambda^o \).

Let \( \{ A_1, A_2, \ldots, A_r \} \subset \mathfrak{h} \) be a basis of \( \mathfrak{h} \) mod \( \mathfrak{h}_r \) that is dual to the minimal spanning set of roots, so that \( \alpha_v(A_w) = 0 \) or 1 according as \( v \neq w \) or \( v = w \). Choosing a basis \( \{ A_{r+1}, \ldots, A_p \} \) for \( \mathfrak{h}_r \), we fix from now on the ordered basis \( \{ A_1, A_2, \ldots, A_p \} \) for \( \mathfrak{h} \). With the ordered Jordan-Hölder basis \( \{ Z_1, Z_2, \ldots, Z_n, A_p, A_{p-1}, \ldots, A_1 \} \) for \( \mathfrak{g} \) in place we apply the stratification procedure to \( \mathfrak{g}^* \) as described above (of course we could rename \( Z_m = A_1, Z_{m-1} = A_2 \), etc.) Let \( \Omega = \Omega_{\mathfrak{e}^o, \mathfrak{j}^o} \) be the minimal, Zariski open, fine layer in \( \mathfrak{g}^* \). Write the defining index sequence pair as \( \mathfrak{i}^o = \{ i_1 < i_2 < \cdots < i_d \}, \mathfrak{j}^o = \{ j_1, j_2, \ldots, j_d \} \) so that \( 2d \) is the dimension of the coadjoint orbits in \( \Omega \). Set

\[
K^o = \{ 1 \leq k \leq d \mid j_k \leq n \} = \{ k_1 < k_2 < \cdots < k_d \}.
\]

**Lemma 1.1.** One has \( p(\Omega) \subset \Omega^o \), and the index sequence pair for \( \Omega^o \) is

\[
\mathfrak{i}^o = \{ i_{k_1} < i_{k_2} < \cdots < i_{k_d} \}, \quad \mathfrak{j}^o = \{ j_{k_1}, j_{k_2}, \ldots, j_{k_d} \}.
\]

**Proof.** By [1, Lemma 2.2], \( p(\Omega) \) is contained in the layer \( \Omega_{\mathfrak{e}^o, \mathfrak{j}^o}^o \) of \( N \)-orbits in \( \mathfrak{n}^* \) whose index data is the above. At the same time we have that \( p(\Omega) \) is open in \( \mathfrak{n}^* \), and since \( \Omega^o \) is dense in \( \mathfrak{n}^* \), it follows that \( \Omega^o = \Omega_{\mathfrak{e}^o, \mathfrak{j}^o}^o \).

The next lemma is proved in [1, Lemma 4.2], and clarifies the relationship between the functions \( \rho_k, 0 \leq k \leq d \) and \( \rho^o_k, 1 \leq r \leq d^o \).

5
Lemma 1.2. Fix $k = k_r \in K^\circ$, $\ell \in \Omega$, and set $f = p(\ell)$. Set $Y_{k_r}(\ell) = \text{span} \{ Y_h(\ell) \mid 1 \leq h \leq k_r - 1, h \notin K^\circ \}$ We have each of the following.

(a) $Y_{k_r}(\ell) \subset n(f)$.
(b) For each $k_{r-1} < h < k_r$,
\[ \rho_h(Z, \ell) = \rho_{k-1}^o(Z, f) \mod Y_{k_r}(\ell) \]
holds for all $Z \in n$.
(c) For any $Z \in n$, $\ell[Z, Y_{k_r}(\ell)] = f[Z, Y_{k_r}^o(f)]$ and $\ell[Z, X_{k_r}(\ell)] = f[Z, X_{k_r}^o(f)]$
(d) $\rho_k(Z, \ell) = \rho_1(Z, f) \mod Y_{k_r}(\ell)$ holds for all $Z \in n$.

We now focus on the special properties of the stratification procedure on $g$ when applied to the elements $\ell \in p^{-1}(\Lambda^o)$.

Lemma 1.3. Let $\ell \in \Omega$ such that $f = p(\ell) \in \Lambda^o$.

(a) One has $\rho_k(\ell, \ell) \in \mathfrak{h}$, $1 \leq k \leq d$.
(b) For each $j \in \mathfrak{e}^o$, $A \in \mathfrak{h}$, one has $\ell[\rho_k(A, \ell), Z_j] = \ell[A, Z_j] = 0$, $1 \leq k \leq d$.

Proof. We proceed by induction on $k$; if $k = 0$, then $\rho_0[\cdot, \ell]$ is the identity map and both statements (a) and (b) are clear. Suppose that $k \geq 1$ and that (a) and (b) hold for $k - 1$.

To prove (a) for $k$, let $A \in \mathfrak{h}$. The assumption that (a) holds for $k - 1$ says that $\rho_{k-1}(A, \ell)$ belongs to $\mathfrak{h}$. Suppose first that $j_k > n$. Then the assumption that (a) and (b) hold for $k - 1$ also gives $X_k(\ell) \in \mathfrak{h}$, and since $\mathfrak{h}$ is abelian, $[A, X_k(\ell)] = 0$. Thus
\[ \rho_k(A, \ell) = \rho_{k-1}(A, \ell) - \frac{\ell[A, Y_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]} X_k(\ell) - \frac{\ell[A, X_k(\ell)]}{\ell[Y_k(\ell), X_k(\ell)]} Y_k(\ell) \]
belongs to $\mathfrak{h}$. On the other hand if $j_k \leq n$, then the assumption that (b) holds for $k - 1$ says that $\ell[A, X_k(\ell)] = \ell[A, Y_k(\ell)] = 0$, hence $\rho_k(A, \ell) = \rho_{k-1}(A, \ell)$ belongs to $\mathfrak{h}$ in this case. This completes the induction step for part (a).

As for (b), let $j \in \mathfrak{e}^o$ and let $A \in \mathfrak{h}$; we need only show that $\ell[A, \rho_k(Z_j, \ell)] = \ell[A, \rho_{k-1}(Z_j, \ell)]$. As before, we suppose first that $j_k > n$, so that we have $X_k(\ell) \in \mathfrak{h}$ and $\ell[A, X_k(\ell)] = 0$. The assumption that (b) holds for $k - 1$ now gives
\[ \ell[Z_j, X_k(\ell)] = \ell[Z_j, \rho_{k-1}(Z_j, \ell)] = 0. \]
Hence
\[ \ell[A, \rho_k(Z_j, \ell)] = \ell[A, \rho_{k-1}(Z_j, \ell)] - \frac{\ell[Z_j, Y_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]} \ell[A, X_k(\ell)] - \frac{\ell[Z_j, X_k(\ell)]}{\ell[Y_k(\ell), X_k(\ell)]} \ell[A, Y_k(\ell)] = \ell[A, \rho_{k-1}(Z_j, \ell)]. \]

For the case $j_k \leq n$, the assumption that (b) holds for $k - 1$ gives immediately $\ell[A, X_k(\ell)] = \ell[A, Y_k(\ell)] = 0$, whence $\ell[A, \rho_k(Z_j, \ell)] = \ell[A, \rho_{k-1}(Z_j, \ell)]$. This completes the proof.
Lemma 1.4. Let $\ell \in \Omega$ such that $f = p(\ell) \in \Lambda^\circ$. Assume that $\{1, 2, \ldots, d\} - K^\circ$ is non-empty and write $\{1, 2, \ldots, d\} - K^\circ = \{h_1 < h_2 < \cdots\}$. Choose an index $h_v \in \{1, 2, \ldots, d\} - K^\circ$.

(a) For $0 \leq k < h_v$, one has $p_k(A_v, \ell) = A_v$.
(b) One has $v \leq r$ and $\{i_1 < i_2 < \cdots < i_v\} = \{u_{a_1} < u_{a_2} < \cdots < u_{a_r}\}$.
(c) One has $\{j_h, j_{h_2} = m - 1, \ldots, j_{h_v} = m - v + 1\}$.

Proof. Suppose that $v = 1$; we repeat the argument for Lemma 1.3 (a) with the additional fact that the case $j_k > n$ cannot occur here, as $h_1 = \min\{1 \leq k \leq d \mid j_k > n\}$. It follows immediately that $p_k(A_1, \ell) = A_1, 1 \leq k < h_1$.

Now set $u = u_{a_1}, i = i_{h_1}$ and $j = j_{h_1}$: we show that $u = i$. First we claim that $u \leq i$. To see this, note that by definition of $u$, $\ell[h, g_{u-1}] = 0$. If $u > i$ were true, then $h \subset g_i^\ell$ and $g_i \subset h^\ell$. The first of these inclusions implies that

$$p_{h_1-1}(\ell) = p_{h_1-1}(\ell) \cap n + h.$$ 

Since $i \notin I^\circ$, $g_i \cap p_{h_1-1}(\ell) \subset (p_{h_1-1}(\ell) \cap n)^\ell$. This together with the second inclusion above gives

$$g_i \cap p_{h_1-1}(\ell) \subset (p_{h_1-1}(\ell) \cap n)^\ell \cap h^\ell$$

$$
\subset (p_{h_1-1}(\ell) \cap n + h)^\ell
$$

$$= (p_{h_1-1}(\ell))^\ell$$

contradicting the definition of $i = i_{h_1}$. Thus the claim is proved. In light of this and the fact that $(e - e^\circ) \cap \{1, 2, \ldots, n\} = i - I^\circ$, it remains to show that $u \in e$. Suppose then that $u \notin e$; then for any $\ell \in \Omega$, we have $Z_u = T(\ell) + W(\ell)$ where $T(\ell) \in g(\ell)$ and $W(\ell) \in g_{u-1}$. But again since $\ell[h, g_{u-1}] = 0$, it follows that

$$\ell(Z_u) = \gamma_u(A_1)\ell(Z_u) = \ell[A_1, Z_u] = \ell[A_1, T(\ell)] + \ell[A_1, W(\ell)] = 0$$

holds for all $\ell \in \Omega$, which is impossible since $\Omega$ is dense in $g^\circ$.

Next we show that $j = m$. Observe that $g_{m-1} = n + h_1$ and $h_1 \subset p_{h_1}(\ell) \subset p_{h_1-1}(\ell)$. On the other hand since $i \in I^\circ$, we have $j > n$ and $p_{h_1-1}(\ell) \cap n \subset p_{h_1}(\ell)$. It follows that

$$p_{h_1-1}(\ell) \cap g_{m-1} = p_{h_1-1}(\ell) \cap n + h_1 \subset p_{h_1}(\ell),$$

which means that $j = m$.

Now suppose that $v > 1$ and that the proposition holds for $1 \leq w \leq v - 1$. To prove part (a) for $v$, we let $0 < k < h_v$. We proceed by induction on $k$, the statement being clear when $k = 0$. If $k \in K^\circ$, then by Lemma 1.3 we have $\ell[A_v, X_k(\ell)] = \ell[A_v, Y_k(\ell)] = 0$, and hence $p_k(A_v, \ell) = p_{k-1}(A_v, \ell)$. If $k \notin K^\circ$, say $k = h_v$, then by our induction hypothesis, $i_k = u_{a_w}, j_k = m - w + 1$, and $X_k(\ell) = A_w$. Hence $\ell[A_v, X_k(\ell)] = \ell[A_v, A_w] = 0$ and

$$\ell[A_v, Y_k(\ell)] = \ell[p_{k-1}(A_v, \ell), Z_{k_{h_v}}] = \ell[A_v, Z_{k_{h_v}}] = 0.$$ 

So $p_k(A_v, \ell) = p_{k-1}(A_v, \ell)$ in this case also. Now by induction on $k$, part (a) is true for $v$.

As for part (b), set $u = u_{a_v}, i = i_{h_v}$, and $j = j_{h_v}$. Then $[h_{v-1}, g_{u-1}] = 0$. Imitating the argument above for the case $v = 1$ we see that the assumption that $u > i$ leads to the inclusions $h_{v-1} \subset g_i^\ell$ and $g_i \subset h_v^\ell$. In the same way as when $v = 1$, I claim that

$$p_{h_{v-1}}(\ell) = p_{h_{v-1}}(\ell) \cap n + h_{v-1}.$$ 

To see this, note that $h_{v-1} \subset p_{h_{v-1}}(\ell)$ so obviously $p_{h_{v-1}}(\ell) \supset p_{h_{v-1}}(\ell) \cap n + h_{v-1}$. Counting dimensions gives equality: $\dim(p_{h_{v-1}}(\ell)) = m - h_v + 1$ and

$$\dim(p_{h_{v-1}}(\ell) \cap n) = n - \{|i_k \in I^\circ \mid k \leq h_v - 1\} = n - (h_v - 1 - (v - 1)) = n - (h_v - v),$$
\[
\dim((p_{h_0 - 1}(\ell) \cap n) + h_{v-1}) = n - (h_v - v) + p - v + 1 = m - h_v + 1 = \dim(p_{h_0 - 1}(\ell)).
\]

Now we follow the same line of reasoning as in the case \(v = 1\) verbatim to arrive at a contradiction, thereby concluding that \(u \leq i\). Since by induction, we already have \(i_{h_w} = u_{a_w}, 1 \leq w \leq v - 1\), then \(i_{h_w} < u\) for \(1 \leq w \leq v - 1\). Now arguing as in the case \(v = 1\), we find that it remains to show that \(u \in \mathcal{e}\). But again, the argument for this is identical to the case \(v = 1\): if \(u \notin \mathcal{e}\), then we find that \(\ell(Z_u) = \ell[A_v, Z_u] = 0\) holds for all \(\ell \in \Omega\), etc.

Finally we show that \(j = m - v + 1\). As in the case \(v = 1\), \(g_{m-v+1} = n + h_{v-1}\) and \(h_v \subset p_{h_v}(\ell) \subset p_{h_v-1}(\ell)\). Also \(i \in i^*\), so \(j > n\) and \(p_{h_v-1}(\ell) \cap n \subset p_{h_v}(\ell)\). It follows that

\[
p_{h_v-1}(\ell) \cap g_{m-v} = p_{h_v-1}(\ell) \cap n + h_v \subset p_{h_v}(\ell).
\]

Since we already have \(j_{h_w} = m - w + 1, 1 \leq w \leq v - 1\), then \(j = m - v + 1\) follows. This completes the proof.

**Lemma 1.5.** Let \(d - d^\circ < w \leq p\). Then for each \(\ell \in \Lambda^\circ\) and \(0 \leq k \leq d\), one has \(p_k(A_w, \ell) = A_w\).

**Proof.** As usual we proceed by induction on \(k\), the case \(k = 0\) being clear. Suppose that \(k \geq 1\) and that the lemma holds for \(k - 1\). If \(k \in K^w\), then Lemma 1.3 gives \(\ell[A_w, X_k(\ell)] = \ell[A_w, Y_k(\ell)] = 0\). If \(k = h_v \in \{1, 2, \ldots, d\} - K^w\), then Lemma 1.4 gives \(X_k(\ell) = A_v\) and \(Y_k(\ell) = p_{k-1}(Z_{u_w}, \ell)\) so that in this case also \(\ell[A_w, X_k(\ell)] = \ell[A_w, Y_k(\ell)] = 0\). In either case then, we have \(p_k(A_w, \ell) = p_{k-1}(A_w, \ell)\).

**Proposition 1.6.** Let \(\mathfrak{g}\) be a completely solvable Lie algebra of the form \(\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}\) where \(\mathfrak{n}\) is a nilpotent ideal and \(\mathfrak{h}\) is an abelian subalgebra such that \(ad(\mathfrak{h})\) is completely reducible. Let \(\{Z_1, Z_2, \ldots, Z_n, A_p, A_{p-1}, \ldots, A_2, A_1\}\) be an ordered Jordan-Holder basis of \(\mathfrak{g}\) with the following properties.

(i) \(\{Z_1, Z_2, \ldots, Z_n\}\) is a basis of \(\mathfrak{n}\) with respect to which \(ad(\mathfrak{h})\) is diagonalized.

(ii) \(\{A_1, A_2, \ldots, A_r\}\) is dual to the minimal spanning set of roots and \(\{A_{r+1}, \ldots, A_p\}\) is a basis for \(\mathfrak{h}_r\) where \(\mathfrak{h}_r\) is defined as above.

Let \(\Omega = \Omega_{e_j}\) be the minimal fine layer in \(\mathfrak{g}^*\) and \(\Omega^\circ = \Omega_{e^w, j^w}\) the minimal fine layer in \(\mathfrak{n}^*\), with respect to the above bases chosen. Write \(K^\circ\) and \(\{1, 2, \ldots, d\} - K^\circ = \{h_1 < h_2 < \cdots < h_{d-d^\circ}\}\) as above. Let \(p_{\mathfrak{g}^w, 1} \leq w \leq d^\circ\) be the Pfaffian polynomials that define \(\Omega^\circ\). Then one has the following.

(a) \(d - d^\circ = r\), and the increasing sequence \(\{i_{h_1}, i_{h_2}, \ldots, i_{h_r}\}\) is precisely the sequence \(\{u_{a_1} < u_{a_2} < \cdots < u_{a_r}\}\) corresponding to the minimal spanning set of roots.

(b) \(j_{h_w} = m - v + 1, 1 \leq v \leq r\).

(c) Let \(\ell \in \Omega \cap p^{-1}(\Lambda^\circ)\) with \(f = p(\ell)\). For each \(1 \leq k \leq d\), let \(u_0 = \max\{1 \leq v \leq r \mid h_v \leq k\}\) and \(w_0 = \max\{1 \leq w \leq d^\circ \mid k_w \leq k\}\). Then

\[
P_{\mathfrak{g}^w, k}(\ell) = \prod_{v=1}^{u_0} \ell(Z_{i_{h_v}}) P_{\mathfrak{g}^w, w_0}(f).
\]

(d) For every \(\ell \in \Omega\), the dilation set \(\varphi(\ell)\) is precisely the set \(\{i_k \mid k \notin K^\circ\} = \{i_{h_v} \mid 1 \leq v \leq r\}\) and hence the minimal fine layer in \(\mathfrak{g}^\circ\) coincides with the minimal ultra-fine layer.

**Proof.** It follows from Lemma 1.4 that the sequence \(\{i_{h_1} < i_{h_2} < \cdots < i_{h_{d-d^\circ}}\}\) coincides with the first \(d - d^\circ\) terms of the sequence \(\{u_{a_1} < u_{a_2} < \cdots < u_{a_r}\}\). Now if \(d - d^\circ < w \leq r\), then Lemma 1.5 implies that \(A_w \in \mathfrak{g}(\ell)\) holds for all \(\ell \in p^{-1}(\Lambda^\circ)\). But this means that

\[
f(Z_{u_w}) = f[A_w, Z_{u_w}] = 0
\]
holds for all \( f \in \Lambda^\circ \). Since \( \Lambda^\circ \) is a dense open subset of \( V = \text{span}\{e_{u_1}, e_{u_2}, \ldots, e_{u_r}\} \), this is impossible. Thus part (a) is proved. Part (b) now follows from Lemma 1.4.

For part (c) we compute using Lemma 1.2, Lemma 1.4, and the properties of \( \rho_k \):

\[
\ell[Y_1(\ell), X_1(\ell)] \ell[Y_2(\ell), X_2(\ell)] \cdots \ell[Y_k(\ell), X_k(\ell)] = \ell[Z_{i_1}, Z_{j_{i_1}}] \ell[Z_{i_2}, \rho_1(Z_{j_{i_2}}, \ell)] \cdots \ell[Z_{i_k}, \rho_{k-1}(Z_{j_{i_k}}, \ell)]
\]

\[
= \prod_{w=1}^{w_0} \ell[Z_{i_{k_w}}, \rho_{w-1}(Z_{j_{k_w}}, \ell)] \prod_{w=1}^{w_0} \ell[Z_{i_{k_w}}, \rho_{w-1}(Z_{j_{k_w}}, \ell)]
\]

\[
= \prod_{w=1}^{w_0} \ell[Z_{i_{k_w}}, A_v] \prod_{w=1}^{w_0} f[Z_{i_{k_w}}, \rho_{w-1}(Z_{j_{k_w}}, f)]
\]

\[
= \left( \prod_{w=1}^{w_0} \ell(Z_{i_{k_w}}) \right) \text{Pf}_{e^v, u_0}(f)
\]

Finally for part (d), Lemma 1.4, part (a) shows that for \( k = h_v \notin \Lambda^\circ \), we have \( X_k(\ell) = A_v \), hence

\[
\varphi(\ell) = \{ i_k \in i \mid d \delta_{i_k}(X_k(\ell)) \neq 0 \} = \{ i_k \in i \mid k \notin \Lambda^\circ \}
\]

holds for each \( \ell \in \Omega \).

**Corollary 1.7.** Let \( \ell \in \Omega \cap p^{-1}(\Lambda^\circ) \) with \( f = p(\ell) \). Then one has

\[
\dim(h/h \cap g(\ell)) = \frac{1}{2} (\dim(g/g(\ell)) - \dim(n/n(f))).
\]

**Proof.** This amounts to showing that

\[
h_r = \cap_{\ell=1}^{\ell} \ker \alpha_v = h \cap g(\ell)
\]

holds for each \( \ell \in \Omega \cap p^{-1}(\Lambda^\circ) \). It is already clear that for such \( \ell \), \( h_r \subset h \cap g(\ell) \). On the other hand, if \( A \in h \cap g(\ell) \), then for each \( 1 \leq v \leq r \),

\[
\alpha_v(A)(Z_{i_{h_v}}) = -\ell[A, Z_{i_{h_v}}] = 0.
\]

From Proposition 1.6 (c), we have \( \ell(Z_{i_{h_v}}) \neq 0 \), hence \( A \in \ker \alpha_v \), and the equation above is proved. Now

\[
\dim(h/h \cap g(\ell)) = r = d - d^\circ = \frac{1}{2} (\dim(g/g(\ell)) - \dim(n/n(f))).
\]

**Corollary 1.8.** With the hypothesis of Proposition 1.6, we have

\[
p(\Omega) \cap \Lambda^\circ = \{ f \in \Lambda^\circ \mid f(Z_{i_{h_v}}) \neq 0, \text{ holds for all } 1 \leq v \leq r \}
\]

and

\[
p(\Lambda) = \{ f \in \Lambda^\circ \mid |f(Z_{i_{h_v}})| = 1, \text{ holds for all } 1 \leq v \leq r \}.
\]

**Proof.** Recall that

\[
\Omega = \{ \ell \in g^* \mid \text{Pf}_{e^j, h}(\ell) \neq 0 \};
\]
and that
\[ \Omega^\circ = \{ f \in n^* \mid Pf_{e^v,j^v}(f) \neq 0 \}. \]

By Proposition 1.6 part (c), if \( f = p(\ell) \in \Lambda^\circ \), then
\[ Pf_{e^v,j^v}(\ell) = R(f)Pf_{e^v,j^v}(f) \]
where \( R(f) \) is a product of the factors \( f(Z_{i_{hv}}), 1 \leq v \leq r \). These observations mean that \( f \in p(\Omega) \cap \Lambda^\circ \) if and only if \( f \in \Lambda^\circ \) and \( R(f) \neq 0 \). The first equation above follows.

As for the second, set \( V = \{ \ell \in g^* \mid \ell(Z_j) = 0 \text{ for all } j \in e - \varphi \} \) and \( V^\circ = \{ f \in n^* \mid f(Z_j) = 0 \text{ for all } j \in e^\circ \} \). Observe that, by virtue of preceding results, we have \( p(V) = V^\circ \) and \( p(\Omega) \cap V^\circ = p(\Omega) \cap \Lambda^\circ \). Now from Proposition 1.6 (d) and the definition of the cross-section \( \Lambda \), we have
\[ \Lambda = \{ \ell \in V \cap \Omega \mid |q_{i_{hv}}(\ell)| = 1, \text{ holds for all } 1 \leq v \leq r \}. \]

Let \( f \in p(\Lambda), f = p(\ell) \) for some \( \ell \in \Lambda \). Then \( f \in p(V) = V^\circ \), and \( f \in p(\Omega) \subset \Omega^\circ \), so \( f \in \Lambda^\circ \). But now an examination of the definition of \( q_j \) together with the observation that \( X_{i_{hv}}(\ell) = \Lambda \) gives \( q_{i_{hv}}(\ell)^{-1} = \ell(Z_{i_{hv}}) \). Hence \( f \) belongs to the right hand side of the above equation.

On the other hand, let \( f \in \Lambda^\circ \) with \( |f(Z_{i_{hv}})| = 1, 1 \leq v \leq r \). Let \( \ell \in p^{-1}(f) \cap V \). By definition of \( \Omega \), we have \( \ell \in \Omega \cap V \), and \( |\ell(Z_{i_{hv}})| = 1, 1 \leq v \leq r \). Hence \( \ell \in \Lambda \) and \( f \in p(\Lambda) \).

2. The Wavelet Transform

In this section we apply the algebraic constructions of Section 1 in order to address the question of admissibility. Denote by \( Irr(N) \) the Borel space of irreducible unitary representations of \( N \), and by \( \tilde{N} \) the Borel space of unitary equivalence classes in \( Irr(N) \). Let \( \kappa : n^*/N \to \tilde{N} \) be the canonical Kirillov correspondence. With the constructions of Section 1 in place, we associate to each linear functional \( f \in n^* \) a specific irreducible representation \( \pi_f \) whose equivalence class is \( \kappa^\circ(Nf) \), as follows. First of all the basis \( \{ Z_1, Z_2, \ldots, Z_n \} \) provides us with global coordinates on \( N \) via the exponential mapping, and Lebesgue measure becomes Haar measure on \( N; d(exp X) = dX, X \in n. \) We denote this measure by \( dx \). Next we partition \( n^* \) by the fine stratification, and let \( \Omega^\circ = \Omega_{e^v,j^v} \) be the fine layer containing \( f \). Then \( p(f) = \sum_j n_j^f \cap n_j = p_d(f) \) is a subalgebra of \( n \) with the property that \( p(f)^f = p(f) \). Rearranging the sequence \( j^v \) in increasing order \( \{ j_1 < j_2 < \cdots < j_d \} \), we have that
\[ (s_1, s_2, \ldots, s_d) \mapsto \exp(s_dZ_{j_d})\exp(s_{d-1}Z_{j_{d-1}})\cdots\exp(s_1Z_{j_1})P(f) \]
is a global chart for \( N/P(f) \), and Lebesgue measure on \( R^d \) is thereby carried to an invariant measure on \( N/P(f) \). Let \( \chi_f \) be the unitary character on \( P(f) = \exp p(f) \) whose differential is \( if \). Then the unitary representation \( \pi_f \), induced from \( P(f) \) to \( N \) by \( \chi_f \), is irreducible. Denoting by \([\pi_f]\) its equivalence class in \( \tilde{N} \), one has \( \kappa^\circ(Nf) = [\pi_f] \). We denote the Hilbert space in which \( \pi_f \) acts by \( \mathcal{H}_f \). Note that the map \( J_f : \mathcal{H}_f \to L^2(R^d) \) defined by
\[ J_f \psi(s) = \psi(\exp(s_1Z_{j_1})\exp(s_2Z_{j_2})\cdots\exp(s_dZ_{j_d})) \]
is an isometric isomorphism.

An algorithm for determination of the Plancherel measure class and the Plancherel formula for nilpotent groups in terms of the orbit method is given in [20]. A similar result for the class of exponential solvable groups is proved in [4], and it is this version, specialized to the nilpotent case, that we use here.

The procedure is implemented as follows. Recall that we have a cross-section \( \Lambda^\circ \) for the coadjoint orbits in \( \Omega^\circ \) and that \( \Lambda^\circ = \Omega^\circ \cap V^\circ \) where \( V^\circ = \{ f \in n^* \mid f(Z_j) = 0 \text{ holds for all } j \in e^\circ \} \). Let \( \Omega \) be the minimal fine
layer in $\mathfrak{g}^*$, and set $\Lambda^1 = \Lambda^* \cap p(\Omega)$. Recall that we have written \(\{1, 2, \ldots, n\} - e^0 = \{u_1 < u_2 < \cdots < u_c\}\), where \(c = n - 2d\). Via the identification
\[
f \rightarrow (f(Z_{u_1}), f(Z_{u_2}), \ldots, f(Z_{u_c})),
\]
we regard $\Lambda^1$ not only as a subset of $\mathfrak{n}^*$, but also as a (dense open) subset of $\mathbb{R}^c$, and we shall henceforth use the notation $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_c)$ for elements of $\Lambda^1$. Accordingly, we shall write $\pi_\lambda$ for the irreducible representation corresponding to $\lambda$ as constructed above; note that each of the Hilbert spaces $\mathcal{H}_\lambda$ is isomorphic with $L^2(\mathbb{R}^d)$ via the map $J_\lambda$. Also, with $\text{PF} = \text{PF}_{e^0, d^0}$ the Pfaffian polynomial on $\mathfrak{n}^*$ as defined in Section 1, we shall write $\text{PF}(\lambda)$, $\lambda \in \Lambda^1$. At the same time we let $d\lambda$ denote Lebesgue measure on $\Lambda^1$. We describe the Fourier transform and Plancherel formula in these terms. For each $\lambda \in \Lambda^1$ and $\psi \in L^1(N) \cap L^2(N)$, set
\[
F(\psi)(\lambda) = \int_N \psi(x) \pi_\lambda(x) \, dx.
\]
Then $F(\psi)(\lambda)$ belongs to the space $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda$ of Hilbert-Schmidt operators on $\mathcal{H}_\lambda$. Now let $\mu$ be the Borel measure on $\Lambda^1$ defined by $d\mu(\lambda) = \frac{1}{(2\pi)^n} |\text{PF}(\lambda)| \, d\lambda$. Then \(\{\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda\}_{\lambda \in \Lambda^1}\) is a measurable field of Hilbert spaces and we set
\[
\mathbf{H} = \int_{\Lambda^1} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda \, d\mu(\lambda).
\]
Now $\lambda \rightarrow \pi_\lambda$ is a Borel function from $\Lambda^1$ to $\text{Irr}(N)$, $F(\psi)$ belongs to $\mathbf{H}$, and the map
\[
F : L^1(N) \cap L^2(N) \rightarrow \mathbf{H}
\]
as defined above extends to all of $L^2(N)$ as a unitary isomorphism.

With the Fourier transform on $N$ in place, we turn to the quasiregular representation of $G$ in $L^2(N)$. From now on we shall use the letter $f$ to refer to elements of $L^2(N)$. Let $\delta : H \rightarrow \mathbb{R}_+^*$ be the character $\delta(h) = \delta_1(h)\delta_2(h) \cdots \delta_n(h)$, and let $G$ have the Haar measure $d\nu_G(xh) = dx \delta(h)^{-1}d\nu_H(h)$. Define the unitary representation $\tau : G \rightarrow \mathbb{U}(L^2(N))$ as follows. For $f \in L^2(N)$, set
\[
(\tau(h)f)(x_0) = f(h^{-1}x_0h)\delta(h)^{-1/2}, \quad h \in H
\]
\[
(\tau(x)f)(x_0) = f(x^{-1}x_0), \quad x \in N.
\]
Recall that $\tau$ is isomorphic with the representation of $G$ induced from $H$ by the trivial character. Fix $\psi \in L^2(N)$ and for each $f \in L^2(N)$, denote by $m_{f, \psi}$ the bounded continuous function on $G$ defined by
\[
m_{f, \psi}(s) = \langle f, \tau(s)\psi \rangle_{L^2(N)}, \quad s \in G.
\]
Recall that $\psi$ is admissible for $\tau$ if $m_{f, \psi}$ is square-integrable for each $f \in L^2(N)$ and $\|m_{f, \psi}\|_{L^2(G)} = \|f\|_{L^2(N)}$. Following [12], we search for admissible vectors by means of the Fourier transform on $L^2(N)$. For $f \in L^2(N)$ set $\tilde{f}(\lambda) = F(f)(\lambda)$, $\lambda \in \Lambda^1$ and let $\tilde{\tau}(s) = F \circ \tau(s) \circ F^{-1}, s \in G$. The representation $\tilde{\tau}$ is described in terms of the usual action of $H$ on $\hat{N}$. Specifically, for $\pi \in \text{Irr}(N)$ and $h \in H$, set $(h \cdot \pi)(x) = \pi(h^{-1}xh), x \in N$. For each $h \in H$, the representation $h \cdot \pi_f$ is equivalent to $\pi_{hf}$ via the intertwining operator $C(h, f) : \mathcal{H}_f \rightarrow \mathcal{H}_{hf}$ defined by
\[
(C(h, f)\phi)(x) = \phi(h^{-1}xh) \delta_{hf}(h)^{-1/2}, \quad \phi \in \mathcal{H}_f,
\]
where $\delta_{hf}(h) = \prod_{j \in \mathfrak{c}^+} \delta_j(h)$. Passing to the quotient $\hat{N}$ and applying the orbit method one sees that the stabilizer $H[\pi_f]$ of $[\pi_f]$ in $H$ coincides with the analytic subgroup \(\{h \in H \mid hf \in Nf\} = \exp(\mathfrak{h} \cap (\mathfrak{n} + \mathfrak{n}^f))\). For
\( \lambda \in \Lambda^1 \), since the action of \( H \) is already diagonalized we have \( h\lambda \in \Lambda^1 \) and since \( \Lambda^1 \) is an orbital cross-section we have that
\[
H_{[\pi_\lambda]} = H_\lambda = \exp(\mathfrak{h} \cap \mathfrak{g}^\lambda) = \exp(\mathfrak{h}_r) = H_r
\]
holds for each \( \lambda \in \Lambda^1 \). For \( h \in H \) and \( \lambda \in \Lambda^1 \), let \( D(h, \lambda) : \mathcal{B}(\mathcal{H}_\lambda) \to \mathcal{B}(\mathcal{H}_{h\lambda}) \) be defined by
\[
D(h, \lambda)(T) = C(h, \lambda) \circ T \circ C(h, \lambda)^{-1}.
\]

Adapting the result [12, Proposition 2.1] to the present context, we have the following description of \( \hat{\tau} \) in terms of the preceding orbital parameters for the Fourier transform.

**Proposition 2.1.** Let \( f \in L^2(N), h \in H, x \in N, \lambda \in \Lambda^1 \). One has

(i) \( (\hat{\tau}(h)f)(\lambda) = D(h, h^{-1}\lambda)(\hat{f}(h^{-1}\lambda)) \delta(h)^{1/2} \)

(ii) \( (\hat{\tau}(x)f)(\lambda) = \pi_\lambda(x) \circ \hat{f}(\lambda) \)

We observe that [12, Proposition 2.2] also restates in the same way.

**Proposition 2.2.** For each \( h \in H \), \( d\mu(h\lambda) = \delta(h)^{-1}d\mu(\lambda) \).

An easy calculation shows that for each \( x \in N \) and \( h \in H \), one has
\[
m_{f, \psi}(xh) = (f * (\tau(h)\psi^*)) (x)
\]
where \( \psi^*(x) = \overline{\psi(x^{-1})} \). We then apply the Fourier transform:
\[
\int_G |m_{f, \psi}|^2 \, d\nu_G = \int_H \int_N \left| (f * (\tau(h)\psi^*)) (x) \right|^2 \, dx \, \delta(h)^{-1}d\nu_H(h)
\]
\[
= \int_H \int_{\Lambda^1} \left\| \hat{f}(\lambda) \circ (\hat{\tau}(h)\hat{\psi})(\lambda)^* \right\|_{HS}^2 \, d\mu(\lambda) \, \delta(h)^{-1}d\nu_H(h)
\]
\[
= \int_{\Lambda^1} \left( \int_H \left\| \hat{f}(\lambda) \circ C(h, h^{-1}\lambda)\hat{\psi}(h^{-1}\lambda)^*C(h, h^{-1}\lambda)^{-1} \right\|_{HS}^2 \, d\nu_H(h) \right) \, d\mu(\lambda)
\]
(2.1)

If \( N \) is abelian, so that the Fourier transform is scalar-valued, then
\[
\left\| \hat{f}(\lambda) \circ C(h, h^{-1}\lambda)\hat{\psi}(h^{-1}\lambda)^*C(h, h^{-1}\lambda)^{-1} \right\|_{HS}^2 = |\hat{f}(\lambda)|^2 |\hat{\psi}(h^{-1}\lambda)|^2
\]
and it becomes apparent from (2.1) that a necessary condition for \( \tau \)-admissibility is that \( H_\lambda \) be compact for \( \mu \)-a.e. \( \lambda \). Note that in the context of this paper that means simply that \( \mathfrak{h}_r = 0 \). Now for the class of groups considered here, it is reasonable to expect that the condition \( \mathfrak{h}_r = 0 \) is necessary for the existence of \( \tau \)-admissible vectors even when \( N \) is not abelian, but that question remains open. Therefore, for the remainder of this paper, we shall just make the assumption that \( \mathfrak{h}_r = 0 \). We observe that, if \( N \) is not abelian, then this means that the irreducible decomposition of \( \tau \) will have infinite multiplicity: we have \( r = \dim(H) = \dim H_\lambda \) holds for all \( \lambda \in \Lambda^1 \) and (since \( \mathfrak{h}_r = 0 \)) it follows that the generic dimension of \( H \)-orbits in \( \mathfrak{h}^\perp \) is \( r \). Now Corollary 1.7 says that \( r = d - d^0 \), where \( 2d \) is the generic dimension of \( G \) orbits (that meet \( \mathfrak{h}^\perp \)) and \( 2d^0 \) is the generic dimension of \( N \) orbits in \( \mathfrak{n}^* \). Combining these observations with the results of [16, 17] , we have \( \tau \) is finite multiplicity if and only if \( r = d \) if and only if \( N \) is abelian.
Recall also that in the case where $N$ is abelian, (2.1) is the starting point for proving the Caldéron condition for admissibility (for quite general groups $H$): $\psi$ is admissible for $\tau$ if and only if

$$\int_H |\psi(h^{-1} \lambda)|^2 \, dv_H(h) = 1$$

holds for $\mu$-a.e $\lambda$ [22, Theorem 2.1]. We shall see below that this result can be generalized to the case where $N$ is not abelian: we shall write $\tau$ as a direct sum of multiplicity-free subrepresentations $\tau^\beta$ so that a Caldéron condition for $\tau^\beta$-admissibility holds.

We begin by describing the action of $H$ on $\Lambda^1$ explicitly. Recall that we have chosen the ordered basis $\{A_v \mid 1 \leq v \leq r\}$ for $\mathfrak{h}$ in conjunction with a sequence $\{1 \leq u_{a_1} < u_{a_2} < \cdots < u_{a_w} \leq n\}$ of indices corresponding to a minimal spanning set of roots, as defined in Section 1. In particular for each $1 \leq v, w \leq r$, $\gamma_{a_{u_v}}(A_w) = \delta_{wv}$, and if $a < a_v$, $\gamma_{a_v}(A_w) = 0$. Write

$$Q(t, \lambda) = \exp(t_1 A_1) \exp(t_2 A_2) \cdots \exp(t_r A_r) \lambda, \quad t \in \mathbb{R}^r, \lambda \in \Lambda^1.$$  

Then for each $\lambda \in \Lambda^1$, $t \rightarrow Q(t, \lambda)$ is a diffeomorphism of $\mathbb{R}^r$ with $H\lambda$. The following notation will be helpful in the descriptions that follow: for each $1 \leq a \leq c$, if $a < a_1$, set $h^a = 1 \in H$, and for $a \geq a_1$ let $h^a(t) = \exp(t_1 A_1) \exp(t_2 A_2) \cdots \exp(t_{u(a)} A_{u(a)})$ where $w(a) = \max\{1 \leq w \leq r \mid a_w \leq a\}$.

For each $1 \leq a \leq c$ we see that $Q_a(t, \lambda) = \delta(h^a(t))^{-1} \lambda_a$. More explicitly, if we set

$$\gamma_{a,v} = \gamma_{a_v}(A_v), \quad 1 \leq a \leq c, \quad 1 \leq v \leq r,$$

then for $a = a_v$ we have $\delta(h^a(t))^{-1} = e^{t_v}$, while if $a \neq a_v, 1 \leq v \leq r$, then

$$\delta(h^a(t))^{-1} = e^{\gamma_{a,1} t_1 + \gamma_{a,2} t_2 + \cdots + \gamma_{a, w(a)} t_{w(a)}}.$$  

Hence

$$Q_a(t, \lambda) = \begin{cases} e^{t_v \lambda_a}, & \text{if } a = a_v \\ e^{\gamma_{a,1} t_1 + \gamma_{a,2} t_2 + \cdots + \gamma_{a, w(a)} t_{w(a)} \lambda_a}, & \text{if } a \neq a_v. \end{cases}$$  

For $1 \leq v \leq r$, set

$$z_v = e^{t_v |\lambda_{a_v}|}, \quad \epsilon_v = \text{sign}(\lambda_{a_v}).$$  

Making these substitutions into the function $Q$, we obtain a function $P(z, \lambda)$ each coordinate of which has the form

$$P_a(z, \lambda) = \begin{cases} z_v \epsilon_v, & \text{if } a = a_v \\ \frac{z_1}{\lambda_{a_1}}^{\gamma_{a,1}} \frac{z_2}{\lambda_{a_2}}^{\gamma_{a,2}} \cdots \frac{z_{w(a)}}{\lambda_{a_{w(a)}}}^{\gamma_{a, w(a)}} \lambda_a, & \text{if } a \neq a_v. \end{cases}$$  

The function $P$ is easily seen to have the following properties.

(a) For each $\lambda \in \Lambda^1$, $P(\cdot, \lambda)$ maps $\mathbb{R}^r_+$ diffeomorphically onto $H\lambda$.

(b) For each fixed $(z_1, z_2, \ldots, z_r) \in (\mathbb{R}^r_+)^r$, $P(z_1, z_2, \ldots, z_r, \cdot)$ maps $\Lambda^1$ into $\Lambda^1$ and is $H$-invariant.

We set $\Sigma = \{P(1, 1, \ldots, 1, \lambda) \mid \lambda \in \Lambda^1\}$; it is easily seen that $\Sigma$ is a submanifold of $\Lambda^1$ having dimension $c - r$, and that $\Sigma$ meets the $H$-orbit of $\lambda$ at the single point $P(1, 1, \ldots, 1, \lambda)$. In fact we have the following.
Lemma 2.4. Let $\Lambda$ be the cross-section in $\Omega$ for the $G$-orbits in $\Omega$. If $\mathfrak{h}_r = (0)$, then $p|\Lambda$ is a bijection of $\Lambda$ onto $\Sigma$.

Proof. By part (b) of Proposition 1.6 and our assumption that $\mathfrak{h}_r = (0)$, we have $\Lambda \subset \mathfrak{h}^\perp = \{ \ell \in \mathfrak{g}^* \mid \ell(\mathfrak{h}) = \{0\} \}$, and hence $p|\Lambda$ is a bijection. By Corollary 1.8, we have $p(\Lambda) = \{ \lambda \in \Lambda^1 \mid |\lambda_a| = 1, 1 \leq v \leq r \}$. An examination of the map $P(z, \lambda)$ above shows that $\Sigma = P(1, \lambda) \subset p(\Lambda)$ and that for each $\ell \in \Lambda$ with $\lambda = p(\ell)$, $\lambda = P(1, \lambda) \in \Sigma$. This completes the proof.

For each $\epsilon \in \{-1, 1\}^r$, set $\Lambda_1 = \{ \lambda \in \Lambda^1 \mid \text{ sign } (\lambda_a) = \epsilon, 1 \leq v \leq r \}$ and $\Sigma_\epsilon = \Sigma \cap \Lambda_1$. In the event that $r = c$, then for each $\epsilon$, $\Sigma_\epsilon$ is the single point $(\epsilon_1, \epsilon_2, \ldots, \epsilon_c)$. In this case we let $d\sigma$ be the counting measure on $\Sigma$, multiplied by $1/(2\pi)^{n+d}$. Otherwise write $\{1, 2, \ldots, c\} \setminus \{a_v \mid 1 \leq v \leq r\} = \{b_1 < b_2 < \cdots < b_q\}$; set $\sigma_w = \lambda_{b_w}, 1 \leq w \leq q$. Then each set $\Sigma_\epsilon$ is identified with an open subset of $\mathbb{R}^d$, and we thereby transfer Lebesgue measure to each $\Sigma_\epsilon$. The resulting measure on $\Sigma$, including the multiple $1/(2\pi)^{n+d}$, will be denoted by $d\sigma = d\sigma_1 d\sigma_2 \cdots d\sigma_q$. At the same time we identify $H$ with $(\mathbb{R}^*_+)^r$ so that

$$\exp(t_1 A_1) \exp(t_2 A_2) \cdots \exp(t_r A_r) = (e^{t_1}, e^{t_2}, \ldots, e^{t_r}) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_r).$$

The natural Haar measure on $H$ is then

$$d\nu_H(z_1, z_2, \ldots, z_r) = \frac{dz_1 dz_2 \cdots dz_r}{z_1 z_2 \cdots z_r}.$$
But part (a) of Proposition 1.6, together with our choice of basis of \( \mathfrak{h} \) dual to the minimal spanning set of roots, insures that
\[
\delta_e(z)^{-1} = z_1 z_2 \cdots z_r \delta_{e^\beta}(z)^{-1}.
\]
On the other hand, observing that \( p(\ell) = z \sigma \), part (c) of Proposition 1.6 gives
\[
Pf_{e, j}(\ell) = \prod_{e=1}^{r} (Z_{u_e}) Pf_{e, j^e}(z \sigma) = z_1 z_2 \cdots z_r Pf_{e, j^e}(z \sigma).
\]
Similarly \( Pf_{e, j}(\ell) = Pf_{e, j^e}(\sigma) \), and hence
\[
z_1 z_2 \cdots z_r Pf_{e, j^e}(z \sigma) = Pf_{e, j}(\ell) = z_1 z_2 \cdots z_r \delta_{e^\beta}(z)^{-1} Pf_{e, j}(\ell) = z_1 z_2 \cdots z_r \delta_{e^\beta}(z)^{-1} Pf_{e, j^e}(\sigma).
\]
The first part of the lemma is proved.
As for the second part, write \( \delta_{u_k}(z) = \prod_{w=1}^{q} \delta_{u_{w_k}}(z) \); again by virtue of our choice of basis for \( \mathfrak{h} \) we have
\[
\delta(z)^{-1} = z_1 z_2 \cdots z_r \delta_{e^\beta}(z)^{-1} \delta_{u_k}(z)^{-1}.
\]
Hence
\[
d\mu(\lambda) = |Pf(z \sigma)| \delta_{u_k}(z)^{-1} d\nu(z) = z_1 z_2 \cdots z_r \delta_{e^\beta}(z)^{-1} \delta_{u_k}(z)^{-1} d\nu_H(z) |Pf(\sigma)| d\sigma = \delta(z)^{-1} d\nu_H(z) |Pf(\sigma)| d\sigma.
\]

Fix an orthonormal basis \( \{ e^\beta \mid \beta \in B \} \) for \( L^2(\mathbb{R}^d) \), (where \( B \) is some index set) and for each \( \lambda = z \sigma \in H \sigma \), set
\[
e^\beta = C(z, \sigma) J^{-1} e^\beta.
\]
so that \( \{ e^\beta \}_\beta \) is an orthonormal basis of \( \mathscr{H}_\Lambda \). For each \( \lambda \in \Lambda^1 \) and each basis index \( \beta \), we have the subspace:
\[
\mathscr{H}_\lambda \otimes e^\beta = \{ T \in \mathscr{H}_\lambda \otimes \mathscr{H}_\lambda \mid Image(T^*) \subset C e^\beta \}.
\]
Recall that \( \mathscr{H}_\lambda \otimes e^\beta \) is the set of maps of the form \( v \mapsto \langle v, e^\beta \rangle w \) where \( w \in \mathscr{H}_\lambda \), and the obvious map \( \mathscr{H}_\lambda \to \mathscr{H}_\lambda \otimes e^\beta \) is an isometric isomorphism. For each basis index \( \beta \), set
\[
H^\beta = \int_{\Lambda^1}^{\oplus} \mathscr{H}_\lambda \otimes e^\beta d\mu(\lambda)
\]
so that \( H = \oplus_{\beta} H^\beta \). Setting
\[
K = \int_{\Lambda^1}^{\oplus} \mathscr{H}_\lambda d\mu(\lambda)
\]
we have an obvious isometric isomorphism of \( K \) onto each \( H^\beta \): \( w = \{ w(\lambda) \}_{\lambda \in \Lambda^1} \in K \) corresponds to the element \( \{ w(\lambda) \otimes e^\beta \}_{\lambda \in \Lambda^1} \in H^\beta \).

For any element \( g = \{ g(\lambda) \}_{\lambda \in \Lambda^1} = \{ w(\lambda) \otimes e^\beta \}_{\lambda \in \Lambda^1} \) of \( H^\beta \), one calculates that \( (\hat{\tau}(x) g)(\lambda) = \pi_\Lambda(x) w(\lambda) \otimes e^\beta \) for \( x \in N \) and
\[
(\hat{\tau}(z) g)(\lambda) = C(z, z^{-1} \lambda) w(z^{-1} \lambda) \otimes e^\beta \delta(z)^{1/2}, \ z \in H.
\]
Thus the subspace \( H^\beta \) of \( H \) is \( \hat{\tau} \)-invariant, and its inverse Fourier image \( L^2(N)^\beta = F^{-1}(H^\beta) \) is \( \tau \)-invariant. Accordingly we write \( \hat{\tau} = \oplus_{\beta} \hat{\tau}^\beta \) and \( \tau = \oplus_{\beta} \tau^\beta \). Now for each basis index \( \beta \), the preceding decomposition of the Plancherel measure \( \mu \) gives a direct integral decomposition of \( H^\beta \):
\[
H^\beta \cong \int_{\Sigma}^{\oplus} H^\beta_\sigma |Pf(\sigma)| d\sigma
\]
where

$$H_\beta = \int_H H_{z\sigma} \otimes e_{z\sigma}^\beta \, d\nu_H(z).$$

For the moment, fix $\sigma \in \Sigma$ and a basis index $\beta$. Define $\hat{\tau}_\sigma^\beta : G \to \mathcal{U}(H_\beta)$ by the same formula as in Proposition 2.1 above: for $g = \{g(z)\}_{z \in H} \in H_\beta$ and $z_0 \in H$ define

(i) $(\hat{\tau}_\sigma^\beta(z)g(z_0)) = D(z, z^{-1}z_0\sigma)(g(z^{-1}z_0)) \, \delta(z)^{1/2}$, $z \in H$

(ii) $(\hat{\tau}_\sigma^\beta(x)g(z_0)) = \pi_{z_0\sigma}(x) \circ g(z_0)$, $x \in N$.

**Proposition 2.6.** For each $\sigma \in \Sigma$ and for each $\beta$, $\hat{\tau}_\sigma^\beta$ is unitarily isomorphic with $\hat{\pi}_\sigma = \text{ind}_N^G(\pi_\sigma)$ (and hence is irreducible.)

**Proof.** Fix $\sigma \in \Sigma$ and let $L$ be the Hilbert space of $\hat{\pi}_\sigma$. For $w \in L$, $\lambda = z\sigma \in \Lambda^1$, set

$$(Tw)(z) = C(z, \sigma)(w(z)) \otimes e_{z\sigma}^\beta \, \delta(z)^{1/2},$$

Then

$$\int_H \| (Tw)(z) \|^2_{HS} \, d\nu_H(z) = \int_H \| C(z, \sigma)(w(z)) \otimes e_{z\sigma}^\beta \|^2_{HS} \, d\nu_H(z) = \int_H \| w(z) \|^2_{\hat{\pi}_\sigma} \, d\nu_H(z) = \| w \|^2_L,$$

hence $T$ is a linear isometry from $L$ into $H_\beta$. It is easily seen that $T$ is invertable. We compute that

$$\hat{\tau}_\sigma^\beta(z)(Tw)(z_0) = \hat{\tau}_\sigma^\beta(z)(C(z_0, \sigma)w(z_0) \otimes e_{z_0\sigma}^\beta \, \delta(z_0)^{1/2})$$

$$= C(z, z^{-1}z_0\sigma)C(z^{-1}z_0, \sigma)w(z^{-1}z_0) \otimes e_{z_0\sigma}^\beta \, \delta(z_0)^{1/2} \, \delta(z)^{1/2}$$

$$= C(z_0, \sigma)w(z^{-1}z_0) \otimes e_{z_0\sigma}^\beta \, \delta(z_0)^{1/2}$$

$$= T(\hat{\pi}_\sigma(z)w)(z_0).$$

It follows that the natural isomorphism (2.2) intertwines the representation $\hat{\tau}^\beta$ with the direct integral of the representations $\hat{\tau}_\sigma^\beta$. To sum up the preceding, we have shown that the Fourier transform, together with the decomposition of the Plancherel measure $\mu$, implements a natural decomposition of $\tau$ into unitary irreducibles:

$$\tau \cong \bigoplus \hat{\tau}_\sigma^\beta |Pf(\sigma)| \, d\sigma.$$

Now fix an index $\beta$ and for $f \in L^2(N)\beta$, write $\hat{f}(\lambda) = w_f(\lambda) \otimes e_\lambda^\beta$ where $w_f \in K$. Note that for each $\lambda \in \Lambda^1$, $\| \hat{f}(\lambda) \|_{HS} = \| w_f(\lambda) \|_{\hat{\pi}_\lambda \nu}$. In the sequel we shall often drop the cumbersome subscripts on norms indicating the Hilbert space, relying on context and other notation to affect the appropriate distinctions.

Fix $\psi \in L^2(N)\beta$ and set $u = u_\psi$ so that $\hat{\psi}(\lambda) = u(\lambda) \otimes e_\lambda^\beta$. One calculates that for each $\lambda \in \Lambda^1$ and $z \in H$,

$$\| \hat{f}(\lambda) \circ (\hat{\tau}(z)\hat{\psi})(\lambda) \|^2 = \| w_f(\lambda) \|^2 \| u(z^{-1}\lambda) \|^2 \, \delta(z) = \| \hat{f}(\lambda) \|^2 \| \hat{\psi}(z^{-1}\lambda) \|^2 \, \delta(z).$$

Define $\Delta_\psi : \Lambda^1 \to [0, +\infty)$ by

$$\Delta_\psi(\lambda) = \int_H \| \hat{\psi}(z^{-1}\lambda) \|^2 \, d\nu_H(z) = \int_H \| \hat{\psi}(z\sigma) \|^2 \, d\nu_H(z).$$
Note that $\Delta_\psi$ is constant on $H$-orbits in $\Lambda^1$. Combining the equations (2.1) and (2.3), we get

$$
\int_G |m_{f,\psi}|^2 \, d\nu_G = \int_H \int_{\Lambda^1} \|w_f(\lambda)\|^2 \|w(z^{-1}\lambda)\|^2 \, d\mu(\lambda) \, d\nu_H(z)
$$

$$
= \int_H \int_{\Lambda^1} \|f(\lambda)\|^2 \|\hat{\psi}(z^{-1}\lambda)\|^2 \, d\mu(\lambda) \, d\nu_H(z)
$$

$$
= \int_{\Lambda^1} \|\hat{f}(\lambda)\|^2 \left( \int_H \|\hat{\psi}(z^{-1}\lambda)\|^2 \, d\nu_H(z) \right) \, d\mu(\lambda)
$$

$$
= \int_{\Lambda^1} \|\hat{f}(\lambda)\|^2 \Delta_\psi(\lambda) \, d\mu(\lambda).
$$

So it is clear that if $\Delta_\psi(\lambda) = 1$ holds $\mu$-a.e., then $m_{f,\psi}$ belongs to $L^2(G)$ and $\|m_{f,\psi}\| = \|f\|$, that is, $\psi$ is admissible for $\tau^\beta$. An easy adaptation of the argument in [22, Theorem 2.1] shows that the converse is true.

**Proposition 2.7.** Let $\psi \in L^2(N)^\beta$. Then $\psi$ is admissible for $\tau^\beta$ if and only if $\Delta_\psi(\lambda) = 1$ holds for $\mu$-a.e. $\lambda \in \Lambda^1$.

**Proof.** The proof is already halfway done; to complete it, suppose that $\|m_{f,\psi}\| = \|f\|$ holds for all $f \in L^2(N)^\beta$. Fix $\lambda_0 \in \Lambda^1$. For $r > 0$ let $B_r(\lambda_0)$ be the ball about $\lambda_0$ of radius $r$, let $\chi_{B_r(\lambda_0)}$ be the characteristic function of the set $B_r(\lambda_0)$, and let $f = f_{\lambda_0, r} \in L^2(N)^\beta$ be defined by

$$
\hat{f}(\lambda) = \mu(B_r(\lambda_0))^{-1/2} \chi_{B_r(\lambda_0)} \cdot e_\lambda^\beta \otimes e_\lambda^\beta.
$$

Then $\|f\|^2 = 1$, so from our assumption and the above calculation, we have

$$
1 = \int_{\Lambda^1} \|\hat{f}(\lambda)\|^2 \Delta_\psi(\lambda) \, d\mu(\lambda) = \frac{1}{\mu(B_r(\lambda_0))} \int_{B_r(\lambda_0)} \Delta_\psi(\lambda) \, d\mu(\lambda).
$$

The result now follows from standard differentiability results.

**Remark 2.8.** Let $\psi \in L^2(N)^\beta$ be admissible for $\tau^\beta$ and let $f \in L^2(N)^\beta$. Write $\hat{\psi}(\lambda) = u(\lambda) \otimes e_\lambda^\beta$ and $\hat{f}(\lambda) = w_f(\lambda) \otimes e_\lambda^\beta$ as above. Then

$$
\hat{\tau}(xz)\hat{\psi}(\lambda) = \pi_{\lambda}(x) \circ C(z, z^{-1}\lambda)u(z^{-1}\lambda) \otimes e_\lambda^\beta \delta(z)^{1/2}
$$

and

$$
W_\psi(f)(xz) = \langle \hat{f}, \hat{\tau}(xz)\hat{\psi} \rangle
$$

$$
= \int_{\Lambda^1} \langle \hat{f}(\lambda), \hat{\tau}(xz)\hat{\psi}(\lambda) \rangle \, d\mu(\lambda)
$$

$$
= \int_{\Lambda^1} \langle w_f(\lambda), \pi_{\lambda}(x) \circ C(z, z^{-1}\lambda)u(z^{-1}\lambda) \rangle \, d\mu(\lambda) \delta(z)^{1/2}.
$$

Hence if $L^\beta : K \to H^\beta$ is the canonical isomorphism and $\hat{\psi}' = L^\beta \circ (L^\beta)^{-1}\hat{\psi}$, then

$$
W_\psi \circ L^\beta = W_{\hat{\psi}'} \circ L^\beta.
$$

We now show how to construct admissible vectors for $\tau^\beta$: suppose that $G$ is not unimodular and that $\eta$ is a unit vector in $L^2(H, \nu_H)$ that also happens to belong to $L^2(H, \delta^{-1}\nu_H)$. Since $\delta \neq 1$, there is $v, 1 \leq v \leq r$,
such that $\delta(0,0,\ldots,z_v,\ldots,0) \neq 1$; write $\delta(0,0,\ldots,z_v,\ldots,0) = z_v^p$, $p \neq 0$. Assume that $q = \dim(\Sigma) > 0$, and for each $\epsilon \in \{ -1,1 \}^r$, let $s_{\epsilon}$ be the identification map from $\Sigma_\epsilon$ onto an open subset of $\mathbb{R}^q$.

We choose a measurable function $\tilde{u} : \mathbb{R}^q \to (0,\infty)$ such that for any polynomial function $P(t)$ on $\mathbb{R}^q$, we have

$$\int_{\mathbb{R}^q} \tilde{u}(t)^p |P(t)| dt < \infty.$$ 

Define $u : \Sigma \to (0,\infty)$ by $u(\sigma) = \tilde{u}(s_{\epsilon}(\sigma)), \sigma \in \Sigma_\epsilon$. Then we have

$$\int_{\Sigma} u(\sigma)^p |Pf(\sigma)| d\sigma < \infty.$$ 

Now for each pair of basis indices $\alpha$ and $\beta$, define $\psi = \psi^{\alpha,\beta}_{\eta,\nu} \in L^2(N)^\beta$ by

$$(2.4.) \quad \hat{\psi}(z\sigma) = \eta(z_1, z_2, \ldots, z_{v-1}, z_v u(\sigma), z_{v+1}, \ldots, z_r) e^{\alpha}_{z\sigma} \otimes e^{\beta}_{z\sigma}.$$ 

With the identification $\lambda = z\sigma$, it will be helpful to abuse notation slightly by writing

$$\eta(\lambda) = \eta(z\sigma) = \eta(z_1, z_2, \ldots, z_{v-1}, z_v u(\sigma), z_{v+1}, \ldots, z_r)$$

so that $\hat{\psi}(\lambda) = \eta(\lambda) e^{\alpha}_{\lambda} \otimes e^{\beta}_{\lambda}$. Now we have that

$$\int_H |\hat{\psi}(z\sigma)|^2 d\nu_H(z) = \int_H |\eta(\sigma)|^2 d\nu_H(z) = 1$$

holds for all $\sigma \in \Sigma$, and the calculation

$$\int_N |\psi(x)|^2 dx = \int_{\Lambda^t} |\hat{\psi}(\lambda)|^2 d\mu(\lambda)$$

$$= \int_{\Sigma} \int_H |\hat{\psi}(z\sigma)|^2 \delta(z)^{-1} d\nu_H(z) |Pf(\sigma)| d\sigma$$

$$= \int_{\Sigma} \int_H |\eta(z_1, z_2, \ldots, z_{v-1}, z_v u(\sigma), z_{v+1}, \ldots, z_r)|^2 \delta(z)^{-1} d\nu_H(z) |Pf(\sigma)| d\sigma$$

$$= \int_{\Sigma} \int_H |\eta(\sigma)|^2 \delta(z_1) \cdots \delta(z_{v-1}) \delta(u(\sigma)^{-1} z_v) \delta(z_{v+1}) \cdots \delta(z_r) \delta(z)^{-1} d\nu_H(z) |Pf(\sigma)| d\sigma$$

$$= \int_H |\eta(\sigma)|^2 \delta(z)^{-1} d\nu_H(z) \int_{\Sigma} u(\sigma)^p |Pf(\sigma)| d\sigma < \infty$$

shows that $\psi \in L^2(N)^\beta$. Hence by Proposition 2.7, $\psi$ is admissible for $r$.

Next, suppose that $\psi = \psi^{\alpha,\beta}_{\eta,\nu}$ and $\psi' = \psi'^{\alpha,\beta}_{\eta,\nu}$ are two such admissible vectors. For $f \in L^2(N)^\beta$ and $f' \in L^2(N)^{\beta'}$ we compute that

$$\langle m_f, \psi(s), m_{f'}, \psi'(s) \rangle_{L^2(G)} = \int_G m_f, \psi(s)m_{f'}, \psi'(s) d\nu_G(s)$$

$$= \int_H \int_N \langle f(\tau(z)\psi)^* (x), (f' \ast (\tau(z)\psi')^*) (x) \rangle dx \delta(z)^{-1} d\nu_H(z)$$

$$= \int_H \int_{\Lambda^t} \langle \hat{f}(\lambda) \circ (\tau(z)\psi)^*(\lambda)^*, \hat{f}'(\lambda) \circ (\tau(z)\psi')^*(\lambda)^* \rangle_{HS} d\mu(\lambda) \delta(z)^{-1} d\nu_H(z)$$

$$= \int_H \int_{\Lambda^t} Trace(\hat{f}'(\lambda)^* \circ \hat{f}(\lambda) \circ (\tau(z)\psi)^*(\lambda)^* \circ (\tau(z)\psi')^*(\lambda)) d\mu(\lambda) \delta(z)^{-1} d\nu_H(z).$$
Now one checks that
\[
\hat{\dot{f}}(\lambda^*) \circ \hat{f}(\lambda) \circ (\tau(z)\psi)\dot{\gamma}(\lambda^*) \circ (\tau(z)\psi')\dot{\gamma}(\lambda) = \langle w_f(\lambda), w_{f'}(\lambda) \rangle \eta(z^{-1}\lambda) \eta'(z^{-1}\lambda) \delta(z) \varepsilon_{\lambda}^\alpha \otimes \varepsilon_{\lambda'}^\alpha' \delta_{\alpha,\alpha'}
\]
where \(\delta_{\alpha,\alpha'} = 1\) or 0 according as \(\alpha = \alpha'\) or \(\alpha \neq \alpha'\). Apply this with the decomposition of \(\mu\) and we get
\[
\langle m_{f,\psi}(s), m_{f',\psi'}(s) \rangle_{L^2(G)} = \int_H \int_{\Lambda^1} \langle w_f(\lambda), w_{f'}(\lambda) \rangle \eta(z^{-1}\lambda) \eta'(z^{-1}\lambda) \ d\mu(\lambda) d\nu(\lambda) \cdot \delta_{\alpha,\alpha'}
\]
which means we have the orthogonality relation
\[(2.5) \quad \langle W_{\psi}(f), W_{\psi'}(f') \rangle_{L^2(G)} = \langle w_f, w_{f'} \rangle \mathbf{K} \frac{1}{\langle \eta, \eta' \rangle_{L^2(H,\nu)}} \delta_{\alpha,\alpha'}.
\]
In particular this shows that if \(\alpha \neq \alpha'\) then the images of \(W_{\psi}\) and \(W_{\psi'}\) are orthogonal in \(L^2(G)\). We are now ready to prove the main result.

**Theorem 2.9.** Let \(G = N \ltimes H\) where \(N\) is a connected, simply connected nilpotent Lie group, and where \(H\) is a vector group such that the Lie algebra \(\text{ad}(\mathfrak{h})\) is \(\mathbb{R}\)-split and completely reducible, and such that \(H[\pi] = (1)\) holds for almost every \([\pi] \in \hat{N}\). Let \(\tau\) be the quasiregular representation of \(G\) in \(L^2(N)\). Then \(\tau\) has an admissible vector if and only if \(G\) is not unimodular.

**Proof.** Suppose first that \(G\) is not unimodular. We need to construct an admissible vector for \(\tau\). To do this, we fix a Jordan-Hölder basis of \(G\) satisfying the conditions of Section 1, and with all notation from Section 1, we conclude that \(\mathfrak{h}_r = (0)\). Recalling the structure of the Fourier transform on \(L^2(N)\) developed in the preceding, and in particular the decomposition \(L^2(N) = \bigoplus_{\beta} L^2(N)^\beta\), we then execute the construction given above for \(\tau^\beta\)-admissible vectors: let \(\eta\) be a unit vector in \(L^2(H,\nu_H)\) that also belongs to \(L^2(H,\delta^{-1}\nu_H)\) and let \(v, 1 \leq v \leq r\), such that \(\delta(0, 0, \ldots, z_v, \ldots, 0) \neq 1\). Write \(\delta(0, 0, \ldots, z_v, \ldots, 0) = z_v^p, p \neq 0\), and assume that \(q = \dim(\Sigma) > 0\). We omit the proof in the case where \(q = 0\); in that case each \(\tau^\beta\) is a finite direct sum of irreducible, square-integrable representations, and the proof is a simplification of what follows. For each \(\epsilon \in \{-1, 1\}^r\), recall that \(s_\epsilon\) is the identification map from \(\Sigma_\epsilon\) onto an open subset of \(\mathbb{R}^q\).

Now for each basis index \(\beta\), we choose a measurable function \(\hat{u}^\beta : \mathbb{R}^q \rightarrow (0, \infty)\) such that for any polynomial function \(P(t)\) on \(\mathbb{R}^q\), we have
\[
\sum_\beta \int_{\mathbb{R}^q} \hat{u}^\beta(t)^p \ |P(t)| dt < \infty.
\]
Define \(u^\beta : \Sigma \rightarrow (0, \infty)\) by \(u^\beta(\sigma) = \hat{u}^\beta(s_\epsilon(\sigma))\), \(\sigma \in \Sigma_\epsilon\), so that we have
\[
\sum_\beta \int_{\Sigma} u^\beta(\sigma)^p \ |Pf(\sigma)| d\sigma < \infty.
\]
Let \(\psi^\beta\) denote the function \(\psi_{\eta,u^\beta}^\beta\) as defined above, so that
\[
\hat{\psi}^\beta(z\sigma) = \eta(z_1, z_2, \ldots, z_{v-1}, z_v u^\beta(\sigma), z_{v+1}, \ldots, z_r) \varepsilon_{\sigma}^\beta \varepsilon_{z\sigma}^\beta.
\]
Then each $\psi^{\beta}$ is admissible for $\tau^{\beta}$ and the images of $W_{\psi^{\beta}}$ are pairwise orthogonal. Set $\psi = \sum_{\beta} \psi^{\beta}$; then $\psi$ belongs $L^{2}(N)$: for each $\beta$,
\[
\int_{N} |\psi^{\beta}(x)|^{2} \, dx = \int_{H} |\eta(z)|^{2} \, \delta(z)^{-1} \, d\nu(z) \int_{\Sigma} w^{\beta}(\sigma)^{p} |\text{Pf}(\sigma)| \, d\sigma
\]
so $\sum_{\beta} \|\psi^{\beta}\|^{2} < \infty$. For any $f \in L^{2}(N)$,
\[
W_{\psi}(f) = (f, \tau(\cdot)\psi) = \sum_{\beta} (f^{\beta}, \tau(\cdot)\psi^{\beta}) = \sum_{\beta} W_{\psi^{\beta}}(f^{\beta})
\]
and $\sum_{\beta} \|W_{\psi^{\beta}}(f^{\beta})\|^{2} = \sum_{\beta} \|f^{\beta}\|^{2} = \|f\|^{2}$. Thus $W_{\psi}(f) \in L^{2}(G)$ and $\|W_{\psi}(f)\| = \|f\|$ holds for all $f \in L^{2}(N)$.

On the other hand, suppose that $\psi \in L^{2}(N)$ is admissible for $\tau$, and fix any basis index $\beta$. Then $\psi^{\beta}$ is admissible for $\tau^{\beta}$, so by Proposition 2.7, $\Delta_{\psi^{\beta}}(\lambda) = 1$ a.e. on $A^{1}$, and hence $\Delta_{\psi^{\beta}}(\sigma) = 1$ a.e. on $\Sigma$. Now if $G$ is unimodular, then $\delta(1) = 1$ for all $z \in H$ so by Lemma 2.5,
\[
\int_{\Sigma} |\text{Pf}(\sigma)| \, d\sigma = \int_{\Sigma} \Delta_{\psi^{\beta}}(\sigma) \, |\text{Pf}(\sigma)| \, d\sigma = \int_{\Sigma} \int_{H} |\psi(z\sigma)|^{2} \, d\nu(z) \cdot |\text{Pf}(\sigma)| \, d\sigma = \int_{\Sigma} \int_{A^{1}} \|\psi(\lambda)\|^{2} \, d\mu(\lambda) = \|\psi\|^{2} < \infty
\]

This is possible only if $d\sigma$ is a finite measure. But by Lemma 2.4, $\Sigma$ is diffeomorphic with the cross-section $\Lambda$ for $G$-orbits in $\Omega$, and it is known [4, Corollary 2.2.2] that $d\sigma$ can only be finite when $q = 0$ and $\Sigma$ is a finite set. By Lemma 2.4, this means that the regular representation of the unimodular group $G$ decomposes into a finite sum of irreducible (square integrable) representations. It is well-known (see for example [11, Proposition 0.4]) that this can only happen when $G$ is discrete.

Next we show that $L^{2}(G)$ can be decomposed by means of the wavelet transforms on each $L^{2}(N)^{\beta}$.

**Lemma 2.10.** There is an orthonormal basis $\{\eta_{j}\}$ for $L^{2}(H, \nu_{H})$ each element of which also belongs to $L^{2}(H, \delta^{-1}\nu_{H})$.

Proof. Write $\delta(z)^{-1} \, d\nu_{H}(z) = z_{1}^{p_{1}} z_{2}^{p_{2}} \cdots z_{r}^{p_{r}} \, dz_{1} \cdots dz_{r}$ where $p_{w} \in \mathbb{R}$, $1 \leq w \leq r$, and choose $\nu \geq 0$ such that $\nu \geq -\min(p_{1}, p_{2}, \ldots, p_{r})$. For $j = (j_{1}, j_{2}, \ldots, j_{r}) \in \{0, 1, \ldots\}^{r}$, set
\[
\eta_{j}(z) = \prod_{w=1}^{r} \left( e^{-z_{w}} (2z_{w})^{\frac{p_{w}}{2}} L_{j_{w}}^{(\nu)}(2z_{w}) c_{\nu,j_{w}}^{-1/2} \right), \quad z = (z_{1}, z_{2}, \ldots, z_{r}) \in H,
\]
where $L_{j}^{(\nu)}$, $l = 0, 1, \ldots$ is the Laguerre polynomial
\[
L_{j}^{(\nu)}(s) = \frac{1}{j!} e^{s} s^{-\nu} \left( \frac{d}{ds} \right)^{l} (e^{-s}s^l)^{\nu}, \quad 0 < s < \infty
\]
and
\[
c_{\nu,l} = \int_{0}^{\infty} e^{-s} s^{\nu} L_{j}^{(\nu)}(s)^{2} \, ds.
\]

As in [18] we see that $\{\eta_{j}\}_{j \in \{0,1,2,\ldots\}^{r}}$ is an orthonormal basis of $L^{2}(H, \nu_{H})$. Also, since $\nu + p_{w} \geq 0$, $1 \leq w \leq r$, we have
\[
\int_{H} |\eta_{j}(z)|^{2} \, \delta(z)^{-1} \, d\nu_{H}(z) = \prod_{w=1}^{r} c_{\nu,j_{w}}^{-1} \int_{0}^{\infty} e^{-2z_{w}} (2z_{w})^{\nu+p_{w}} L_{j_{w}}^{(\nu)}(2z_{w})^{2} \, 2^{1-p_{w}} \, dz_{w} < \infty.
\]
Assume that \( G \) is not unimodular, and that \( q = \dim(\Sigma) > 0 \). Let \( \{ \eta_j \} \) be the basis of \( L^2(H, \nu) \) as in Lemma 2.10, and let \( u : \Sigma \to (0, \infty) \) a measurable function such that \( \int_\Sigma u(\sigma)^p |Pf(\sigma)| d\sigma \) where \( p \) is chosen appropriately as above. Fix a basis index \( \beta_0 \), set

\[
W_{J, \alpha}^\beta = W_{J, \alpha}(L^2(N)^{\beta_0}).
\]

From (2.5) we see that the subspaces \( J_{\beta}^\alpha \) are pairwise orthogonal in \( L^2(G) \), and that each is isomorphic with \( K \).

**Theorem 2.11.** We have

\[
L^2(G) = \bigoplus_{\alpha \in B} J_{\beta}^\alpha.
\]

**Proof.** We must show that \( L^2(G) \) is contained in the direct sum. Let \( Y \in L^2(G) \) and for \( z \in H \) set \( Y_z(x) = Y(xz), x \in N \). We have

\[
\|Y\|^2 = \int_H \left( \int_N |Y_z(x)|^2 dx \right) \delta(z)^{-1} dv(z) = \int_H \left( \int_{\Lambda^1} \|\hat{Y}_z(\lambda)\|^2 d\mu(\lambda) \right) \delta(z)^{-1} dv(z).
\]

Since \( \|\hat{Y}_z(\lambda)\|^2 = \sum_{\alpha, \beta} |\langle \hat{Y}_z(\lambda)e^\alpha, e^\beta \rangle|^2 \), then for each pair of indices \( \alpha \) and \( \beta \),

\[
\int_H |\langle \hat{Y}_z(\lambda)e^\alpha, e^\beta \rangle|^2 \delta(z)^{-1} dv(z) < \infty
\]

holds for \( \mu \)-a.e. \( \lambda \). Let \( y_{\lambda}^{\alpha, \beta} \) denote the mapping \( z \mapsto \langle \hat{Y}_z(\lambda)e^\alpha, e^\beta \rangle \delta(z)^{-1/2} \); then there is a co-null subset \( \Lambda_0^1 \) of \( \Lambda^1 \), such that \( y_{\lambda}^{\alpha, \beta} \in L^2(H, \nu_H) \) holds for each \( \lambda \in \Lambda_0^1, \alpha, \beta \in B \). Now for each \( \lambda \in \Lambda_0^1 \), write \( z = z_\lambda \sigma \) and set \( \eta_j(\lambda) = \eta_j(z_\lambda^{-1} z_\lambda), z \in H \). Observe that \( \{ \eta_j, \lambda \mid j \in \{0, 1, 2, \ldots \}^r \} \) is an orthonormal basis of \( L^2(H, \nu_H) \). Hence for each \( \lambda \in \Lambda_0^1, \alpha, \beta \in B \), we have complex numbers \( \{ a_j(\lambda, \alpha, \beta) \mid j \in \{0, 1, 2, \ldots \}^r \} \) such that

\[
y_{\lambda}^{\alpha, \beta} = \sum_{j \in \{0, 1, 2, \ldots \}^r} a_j(\lambda, \alpha, \beta) \eta_j, \lambda.
\]

This means that

\[
\hat{Y}_z(\lambda) \delta(z)^{-1/2} = \sum_{\alpha, \beta} \sum_{j} a_j(\lambda, \alpha, \beta) \eta_j, \lambda(z) e^\beta \otimes e^\alpha
\]

Now for each \( \alpha \in B, j \in \{0, 1, 2, \ldots \}^r \) set

\[
g_{\lambda}^\alpha(\lambda) = \sum_{\beta} a_j(\lambda, \alpha, \beta) e^\beta \otimes e^\beta_0.
\]
I claim that $g_j^\alpha \in H^{\beta_0}$ for all $\alpha$. To see this we observe that

$$\|Y\|^2 = \int_H \int_{\Lambda^1} \|\hat{Y}_z(\lambda)\|^2 d\mu(\lambda) \delta(z)^{-1} d\nu(z)$$

$$= \int_{\Lambda^1} \sum_{\alpha,\beta} \int_H |(\delta(z)^{-1/2}\hat{Y}_z(\lambda)e^\alpha_{\lambda}, e^\beta_{\lambda})|^2 d\nu(z) d\mu(\lambda)$$

$$= \int_{\Lambda^1} \sum_{\alpha,\beta} \|g^\alpha_{\lambda}\|^2 d\mu(\lambda)$$

$$= \int_{\Lambda^1} \sum_{\alpha,\beta} \sum_j |a_j(\lambda, \alpha, \beta)|^2 d\mu(\lambda)$$

$$\geq \int_{\Lambda^1} \sum_{\alpha,\beta} |a_j(\lambda, \alpha, \beta)|^2 d\mu(\lambda)$$

$$= \|g^\alpha_j\|^2.$$

Denote by $f_j^\alpha$ the inverse Fourier transform of $g_j^\alpha$, and set $\psi = \psi_j^{\alpha,\beta_0}$. Then for a.e. $z \in H$,

$$\left(W_{j,u}^\alpha(f_j^\alpha)\right)_z = (f_j^\alpha * (\tau(z)\psi)^*)$$

belongs to $L^2(N)$, and for such $z$,

$$\left(W_{j,u}^\alpha(f_j^\alpha)\right)_z^* (\lambda) = g_j^\alpha(\lambda) \circ (\eta_j(z^{-1}z_{\lambda}) e^\alpha_{\lambda} \otimes e^\beta_{\lambda})^* \delta(z)^{1/2}$$

$$= \delta(z)^{1/2} \left( \sum_{\beta} a_j(\lambda, \alpha, \beta) e^\beta_{\lambda} \otimes e^\alpha_{\lambda} \right) \circ \eta_j(z^{-1}z_{\lambda}) e^\beta_{\lambda} \otimes e^\alpha_{\lambda}$$

$$= \delta(z)^{1/2} \sum_{\beta} a_j(\lambda, \alpha, \beta) \eta_j(z^{-1}z_{\lambda}) e^\beta_{\lambda} \otimes e^\alpha_{\lambda}.$$

Summing over all $\alpha$ and $j$, we find

$$\sum_{\alpha,j} \left(W_{j,u}^\alpha(f_j^\alpha)\right)_z^* (\lambda) = \delta(z)^{1/2} \sum_{\alpha,\beta} \left( \sum_j a_j(\lambda, \alpha, \beta) \eta_j(z^{-1}z_{\lambda}) e^\beta_{\lambda} \otimes e^\alpha_{\lambda} \right)$$

$$= \sum_{\alpha,\beta} \hat{Y}_z(\lambda)e^\alpha_{\lambda} \otimes e^\beta_{\lambda} \right)$$

$$= \hat{Y}_z(\lambda)$$

Taking the inverse Fourier transform we obtain $Y_z = \sum_{\alpha,j} W_{j,u}^\alpha(f_j^\alpha)_z$ for a.e. $z \in H$, and hence

$$Y = \sum_{\alpha,j} W_{j,u}^\alpha(f_j^\alpha).$$
References

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Department of Mathematics and Computer Science, Saint Louis, MO 63103
E-mail address: curreybn @ slu.edu