REGULARITY OF ABELIAN LINEAR ACTIONS

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Abstract. We study the regularity of orbits for the natural action of a Lie subgroup \( G \) of \( GL(V) \), where \( V \) is a finite dimensional real vector space. When \( G \) is connected, abelian, and satisfies a certain rationality condition, we show that there are two possibilities: either there is a \( G \)-invariant Zariski open set \( \Omega \) in which every orbit is regular, or there is a \( G \)-invariant conull \( G_\delta \) set in which every orbit is not regular. Moreover, under the rationality condition, an explicit characterization of almost everywhere regularity is proved.

1. Introduction

For the continuous action of a group on a topological space, the orbit of a point is said to be regular if the relative topology on the orbit coincides with the quotient topology that the orbit carries as a homogeneous space. Given a finite dimensional real vector space and a subgroup of \( GL(V) \) that is discretely or continuously generated by a finite subset of \( gl(V) \), we consider the broad and somewhat vague question: are there general conditions on the generators that will ensure that the orbit of almost every point is regular? In the case of one generator, when the group is \( \exp \mathbb{R}A \), or \( \exp \mathbb{Z}A \), then it is relatively easy to describe necessary and sufficient conditions on \( A \) in order that the orbits are almost everywhere regular. In fact, one finds that

1. If \( A \) is not diagonalizable then almost all orbits are regular,
2. If \( A \) is diagonalizable, and there is at least one non-purely imaginary eigenvalue or if all the eigenvalues are purely imaginary and they generate a discrete additive group, then almost all orbits are regular,
3. If \( A \) is diagonalizable, all the eigenvalues are purely imaginary and they generate a dense additive subgroup in \( i\mathbb{R} \), then almost all orbits are singular.

Moreover, in the case of one generator exactly one of the following obtains: either there is a \( G \)-invariant Zariski open subset \( \Omega \) in \( V \) in which all orbits are regular, or there is a \( G \)-invariant, conull, \( G_\delta \) subset \( U \) of \( V \) in which every orbit is not regular. When there is more than one generator, the problem becomes much more difficult.

The motivation for the study of this question arises in part from its close relation to admissibility. Let \( G \) be a subgroup of \( GL(n, \mathbb{R}) \) and let \( \tau \) be the unitary representation

Date: February 27, 2013.

2000 Mathematics Subject Classification. 57Sxx, 22Exx, 22E25, 22E27, 17B45, 17B08, 58E40.

Key words and phrases. regular orbit, Lie algebra root, linear Lie group action.
of $\mathbb{R}^n \rtimes G$ induced by the trivial character of $G$, acting in $L^2(\mathbb{R}^n)$. Fix $\psi \in L^2(\mathbb{R}^n)$ and for $f \in L^2(\mathbb{R}^n)$ define $W_\psi f$ by

$$W_\psi f(x,s) = \langle f, \tau(x,s)\psi \rangle, \ x \in \mathbb{R}^n, s \in G.$$ 

We say that $\psi$ is \textit{weakly admissible} if $W_\psi$ is a well-defined, bounded, injective map from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n \rtimes G)$, and $\psi$ is \textit{admissible} if $W_\psi$ is an isometry. Now put $V = \hat{\mathbb{R}}^n$; and (regarding $G$ as a subgroup of $GL(V)$) for $\psi \in L^2(\mathbb{R}^n)$ put

$$c_\psi(v) = \int_G |\hat{\psi}(s^{-1} \cdot v)|^2 ds, \ v \in V.$$ 

Then $\psi$ is weakly admissible if and only if $c_\psi$ is bounded and non-vanishing almost everywhere, and $\psi$ is admissible if and only $c_\psi = 1$ almost everywhere. We say that $G$ is (weakly) admissible if $\tau$ has a (weakly) admissible vector.

It is well-known that if $G$ is weakly admissible then $G$ is necessarily closed and almost all of the stabilizers $G(v), v \in V$ are compact. In [9] it is shown that if these necessary conditions hold and in addition for almost every $v \in V$, the $\epsilon$-stabilizer

$$G_\epsilon(v) = \{ s \in G : |s \cdot v - v| \leq \epsilon \}$$

is compact for some $\epsilon > 0$, then $G$ is weakly admissible. But if for some $v$ the stabilizer $G(v)$ is compact, then $v$ has a compact $\epsilon$-stabilizer if and only if the orbit is regular [5]. Thus, a sufficient condition for almost everywhere regularity provides a method for finding weakly admissible groups. The necessity of the $\epsilon$-stabilizer condition for weak admissibility remains open, but in [6], it is shown that, if $G$ is closed and $G(v)$ is compact almost everywhere, then $G$ is weakly admissible if and only if there is a conull, $G$-invariant Borel subset $\Omega$ of $V$ that admits a Borel cross-section for its orbits.

Existence of a Borel cross-section is only provisionally related to regularity: by the fundamental results in [4, 7], if $\Omega$ is a \textit{locally compact} $G$-space, then $\Omega$ admits a global Borel cross-section for its orbits if and only if all orbits in $\Omega$ are regular. We say that a subgroup $G$ of $GL(V)$ is regular if there is a locally compact, $G$-invariant, conull, Borel subset $\Omega$ of $V$ in which every orbit is regular. By the above, if a closed subgroup of $GL(V)$ is regular and has almost everywhere compact stabilizers, then it is weakly admissible. We remark that it is trivial to check whether a weakly admissible group is admissible: if $G$ is weakly admissible, then it is admissible if and only if $\mathbb{R}^n \rtimes G$ is not unimodular.

In the present work we consider the case where $G$ is the exponential of the real linear span $\mathfrak{g}$ of a finite subset of commuting elements in $\mathfrak{gl}(V)$, or in other words, $G$ is a connected abelian Lie subgroup of $GL(V)$. Here, as is described below and also in [3], $G$ can be identified with a group of complex, lower triangular matrices in block form, so that the semisimple part of each block is a scalar multiple of the identity. On each block these scalar values are given by characters $\Lambda$ on $G$ of the form $\Lambda(\exp X) = e^{\lambda(X)}$ where $\lambda$ is a linear form on $\mathfrak{g}$; we call such linear forms \textit{roots} and the corresponding characters \textit{root characters}. 
Now suppose that we are given such a group $G$ and we want to determine whether $G$ is regular. If the imaginary part of each root is a scalar multiple of the real part, then (since $G$ is connected and the action of $G$ is of exponential type) all of its orbits are regular [2]. Otherwise, let $G_1$ be the set of all elements of $G$ that lie in the kernel of all root characters. Since some of the roots have independent imaginary parts, $G_1$ is not necessarily connected, and so even though $G_1$ is unipotent, it may not be regular. Roughly speaking irregular orbits can arise as a result of two distinct characteristics: either from an irrational relation between imaginary parts of the various roots, or from an irregular action of the discrete part of the group $G_1$.

Much of the work in this paper is devoted to understanding the action of $G_1$ and its relation to that of $G$. We show that if $G$ is regular, then $G_1$ is regular, and that if $G_1$ is not regular then there is a $G$-invariant, conull, $\mathfrak{g}(V)$ subset $U$ of $V$ in which all $G$-orbits are not regular. Moreover, we present an explicit procedure for computing a Zariski open subset $\Omega_0$ of $V$ and for each $v \in \Omega_0$, a subset $\Delta(v)$ of $\text{gl}(V)$, so that $G_1$ is regular if and only if the function $\Delta$ is constant and its value is a closed set. We define precisely a rationality condition on (the imaginary parts of) the roots and prove that if $G$ satisfies the rationality condition, then $G$ is regular if and only if $G_1$ is regular. Thus we obtain the following method for “finding” regular and admissible connected abelian groups:

1. Check the rationality condition for $G$. If yes, then
2. compute $G_1$, $\Omega_0$, and for $v \in \Omega_0$ compute $\Delta(v)$.
3. Determine whether $\Delta(v)$ is constant. If yes, then put $\Delta = \Delta(v)$, and
4. determine whether the set $\Delta$ is closed. If yes, then $G$ is regular.
5. Compute the stabilizers $G(v)$. If they are compact almost everywhere and $G$ is closed, then $G$ is weakly admissible.

In Section 2 we provide topological background and prove several characterizations of regularity for an orbit of an analytic group action. The projective action of a Lie group on a fiber bundle is considered in Section 3, and there we prove a crucial result (Proposition 3.1) concerning the relations between regularity of actions on the base space, fibers, and total space. We consider linear actions on a finite dimensional vector space $V$ in Section 4, where we define the rationality condition, and if this condition is satisfied, we give an algorithm for computing an explicit presentation of the unipotent group $G_1$. We then study the action of $G_1$ by first studying a unipotent connected action: we construct the smallest analytic unipotent subgroup $N$ of $GL(V)$ containing $G_1$, and using a standard algorithm, we compute an $N$-invariant Zariski open subset $\Omega_0$ of $V$ where the $N$-action is simply described. At this point the $G_1$-orbits in $\Omega_0$ are embedded in the $N$-orbits in a very transparent way and this permits the computation of $\Delta$ (Lemma 4.13) and the main criterion for regularity of $G_1$. It turns out that $\Omega_0$ is also $G$-invariant, and we use Proposition 3.1 to obtain an explicit Zariski open set $\Omega$ in $V$, included in $\Omega_0$, such that if $G$ satisfies the rationality
condition, then every $G$-orbit in $\Omega$ is regular if and only if every $G_1$-orbit in $\Omega_0$ is regular.

2. ANALYTIC GROUP ACTIONS AND REGULAR ORBITS

Much of this section is issued from [10], essentially Sections 1.1 and 2.9.

A smooth map $\varphi : M \longrightarrow N$ from a $m$-dimensional $C^\infty$ manifold $M$ into a $n$-dimensional manifold $N$ is an immersion if $d\varphi_x$ is injective for any $x$ in $N$, $\varphi$ is an embedding if it is an one-to-one immersion, it is a regular embedding if it is an embedding and a homeomorphism between $M$ and $\varphi(M)$, equipped with the relative topology coming from the $N$ topology.

A subset $A$ in $N$ is locally closed if it is open in its closure or if it is the intersection of a closed and an open subset in $N$.

**Proposition 2.1.** If $\varphi$ is a regular embedding, then $\varphi(M)$ is locally closed in $N$.

**Proof.** First, since $\varphi$ is an immersion, for any $x$ in $M$, if $y = \varphi(x)$, by the maximal rank theorem, we can find open neighborhoods $U$ for $x$, $V$ for $y$, equipped with local coordinates $\xi : U \ni z \mapsto (x_1(z), \ldots, x_m(z)) \in [-a, a]^m \subset \mathbb{R}^m$ and $\eta : V \ni z' \mapsto (y_1(z'), \ldots, y_n(z')) \in [-a, a]^n \subset \mathbb{R}^n$ such that $\xi(x) = 0$, $\eta(y) = 0$ and $\eta \circ \varphi \circ \xi^{-1}(t_1, \ldots, t_m) = (t_1, \ldots, t_m, 0, \ldots, 0)$.

Since $\varphi$ is a homeomorphism, $\varphi(U)$ is open in $\varphi(M)$, for the relative topology. There exists an open subset $W$ in $N$ such that $\varphi(U) = \varphi(M) \cap W$, thus $\varphi(U) = \varphi(M) \cap (W \cap V)$. Now $W \cap V$ is open in $N$ and contains $y = \varphi(x)$, thus there is $0 < a' < a$ such that $V' = \eta^{-1}([-a', a']^n) \subset W \cap V$; by construction, $V'$ is open in $N$. Put $U' = \xi^{-1}([-a', a']^m)$; $U'$ is open in $M$ and, with the above expression of $\varphi$, $\varphi(U') = \varphi(M) \cap V'$.

It follows that $\varphi(M)$ is locally closed: for each $x \in M$ choose such open subsets $U'_x$, $V'_x$ for any $x$ in $M$ as above, and put $N = \bigcup_{x \in M} V'_x$. Then $\varphi(M) = \bigcup_{x \in M} \varphi(U'_x) \subset \bigcup_{x \in M} V'_x = N$, and for any $z$ in $\varphi(M) \cap N$ there is $x$ such that $z$ is in $V'_x$, and a sequence $(z_n)$ in $\varphi(M)$ converging to $z$. If $n$ is larger than some $p$, $z_n$ is in the closed subset $\varphi(M) \cap V'_x$ of $V'_x$, thus $z$ also belongs to $\varphi(M) \cap V'_x$, and hence $\varphi(M) \cap N = \varphi(M)$.

□
Suppose now that $G$ is a Lie group acting analytically on an analytic manifold $N$. Given $x \in N$, we denote by $G(x)$ the stability subgroup of $x$. It is closed in $G$ and we let $G/G(x)$ have the quotient topology, which is Hausdorff, locally compact, and second countable. The natural map $\varphi_x : G/G(x) \to N$ defined by $gG(x) \mapsto g \cdot x$ is well-defined, one-to-one, and $G$-equivariant. Since $G/G(x)$ has the quotient topology, $\varphi_x$ is continuous. Moreover, there is a unique analytic structure on $G/G(x)$ such that it becomes an analytic manifold, the natural action of $G$ on $G/G(x)$ is analytic, and $\varphi_x$ is an immersion. We say that the orbit $O = G \cdot x$ of $x$ is a regular orbit if the map $\varphi_x$ is a regular embedding. This property does not depend upon the choice of $x$ in $O$.

A topological space $X$ is a Baire space if the Baire’s lemma holds for $X$: if $(U_n)$ is a sequence of dense open subset of $X$, then $\bigcap_{n=1}^{\infty} U_n$ is still dense in $X$. Any locally compact space is a Baire space.

**Theorem 2.2.** Let $O = G \cdot x$ an orbit in $N$, equipped with the relative topology. Then the following are equivalent:

1. $O$ is a regular orbit,
2. $O$ is locally closed,
3. $O$ is locally compact,
4. $O$ is a Baire space.

*Proof.*

1. $\implies$ 2. By the preceding result, if $O$ is regular, then $O$ is locally closed.
2. $\implies$ 3. If $O$ is locally closed, $O = \overline{O} \cap W$, with $W$ open, and hence $O$ is a closed subspace of the open subset $W$ in $N$. Since $W$ is locally compact, then for the relative topology, $O$ is locally compact.
3. $\implies$ 4. Since $O$ is locally compact, it is a Baire space.
4. $\implies$ 1. Now let $K$ be a compact subset in $G/G(x)$, with non-empty interior. Thus there is a sequence $(g_n)$ in $G$ such that $G/G(x) = \bigcup_{n=1}^{\infty} g_n K$ (since $G/G(x)$ is second countable).

We get $O = \bigcup_{n=1}^{\infty} g_n \varphi_x(K)$ and each $g_n \varphi_x(K)$ is compact thus closed. By the Baire lemma, one of these subsets has a non-empty interior. Since, they are all homeomorphic to $\varphi_x(K)$ through the map $z \mapsto g_n^{-1} \cdot z$, $\varphi_x(K)$ has a non empty interior $W$ and we put $V_1 = \varphi_x^{-1}(W)$. The restriction $\varphi_x|_{V_1}$ of $\varphi_x$ to $V_1$ is a continuous bijection from $V_1$ onto $W$, but $V_1 \subseteq K$. Thus $V_1$ is compact in $G/G(x)$ and $\varphi_x|_{V_1}$ is a homeomorphism, and $\varphi_x|_{V_1}$ is a homeomorphism from $V_1$ onto $W$. 
Let $\mathcal{U}$ be any open subset in $G/G(x)$; then $\mathcal{U}$ is the union of a family of subsets $g_i\cdot\mathcal{U}_i$, with $g_i \in G$ and $\mathcal{U}_i$ open subset in $\mathcal{V}_1$. Thus
\[
\varphi_x(\mathcal{U}) = \bigcup_i g_i \cdot \varphi_x(\mathcal{U}_i)
\]
is open in $\mathcal{O}$. This shows that $\varphi_x$ is an open map; hence $\varphi_x$ is homeomorphism and $\mathcal{O}$ is regular.

\[\square\]

Let $q : G \rightarrow G/G(x)$ be the canonical mapping; in what follows we will also write $[g] = q(g), g \in G$, and $[K] = q(K)$ for $K \subseteq G$. The map $q$ is not only continuous, but open: for any open subset $\mathcal{V}$ of $G$, $q^{-1}(\mathcal{V}) = \mathcal{V} \cdot G(x) = \bigcup_{t \in G(x)} \mathcal{V} \cdot t$ is open in $G$ and hence $\mathcal{V}$ is open in $G/G(x)$.

Since $G(x)$ is a closed Lie subgroup of $G$, then $G/G(x)$ has a structure of a differentiable manifold, with a local chart around the base point $[1] \in G/G(x)$ given as follows. Let $\mathfrak{g}(x)$ be the Lie algebra of $G(x)$ and $\mathfrak{m}$ a supplementary space for $\mathfrak{g}(x)$ in the Lie algebra $\mathfrak{g}$ of $G$, there is a sufficiently small open neighborhood $\mathcal{U}$ of 0 in $\mathfrak{m}$, so that $\mathcal{V} = \exp \mathcal{U} \cdot G(x)$ is open in $G$ and for which the map $\theta : [\exp X] \mapsto X (X \in \mathcal{U})$ is well-defined on the open set $\mathcal{V}$ in $G/G(x)$, and $([\mathcal{V}], \theta)$ is a local chart about $[1]$. As a consequence, any neighborhood of $[1]$ in $G/G(x)$ contains a subset of the form $[\exp(M)]$ with $M$ a compact neighborhood of 0 in $\mathfrak{m}$, included in $\mathcal{U}$, and if $K_x$ is a compact neighborhood of 1 in $G(x)$, then $K = \exp(M)K_x$ is a compact neighborhood of 1 in $G$ such that $[K] = [\exp(M)]$. Since $K$ contains an open neighborhood of 1 in $G$ and $q$ is open, then $[K]$ is a neighborhood of $[1]$ in $G/G(x)$.

To simplify a proof in the next section, we express the preceding proposition in a little bit different way:

**Corollary 2.3.** The orbit $\mathcal{O} = G \cdot x$ is regular if and only if, for any compact neighborhood $K$ of 1 in $G$, there is a neighborhood $\mathcal{V}$ of $x$ in $N$ such that $\mathcal{O} \cap \mathcal{V} \subset K \cdot x$.

**Proof.** Suppose $\mathcal{O}$ regular and let $K$ be a compact neighborhood of 1 in $G$. Then $[K]$ is a neighborhood of $[1]$ in $G/G(x)$, $K \cdot x$ a neighborhood of $x$ in $\mathcal{O}$, in the relative topology. There is $\mathcal{V}$ open in $N$, containing $x$ such that $\mathcal{O} \cap \mathcal{V} \subset K \cdot x$.

Conversely, suppose that the condition of the corollary holds. To show that $\mathcal{O}$ is regular it is enough (as in the preceding lemma) to show that $\varphi_x : G/G(x) \rightarrow \mathcal{O}$ is an open map. Indeed, let $\mathcal{U}$ be any open subset of $G/G(x)$ and choose any $s \in G$ such that $[s] \in \mathcal{U}$. Then $s^{-1}\mathcal{U}$ is a neighborhood of $[1]$, so by the remark preceding the corollary, $s^{-1}\mathcal{U}$ contains some subset $[K]$ with $K$ compact neighborhood of 1 in $G$. Therefore we have a neighborhood $\mathcal{V}$ of $x$ in $N$ such that $K \cdot x = \varphi_x([K])$ contains $\mathcal{O} \cap \mathcal{V}$. Now $\varphi_x([s]) = s \cdot x$ belongs to $\mathcal{O} \cap s \cdot \mathcal{V}$ and $\mathcal{O} \cap s \cdot \mathcal{V}$ is included in $\varphi_x(\mathcal{U})$, and hence $\varphi_x$ is open. \[\square\]
Let $\Omega$ be a $G$-invariant subset of $N$. We say that $\Omega$ is $G$-regular if all the orbits in $\Omega$ are regular. The following is a consequence of the fundamental work of Glimm and Effros.

**Theorem 2.4.** [7, Theorem 1], [4, Theorem 2.9] Suppose that $\Omega$ is locally compact; then the following are equivalent.

1. $\Omega$ is regular.
2. $\Omega/G$ is countably separated.
3. $\Omega/G$ possesses the $T_0$ separation property.
4. There is a Borel subset $\Sigma$ of $\Omega$ that meets each $G$-orbit in exactly one point.

### 3. Group action on a fiber bundle

Suppose that $\pi : X \rightarrow W$ is a fibre bundle with fiber $Z$, and that $G$ is a Lie group acting smoothly and projectively on $X$, through $(s, x) \mapsto s \cdot x$. Then there is a smooth $G$-action $(s, w) \mapsto b(s)w$ such that, for any $s \in G$ and $x \in X$,

$$
\pi(s \cdot x) = b(s)\pi(x).
$$

For each $w \in W$, denote by $i_w$ the homeomorphism from $\pi^{-1}(w)$ onto $Z$; the group action on $X$ gives rise to a collection of mappings $\phi_w(s) : Z \rightarrow Z$, $s \in G$, such that, for any $s$ and $x$,

$$
i_{b(s)w}(s \cdot x) = \phi_w(s)(i_w(x)),
$$

or

$$
\phi_w(s)z = i_{b(s)w}(s \cdot i_w^{-1}z), \quad s \in G, \quad z \in Z.
$$

Clearly $(s, w, z) \mapsto \phi_w(s)z$ is smooth, $\phi_w(1) = Id$, and for $s_1, s_2 \in G$,

$$
\phi_w(s_1 s_2) = \phi_w(s_2) \circ \phi_w(s_1).
$$

In the present paper, we consider only two simple cases of this situation:

1. Smooth action on product manifold.

   In this case, $X$ is the trivial bundle $W \times Z$, and the action has the form $(s, w, z) \mapsto (b(s)w, \phi(s)z)$, where $\phi$ is a $G$-action on $Z$. The typical example is the action of $\mathbb{R}$ on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, defined by $(t, z) \mapsto \Lambda(t)z = e^{at}z$ ($a \in \mathbb{C}$). Identifying $\mathbb{C}^\times$ with the product manifold $(\mathbb{R}^+ \times \mathbb{T})$ where $\mathbb{T}$ is the one dimensional torus, through $z = re^{i\theta} \mapsto (r, e^{i\theta}) = (|z|, \text{sign}(z))$, the action becomes

$$
(t, z) = (t, r, e^{i\theta}) \mapsto (e^{i\Re(a)t}, e^{i(\theta + 3\text{Im}(a)t)}) = (|\Lambda(t)||z|, \Lambda^\epsilon(t)\text{sign}(z)),
$$

where $\Lambda^\epsilon(t)$ is the character $\text{sign}(\Lambda(t))$. 

2. Linear action with an invariant subspace.

In this case, the total space is a finite dimensional vector space $V$, and we suppose that $G$ acts linearly on $V$, and that $V$ contains a subspace $Z$ which is $G$-invariant. Then there is a natural linear $G$-action on the quotient space $W_0 = W/Z$, defined by $b(s)(v + Z) = s \cdot v + Z$. Let us fix a supplementary space $W$ for $Z$ in $V$. We can identify $W_0$ with $W$ and the canonical map $\pi : V \to W_0$ becomes the projection onto $W$ parallel to $Z$. If $p$ is $Id - \pi$, the projection onto $Z$ parallel to $W$, then $G$-action on $v = w + z$ is

$$s \cdot (w + z) = s \cdot w + s \cdot z = b(s)w + p(s \cdot w) + s \cdot z,$$

and the map $\phi_w(s)$ is the affine map

$$z \mapsto \phi_w(s)z = p(s \cdot w) + s \cdot z.$$

In these two cases the fiber bundle is trivial: the total space is identified with a product $W \times Z$ and the action is written as $(s, w, z) \mapsto (b(s)w, \phi_w(s)z)$. For simplicity of notation, we assume that the bundle is trivial in the following proposition.

For $w \in W$, put $G(w) = \{s \in G : b(s)w = w\}$; then the relation (3.1) shows that the action of $G(w)$ on $X$ naturally induces an action on $Z$ by

$$(s, z) \mapsto \phi_w(s)z, \ s \in G(w), \ z \in Z.$$

Fix $x_0 = (w_0, z_0) \in X$, put $O = G \cdot x_0$, $O_0 = b(G)w_0$, and $\omega = \phi_{w_0}(G(w_0))z_0$. We have the following:

**Proposition 3.1.** If $O$ is regular, then $\omega$ is regular. On the other hand if $O_0$ and $\omega$ are both regular, then $O$ is regular.

**Proof.** Suppose $O$ regular. Let $g(w_0)$ be the Lie algebra of $G(w_0)$, and $m$ a supplementary space to $g(w_0)$ in the Lie algebra $g$ of $G$. Fix a compact neighborhood $M$ of 0 in $m$, sufficiently small such that the map $M \to \exp MG(w_0) \subset G/G(w_0)$ defines a local chart for the manifold $G/G(w_0)$.

Let $K_{w_0}$ be any compact neighborhood of 1 in $G(w_0)$, denote $K = \exp MK_{w_0}$. Then $K \cdot x_0 \cap \{w_0\} \times Z = K_{w_0} \cdot x_0 = \{w_0\} \times \phi_{w_0}(K_{w_0})z_0$. Indeed, for any $X \in M$, $k \in K_{w_0}$, we have $x = k \cdot x_0$ is in $\{w_0\} \times Z$ if and only if $b(\exp X)w_0 = w_0$, if and only if $X = 0$. Similarly, $O \cap (\{w_0\} \times Z) = G(w_0) \cdot x_0 = \{w_0\} \times \omega$, since $s \cdot x_0 \in \{w_0\} \times Z$ if and only if $s \in G(w_0)$.

Since $O$ is regular, there is a neighborhood $V$ of $x_0$ in $X$ such that $O \cap V \subset K \cdot x_0$, thus $\{w_0\} \times \phi(K_{w_0})z_0 = Kx_0 \cap (\{w_0\} \times Z)$ contains $O \cap (\{w_0\} \times Z) \cap V = V \cap (\{w_0\} \times \omega)$. Since $W = p(V \cap (\{w_0\} \times Z))$ is a neighborhood of $z_0$ in $Z$, the corollary implies that $\omega$ is regular.

Now suppose that both $O_0$ and $\omega$ are regular. Let $K$ be any compact neighborhood of 1 in $G$; $K$ contains a compact neighborhood of 1 of the form $\exp(M)K_{w_0}$ ($M$ and $K_{w_0}$ are as above).
Since $\omega$ is regular, we have a neighborhood $\mathcal{Z}$ of $z_0$ in $Z$ such that

$$\mathcal{Z} \cap \omega \subset \phi_{w_0}(K_{w_0})z_0.$$  

Define $f : m \times W \times Z \rightarrow Z$ by

$$f(X, w, z) = \phi_w(\exp X)z.$$  

By continuity of $f$, we have a compact neighborhood $M'$ of $0$ in $m$, a neighborhood $\mathcal{W}$ of $w_0$ in $W$, and a neighborhood $\mathcal{Z}'$ of $z_0$ in $Z$ such that

$$f(M' \times \mathcal{W} \times \mathcal{Z}') \subset \mathcal{Z}.$$  

We may assume that $M' \subset M$, and moreover, $-M' = M'$.  

Since $O_0$ is regular, we have a neighborhood $\mathcal{W}'$ of $w_0$ in $W$ such that

$$\mathcal{W}' \cap O \subset b(\exp M')w_0.$$  

We can choose $\mathcal{W}' \subset \mathcal{W}$.  

We claim that $(\mathcal{W}' \times \mathcal{Z}') \cap O \subset K \cdot x_0$. Let $x = (w, z) \in (\mathcal{W}' \times \mathcal{Z}') \cap O$. Then $w \in \mathcal{W}' \cap O_0$, and so we have $X \in M'$ such that $w = b(\exp X)w_0$. Now $(-X, w, z)$ belongs to $M' \times \mathcal{W} \times \mathcal{Z}'$, and hence

$$\exp -X \cdot x = (b(\exp X)w, \phi_w(\exp X)z) = (w_0, \phi_w(\exp -X)z) = (w_0, f(-X, w, z))$$

belongs to $O \cap (\{w_0\} \times \mathcal{Z})$. But

$$O \cap (\{w_0\} \times \mathcal{Z}) = \{w_0\} \times (\omega \cap \mathcal{Z}) \subset \{w_0\} \times \phi_{w_0}(K_{w_0})z_0.$$  

Hence $\exp -X \cdot x$ belongs to $\{w_0\} \times \phi_{w_0}(K_{w_0})z_0$ and so

$$x \in \exp X(\{w_0\} \times \phi_{w_0}(K_{w_0})z_0) = \exp XK_{w_0} \cdot x_0 \subset K \cdot x_0.$$  

Thus the claim is proved. Now by Corollary 2.3, $O$ is regular.  

\[ \square \]

4. Linear action of a connected abelian group  

Let $V$ be a finite dimensional real vector space and let $G$ be a Lie subgroup of $GL(V)$; we say that $G$ is regular if there is a locally compact, invariant, conull Borel subset $\Omega$ of $V$ such that $\Omega$ is $G$-regular for the natural action of $G$ on $V$. Now suppose that $G$ is connected and abelian; we are looking for those $G$ which are regular. We remark that we do not assume that $G$ is closed in $GL(V)$.

**Example 4.1.** Let us recall a very well-known example: the Mautner group is the semidirect product $\mathbb{C}^2 \rtimes \mathbb{R}$, where the action of $s \in \mathbb{R}$ is given by the complex $2 \times 2$ matrix

$$\exp s \begin{bmatrix} i & 0 \\ 0 & \sqrt{2i} \end{bmatrix}.$$
In this example $V = \mathbb{C}^2 \simeq \mathbb{R}^4$ and $G$ is the one-parameter group above. The group $G$ is therefore not closed in $GL(V)$ (it is dense in the set of unitary diagonal matrices). Let $v = [v_1, v_2]^t$ be in $V$, if $v_1 v_2 \neq 0$, then $G \cdot v$ is also a non locally closed, dense subset inside the 2-dimensional torus \( \{u = [u_1, u_2]^t : |u_j| = |v_j|, \ j = 1, 2\} \). Since the union of these orbits is conull, $G$ is not regular.

Throughout the paper, we use the following notation: given $e_1, e_2, \ldots, e_q$ elements of a vector space over a field $\mathbb{K}$, denote the $\mathbb{K}$-span of these elements by $(e_1, e_2, \ldots, e_q)_{\mathbb{K}}$.

Let $\mathfrak{g}$ be a commutative Lie subalgebra of $\mathfrak{gl}(V)$ and $G = \exp(\mathfrak{g})$. Using the complexification $V_C$ of $V$ and the conjugation $\sigma : u \mapsto \overline{u}$, we can write a convenient matrix representation for elements of $\mathfrak{g}$.

A complex form $\lambda$ is a root for the action of $\mathfrak{g}$ if, for each $X \in \mathfrak{g}$, $\lambda(X)$ is an eigenvalue for $X$. If $\lambda$ is a root, the corresponding generalized eigenspace for $\lambda$ is

$$E_\lambda = \cap_{X \in \mathfrak{g}} \ker_{V_C} (X - \lambda(X)Id)^{\dim V_C}.$$ 

Since $\mathfrak{g}$ is abelian, $V_C = \oplus E_\lambda$ and for each $\lambda$, $XE_\lambda \subseteq E_\lambda$ for all $X \in \mathfrak{g}$. If $\lambda$ is real, $\sigma(E_\lambda) = E_\lambda$; otherwise there is $\lambda'$ such that $\lambda' = \overline{\lambda}$ and $E_{\lambda'} = \sigma(E_\lambda)$. From now on, fix an ordering for the roots such that $\lambda_1, \ldots, \lambda_r$ are real, $\lambda_{r+1}, \ldots, \lambda_p$ are not real, and

$$\lambda_{p+1} = \overline{\lambda_{r+1}}, \ldots, \lambda_{2p-r} = \overline{\lambda_r},$$

where it is understood that if there are no real roots, then $r = 0$. Put $E_j = E_{\lambda_j}$ for all $j$.

Now $V$ is the following real subspace in $V_C$:

$$V = \bigoplus_{j=1}^r (E_j \cap V) \oplus \bigoplus_{j=r+1}^p \left( (E_j + \sigma(E_j)) \cap V \right)$$

$$= \left\{ \sum_{j \leq r} v_j + \sum_{j=r+1}^p (w_j + \overline{w_j}), \ v_j \in E_j \cap V \ (1 \leq j \leq r), \ w_j \in E_j \ (r < j \leq p) \right\}.$$ 

Then the map $\iota : V \longrightarrow V_C$, defined by $\iota(v) = \sum_{j \leq r} v_j + \sum_{r < j \leq p} w_j$, is a bijection between $V$ and the space $\bigoplus_{j \leq r} (E_j \cap V) \oplus \bigoplus_{r < j \leq p} E_j$. Put $V_j = E_j \cap V \ (1 \leq j \leq r)$, and $V_j = E_j \ (r < j \leq p)$, then $\iota(V) = \bigoplus_{j=1}^r V_j$. Since $\iota$ is a homeomorphism, the orbit of $x$ in $V$ is regular if and only if the orbit of $\iota(x)$ is regular in $\iota(V)$. Therefore we identify $V$ with $\bigoplus_{j=1}^r V_j$ and each $X$ in $\mathfrak{gl}(V)$ with $\iota \circ X \circ \iota^{-1}$. 
From now on, we fix a basis \( \{ f^1_j, \ldots, f^r_j \} \) for each \( V_j, 1 \leq j \leq p \), so that for each \( X \in g \), the resulting block-form matrix is lower triangular, and we identify each \( X \) with this matrix.

**Example 4.2.** Define an action of \( g = \mathbb{R}^2 \) on a 5 dimensional space \( V = \sum_{j=1}^5 \mathbb{R}e_j \) by putting

\[
A_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & i & 0 \\
0 & 1 & i \\
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & i & 1 \\
\end{bmatrix}.
\]

This means that the roots are:

\[
\lambda_1(A_1) = 1, \quad \lambda_1(A_2) = 0, \\
\lambda_2(A_1) = i, \quad \lambda_2(A_2) = 1, \\
\lambda_3 = \bar{\lambda}_2,
\]

and that we identify \( V \) with \( \mathbb{R} \oplus \mathbb{C}^2 \) as follows. Define elements of \( V_C \) by \( f_1^1 = e_1, f_2^1 = e_2 - ie_3, f_2^2 = e_4 - ie_5, f_3^1 = f_1^2 \) and \( f_3^2 = f_2^2 \). Then \( \{ f_1^1, f_2^1, f_2^2, f_3^1, f_3^2 \} \) is a basis for \( V_C \), \( V_1 = \mathbb{R}e_1, V_2 = \mathbb{C}f_1^1 + \mathbb{C}f_2^2, \) and \( \iota(V) = V_1 \oplus V_2 \). On the other hand, the matrices in \( \mathfrak{gl}(5, \mathbb{R}) \) for \( A_1 \) and \( A_2 \) with respect to the basis \( e_1, \ldots, e_5 \) are:

\[
A_1' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}, \quad A_2' = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Write now \( \lambda_j = \alpha_j + i\beta_j \), where \( \alpha_j, \beta_j : g \rightarrow \mathbb{R} \). Denote by \( \Lambda_j \) the character

\[
\Lambda_j(\exp A) = e^{\lambda_j(A)}
\]

and put \( \Lambda_j(s) = |\Lambda_j(s)|\Lambda_j^\epsilon(s), \ s \in G \). It is natural to consider two subgroups of \( G \): the subgroup

\[
G_0 = \cap_{j=1}^p \ker \Lambda_j
\]

is connected and in fact \( G_0 = \exp g_0 \) where \( g_0 = \cap_{j=1}^p \ker \alpha_j \). On the other hand,

\[
G_1 = \cap_{j=1}^p \ker \Lambda_j
\]

may not be connected. Though both are closed subgroups of \( G \), they may not be closed in \( GL(V) \). Roughly speaking, the action of \( G \) is a combination of the action of the characters \( \Lambda_j \), and the unipotent action of \( G_1 \). In what follows we shall make this statement precise.
First we describe both $G_0$ and $G_1$ more explicitly. Let $(A_1, \ldots, A_q)$ be a basis of $\mathfrak{g}$ and consider the system of equations:

\[
\begin{align*}
& t_1 \alpha_1 (A_1) + t_2 \alpha_1 (A_2) + \cdots + t_q \alpha_1 (A_q) = 0 \\
& t_1 \alpha_2 (A_1) + t_2 \alpha_2 (A_2) + \cdots + t_q \alpha_2 (A_q) = 0 \\
& \vdots \\
& t_1 \alpha_p (A_1) + t_2 \alpha_p (A_2) + \cdots + t_q \alpha_p (A_q) = 0
\end{align*}
\]

(4.1)

The Lie algebra $\mathfrak{g}_0$ is obtained as follows:

\[
\mathfrak{g}_0 = \left\{ t_1 A_1 + \cdots + t_q A_q \in \mathfrak{g} : (t_1, t_2, \ldots, t_q) \in \mathbb{R}^q \right\}
\]

is a solution for the system of equations in (4.1).

On the other hand, the group $G_1$ can be described as

\[
G_1 = \{ \exp (t_1 A_1 + \cdots + t_q A_q) \in G : (t_1, t_2, \ldots, t_q) \in \mathbb{R}^q \text{ is a solution for (4.2)} \}
\]

where

\[
\begin{align*}
& t_1 \alpha_1 (A_1) + t_2 \alpha_1 (A_2) + \cdots + t_q \alpha_1 (A_q) = 0 \\
& t_1 \alpha_2 (A_1) + t_2 \alpha_2 (A_2) + \cdots + t_q \alpha_2 (A_q) = 0 \\
& \vdots \\
& t_1 \alpha_p (A_1) + t_2 \alpha_p (A_2) + \cdots + t_q \alpha_p (A_q) = 0 \\
& t_1 \beta_1 (A_1) + t_2 \beta_1 (A_2) + \cdots + t_q \beta_1 (A_q) \in 2\pi \mathbb{Z} \\
& t_1 \beta_2 (A_1) + t_2 \beta_2 (A_2) + \cdots + t_q \beta_2 (A_q) \in 2\pi \mathbb{Z} \\
& \vdots \\
& t_1 \beta_p (A_1) + t_2 \beta_p (A_2) + \cdots + t_q \beta_p (A_q) \in 2\pi \mathbb{Z}
\end{align*}
\]

(4.2)

Observe that $\Gamma_1 := \exp^{-1}(G_1)$ is an additive subgroup of $\mathfrak{g}_0$; it is crucial that we describe $\Gamma_1$ explicitly, and to do so, we introduce some terminology. A lattice in a real vector space is a discrete additive subgroup; observe that every additive subgroup of a lattice is itself a lattice. Given a lattice $L$, a lattice basis for $L$ is a linearly independent subset $\{B_1, B_2, \ldots, B_m\}$ of $L$ such that $L = \mathbb{Z}B_1 + \mathbb{Z}B_2 + \cdots + \mathbb{Z}B_m$. It is not difficult to construct a lattice basis, but construction of lattice bases with additional properties is more difficult (see for example [8].)

Put $\mathfrak{k} = \mathfrak{g}_0 \cap (\bigcap_j \ker \beta_j)$; then $\mathfrak{k}$ is the Lie algebra of $G_1$ and the space $\mathfrak{k}^\perp$ of linear forms on $\mathfrak{g}_0$ vanishing on $\mathfrak{k}$ is exactly $(\beta_1|_{\mathfrak{g}_0}, \ldots, \beta_p|_{\mathfrak{g}_0})_{\mathbb{R}}$. Take a basis $(\beta_1|_{\mathfrak{g}_0}, \ldots, \beta_k|_{\mathfrak{g}_0})$ for this space and choose $D_1, \ldots, D_k$ in $\mathfrak{g}_0$ so that $\mathfrak{g}_0 = \mathfrak{k} + \sum_{s=1}^k \mathbb{R} D_s$ and so that $D_1 + \mathfrak{k}, \ldots, D_k + \mathfrak{k}$ is the dual basis in $\mathfrak{g}_0/\mathfrak{k}$.

Let $X = Y + \sum u_s D_s \in \mathfrak{g}_0$ where $Y \in \mathfrak{k}$. Then $\exp X$ belongs to $G_1$ if and only if, for each $1 \leq j \leq p$:

$$
\sum_{s=1}^k u_s \beta_j(D_s) \in 2\pi \mathbb{Z}.
$$
Especially, for \( j = i_s \), this gives \( u_s \in 2\pi \mathbb{Z} \), and hence

\[
\Gamma_1 \subset \mathfrak{k} + \sum_{s=1}^{k} 2\pi \mathbb{Z} D_s.
\]

Now \( L = \mathfrak{k} + \sum_{s=1}^{k} 2\pi \mathbb{Z} D_s / \mathfrak{k} \) is a lattice in the vector space \( \mathfrak{g}_0 / \mathfrak{k} \), and hence the closed additive subgroup \( \Gamma_1 / \mathfrak{k} \) of \( L \) is a lattice. Choosing a basis for \( \Gamma_1 / \mathfrak{k} \), we thereby obtain a linearly independent subset \( \{ B_1, B_2, \ldots, B_m \} \) of \( \mathfrak{g}_0 \) such that

\[
\Gamma_1 = \mathfrak{k} + \sum_{s=1}^{m} 2\pi \mathbb{Z} B_s.
\]

We will see below that if the roots \( \beta_j \) are rationally related, then there is a relatively simple algorithm for constructing the generators \( B_1, B_2, \ldots, B_m \).

Next we define an invariant subspace \( Z \) of \( V \) for which \( G_1 \) is the stabilizer of the action on \( V/Z \) for almost all points in \( V/Z \). For each \( j = 1, 2, \ldots, p \), define the subspace \( Z_j \) of \( V_j \) by

\[
Z_j = (f^j_2, \ldots, f^j_n)_\mathbb{R}, \quad \text{if} \ j \leq r, \quad Z_j = (f^j_2, \ldots, f^j_n)_\mathbb{C}, \quad \text{if} \ r < j
\]

put \( Z := \oplus_j Z_j \), and \( W = (f^1_1, \ldots, f^1_n)_\mathbb{R} \oplus (f^{r+1}_1, \ldots, f^p_1)_\mathbb{C} \). Then \( Z \) is \( G \)-invariant and \( W \) is naturally isomorphic with \( V/Z \). For \( w = w_1 f^1_1 + w_2 f^1_2 + \cdots + w_p f^p_1 \in W \) write \( w = (w_1, w_2, \ldots, w_p) \).

As described in Section 3, we now regard \( V \) as a (trivial) fiber bundle over \( W \) with fiber \( Z \). By construction of \( W \) and \( Z \), the quotient action \( (s, w) \mapsto b(s)w \) of \( G \) on \( W \) is diagonal: for each \( s \in G, w \in W \),

\[
(4.3) \quad b(s)w = (\Lambda_1(s)w_1, \Lambda_2(s)w_2, \ldots, \Lambda_p(s)w_p).
\]

Recall that for \( v \in V, w = \pi(v), z = p(v), \)

\[
s \cdot v = s \cdot (w, z) = (s \cdot w, \phi_w(s)z), \quad s \in G
\]

where \( \phi_w(s)z = s \cdot z + p(s \cdot w) \). Put

\[
\Omega_W = \{ w = (w_1, w_2, \ldots, w_p) \in W : w_j \neq 0, 1 \leq j \leq p \}
\]

and \( \Omega_1 = \pi^{-1}(\Omega_W) \), so that \( \Omega_1 = \Omega_W \times Z \). Clearly \( \Omega_W \) is invariant for the \( G \)-action (4.3) and hence \( \Omega_1 \) is \( G \)-invariant. Note that for each \( w \in \Omega_W, G(w) = G_1 \).

Let \( v = (w, z) \in \Omega_1 \). Observe that by Proposition 3.1, if \( G \cdot v \) is regular, then \( \phi_w(G_1)z \) is regular, and if both \( \phi_w(G_1)z \) and \( b(G)w \) are regular, then \( G \cdot v \) is regular. For the moment we consider the regularity of the action \( (s, w) \mapsto b(s)w \) on \( \Omega_W \).

To study this action we decompose \( \Omega_W \) as a product using polar coordinates: for \( w = (w_1, w_2, \ldots, w_p) \in \Omega_W \), put \( |w| = (|w_1|, |w_2|, \ldots, |w_p|) \) and \( \text{sign}(w) = \).
It is clear that the action of $G$ on $|w|$ given by the characters $|\Lambda_j|$ is regular. Moreover, for each $w \in \Omega_W$, the stabilizer $G(|w|)$ for this action is $G_0$. Now apply Proposition 3.1 to the action defined by $b$ on the fiber bundle $\Omega_W$ with base space $B = \mathbb{R}_+^r \times \cdots \times \mathbb{R}_+^r$ ($p$ times) and fiber $\{-1, 1\}^r \times \mathbb{T}^{p-r}$, and it follows that the orbit $b(G)w$ in $\Omega_W$ is regular if and only if the orbit $b(G_0)w = (|w|, \text{sign}(b(G_0)w))$ is regular. But since $w$ has non-zero coordinates, it is well-known that

$$b(G_0)w = \{(|w|, \Lambda_1(s)\text{sign}(w_1), \ldots, \Lambda_p(s)\text{sign}(w_p)) : s \in G_0\}$$

is locally closed if and only if

$$(4.4) \quad \dim(\beta_1|_{g_0}, \ldots, \beta_p|_{g_0})_\mathbb{Q} = \dim(\beta_1|_{g_0}, \ldots, \beta_p|_{g_0})_\mathbb{R}.$$ 

Similarly, for $v = (w, z) \in \Omega_1$ we can write $v = (|w|, \text{sign}(w), z)$, and thus regard $\Omega_1$ as a fiber bundle over the same base space $B$, but now with fiber $\{-1, 1\}^r \times \mathbb{T}^{p-r} \times \mathbb{Z}$. Again by Proposition 2.1 we conclude that for each $v \in \Omega_1$, $Gv$ is regular if and only if $G_0v$ is regular. We sum up these observations in the following.

**Lemma 4.3.** One has $\Omega_W$ is $G$-regular if and only if $\Omega_W$ is $G_0$-regular if and only if

$$(4.5) \quad \dim(\beta_1|_{g_0}, \ldots, \beta_p|_{g_0})_\mathbb{Q} = \dim(\beta_1|_{g_0}, \ldots, \beta_p|_{g_0})_\mathbb{R}.$$ 

Moreover, for any $v \in \Omega_1$, $Gv$ is regular if and only if $G_0v$ is regular.

We say that $G$ satisfies the rationality condition if (4.5) holds.

Recall that we are looking for regular groups $G$: those for which almost all orbits are regular. Now for each $v = (w, z) \in \Omega_1$ and for $s \in G$,

$$s \cdot v = (b(s)w, \phi_w(s)z),$$

and the stabilizer of $w$ is $G_1$. Hence by Proposition 3.1, the $G$-orbit of $v$ will be regular if both $b(G)w$ and $\phi_w(G_1)z$ are regular. Having established a criterion for the regularity of $b(G)w$ in $\Omega_W$, we turn now to the group $G_1$.

Recall that $G_1 = \exp \Gamma_1$ where $\Gamma_1 = \mathfrak{t} + \sum s 2 \pi \mathbb{Z} B_s$ and where $\{B_1, B_2, \ldots, B_m\}$ is linearly independent modulo $\mathfrak{t}$. The following lemma gives an algorithm for constructing the elements $\{B_1, B_2, \ldots, B_m\}$, in the case where $G$ satisfies the rationality condition. This result is essentially the same as that of [1, Lemma 5.1]; we give its proof here for completeness. For any subset $S$ of $\mathbb{Z}$, denote by $\gcd'(S)$ the greatest common divisor of the nonzero elements of $S$.
Let \( \mathbf{v} \) be a real vector space, let \( \beta_1, \ldots, \beta_p \) be linear forms on \( \mathbf{v} \), and let \( \mathfrak{k} = \bigcap_{j=1}^p \ker \beta_j \). Put \( \Lambda = \{ u \in \mathbf{v} : \beta_j(u) \in 2\pi \mathbb{Z} \} \); \( \Lambda \) being closed, \( \Lambda / \mathfrak{k} \) is a lattice in \( \mathbf{v} / \mathfrak{k} \). Now assume that the linear forms \( \beta_1, \ldots, \beta_p \) are rationally related:

\[
\dim(\beta_1, \ldots, \beta_p)_\mathbb{Q} = \dim(\beta_1, \ldots, \beta_p)_{\mathbb{R}}.
\]

Let \( B_1, \ldots, B_m \) be a basis of \( \mathbf{v} \) modulo \( \mathfrak{k} \) such that \( \beta_j(B_s) \in \mathbb{Z} \) holds for all \( j \) and \( s \), and suppose that \( f_1, \ldots, f_m \) are linear forms on \( \mathbf{v} \) such that for each \( 1 \leq s \leq m \), \( f_s \in \sum_{j=1}^p \mathbb{Z} \beta_j \) and such that \( f_s(B_t) = \delta_{st} \) holds for all \( 1 \leq t \leq m \). Then it is easy to check that \( \Lambda = \mathfrak{k} + \sum_{s=1}^m 2\pi \mathbb{Z} B_s \). This is the motivation for the following formulation.

**Lemma 4.4.** Let \( \mathbf{v} \) be a real vector space, let \( \beta_1, \ldots, \beta_p \) be rationally related linear forms on \( \mathbf{v} \), put \( \mathbf{v}_0 = \mathbf{v}, \mathbf{v}_i = \bigcap_{j=1}^i \ker \beta_j, 1 \leq i \leq p \), and

\[
\mathbf{i} = \{ 1 \leq i \leq p : \mathbf{v}_i \neq \mathbf{v}_{i-1} \} = \{ i_1 < i_2 < \cdots < i_m \},
\]

so that \( \{ \beta_{i_s} : 1 \leq s \leq m \} \) is a basis for \( (\beta_j : 1 \leq j \leq p)_\mathbb{R} \). Then there is an explicit construction of elements \( B_m, B_{m-1}, \ldots, B_1 \in \mathbf{v} \) and linear forms \( f_m, f_{m-1}, \ldots, f_1 \) on \( \mathbf{v} \) such that for any \( s = 1, 2, \ldots, m \),

1. \( B_s \in \mathbf{v}_{i_{s-1}} \setminus \mathbf{v}_{i_{s}} \),
2. \( \beta_j(B_s) \in \mathbb{Z} \) for all \( 1 \leq j \leq p \), and \( \gcd^* \{ \beta_j(B_s) : 1 \leq j \leq p \} = 1 \),
3. \( f_s \in \sum_{j \geq i_s} \mathbb{Z} \beta_j, \ f_s(B_t) = \delta_{i_s,t} \),
4. \( \mathbf{v}_{i_{s-1}} \cap \left( \bigcap_{t \geq s} \ker f_t \right) = \mathfrak{k} \).

**Proof.** We proceed by induction on \( s = m, m-1, \ldots, 1 \). For \( s = m \), it is possible to choose \( B_m \in \mathbf{v}_{i_{m-1}} \setminus \mathbf{v}_{i_{m}} \), such that \( \beta_j(B_m) \in \mathbb{Z} \) for any \( j \), and \( \gcd^* \{ \beta_j(B_m) : j \} = 1 \). Therefore there are natural numbers \( a_{m,j} \) (\( a_{m,j} = 0 \) if \( \beta_j(B_m) = 0 \)) such that

\[
1 = \sum_j a_{m,j} \beta_j(B_m).
\]

Put \( f_m = \sum_j a_{m,j} \beta_j \).

Then \( f_m(B_m) = 1, \mathbf{v}_{i_{m-1}} = \mathbf{v}_{i_{m}} \oplus \mathbb{R} B_m, \) and \( \mathbf{v}_{i_{m-1}} \cap \ker f_m = \mathbf{v}_{i_{m}} \). Properties 1, \ldots, 4 are holding for \( B_m \) and \( f_m \).

Suppose now that we have built \( B_m, \ldots, B_s \) and \( f_m, \ldots, f_s \) such that properties 1, \ldots, 4 hold for any \( s \leq t \leq m \). Then \( (\beta_{i_1}, \ldots, \beta_{i_{s-1}}, f_s, \ldots, f_m) \) is a free system in \( (\beta_j : 1 \leq j \leq p)_{\mathbb{R}} \), belonging to \( (\beta_j : 1 \leq j \leq p)_{\mathbb{Q}} \). Thus it is a basis for this \( \mathbb{Q} \)-vector space. It is then possible to find

\[
B_{s-1} \in \left( \mathbf{v}_{i_{s-2}} \cap \left( \bigcap_{t \geq s} \ker f_t \right) \right) \setminus \left( \mathbf{v}_{i_{s-1}} \cap \left( \bigcap_{t \geq s} \ker f_t \right) \right).
\]

We can choose \( B_{s-1} \) such that \( \beta_j(B_{s-1}) \in \mathbb{Z} \) for all \( 1 \leq j \leq p \), and \( \gcd^* (\beta_j(B_{s-1}) : j) = 1 \). Hence for \( s-1 \), \( B_{s-1} \) satisfies the conditions 1 and 2. Now there are natural numbers \( a_{s-1,j} \) such that

\[
1 = \sum_j a_{s-1,j} \beta_j(B_{s-1}).
\]

Put \( f_{s-1} = \sum_j a_{s-1,j} (\beta_j - \sum_{t \geq s} \beta_j(B_t)f_t) \).
By construction $f_{s-1}(B_t) = 0$ for $t > s - 1$, and since $B_{s-1} \in \cap_{t \geq s} \ker f_t$, then

$$f_{s-1}(B_{s-1}) = \sum_j a_{s-1,j} \beta_j(B_{s-1}) = 1.$$  

Since also $f_t(B_{s-1}) = 0$ for $t \geq s$, we see that the elements $B_m, \ldots, B_{s-1}$ and linear forms $f_m, \ldots, f_{s-1}$ satisfy condition 3. As for condition 4, first note that it is clear that $\mathfrak{k} \subseteq \mathfrak{v}_{i_{s-2}} \cap \cap_{t \geq s-1} \ker f_t$, and moreover, that $\mathfrak{k} + \sum_{t=s-1}^{m} \mathbb{R} B_t \subseteq \mathfrak{v}_{i_{s-2}}$. But

$$\dim(\mathfrak{k} + \sum_{t=s-1}^{m} \mathbb{R} B_t) = \dim \mathfrak{v} - s + 2 = \dim \mathfrak{v}_{i_{s-2}}$$

so in fact $\mathfrak{k} + \sum_{t=s-1}^{m} \mathbb{R} B_t = \mathfrak{v}_{i_{s-2}}$. Therefore if $X$ belongs to $\mathfrak{v}_{i_{s-2}} \cap (\cap_{t \geq s-1} \ker f_t)$, there is $Y$ in $\mathfrak{k}$ such that

$$X = Y + \sum_{t=s-1}^{m} u_t B_t,$$

but then $0 = f_t(X) = u_t$, for any $t \geq s - 1$, showing that $X = Y \in \mathfrak{k}$. Thus condition 4 is verified for $s - 1$, and the induction step is complete.

\[\square\]

The following is immediate.

**Corollary 4.5.** Suppose that $G$ satisfies the rationality condition. Then there is an explicit algorithm to construct elements $B_m, B_{m-1}, \ldots, B_1$ such that $\Gamma_1 = \mathfrak{k} + 2\pi \mathbb{Z} B_1 + 2\pi \mathbb{Z} B_2 + \cdots + 2\pi \mathbb{Z} B_m$.

Observe that, although the elements $B_1, \ldots, B_m$ are not nilpotent, the group $G_1$ is unipotent. Recall that given any $T$ in the Lie algebra $\mathfrak{gl}(V_C)$, we have unique elements $s(T)$ and $n(T)$ belonging to $\mathfrak{gl}(V_C)$ such that $s(T)$ is semisimple, $n(T)$ is nilpotent, $T = s(T) + n(T)$, and $s(T)$ and $n(T)$ commute with any elements commuting with $T$. If $T \in \mathfrak{g}$ is identified with the complex, lower triangular block matrix as described above, then $n(T)$ is just the strictly lower triangular part of $T$. For $T \in \Gamma_1$, we have $\exp s(T) = 1$ and hence $\exp T = \exp n(T)$. Now we set $X_s = n(B_s), 1 \leq s \leq m$, and we have $G_1 = \exp \Gamma$ where

$$\Gamma = n(\Gamma_1) = \mathfrak{k} + \sum_{s=1}^{m} 2\pi \mathbb{Z} X_s.$$  

Put $\mathfrak{n} = \text{Vect}_\mathbb{R}(\Gamma)$; then $\mathfrak{n}$ consists of nilpotent elements and $N = N(G_1) = \exp \mathfrak{n}$ is an analytic unipotent group containing $G_1$. We emphasize that both $\Gamma_1$ and $\mathfrak{n}$ are intrinsically defined and do not depend upon the generators $B_s$. Note also that $N$ is
abelian. If $H$ is any analytic unipotent group containing $G_1$ with Lie algebra $\mathfrak{h}$, then for each $1 \leq s \leq m$, there is $Y \in \mathfrak{h}$ such that

$$\exp 2\pi X_s = \exp 2\pi B_s = \exp Y.$$  

Since both $X_s$ and $Y$ are nilpotent, then $X_s = Y$. Hence $\exp \mathbb{R}X_s \subset H$ and it follows that $N \subset H$. Thus $N$ is the smallest analytic unipotent subgroup of $GL(V)$ containing $G_1$.

We remark that $N$ is a closed Lie subgroup in $GL(V)$, $G_1$ is a closed Lie subgroup in $G$ and a subgroup in $N$, but $G_1$ may not be a closed subgroup in $N$, as the following example shows.

**Example 4.6.** Take $V = \mathbb{C}^4$ and let $\mathfrak{g}$ be the commutative subalgebra with basis $\{A_1, \ldots, A_4\}$ where:

$$A_1 = \begin{bmatrix} i & i \\ i & i \\ 1 & i \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2i \\ 2i & 2i \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}.$$

Then we have, up to complex conjugation, two purely imaginary roots $\lambda_j = i\beta_j$, with

$$\beta_1(A_1) = 1, \quad \beta_1(A_2) = 0, \quad \beta_1(A_3) = 0, \quad \beta_1(A_4) = 0, \quad \beta_2(A_1) = 1, \quad \beta_2(A_2) = 2, \quad \beta_2(A_3) = 0, \quad \beta_2(A_4) = 0.$$

Since

$$\sqrt{2}A_1 - A_2 + A_3 = \begin{bmatrix} i\sqrt{2} & i(\sqrt{2} - 2) \\ i(\sqrt{2} - 2) & i(\sqrt{2} - 2) \end{bmatrix},$$

we see that $G$ is not a closed Lie subgroup of $GL(V)$.

Observe that $\mathfrak{k} = \mathbb{R}A_3 + \mathbb{R}A_4$. Following the algorithm of Lemma 4.4, we obtain $B_2 = \frac{1}{2}A_2$ and $B_1 = A_1 - \frac{1}{2}A_2$, and hence

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 - \sqrt{2} & -\frac{1}{2} & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$
Now \( \Gamma = \mathfrak{t} + 2\pi \mathbb{Z} X_1 + 2\pi \mathbb{Z} X_2 \) and so \( G_1 = \exp \Gamma \) is the subgroup of \( GL(4, \mathbb{C}) \) defined by

\[
G_1 = \left\{ \begin{bmatrix} 1 & 1 & 1 & \pi(2k_1 + \sqrt{2}(k_2 - k_1)) \ z + \pi(k_2 - k_1) & 1 \\ \pi(2k_1 + \sqrt{2}(k_2 - k_1)) & z + \pi(k_2 - k_1) & 1 & 1 & 1 \\ \pi(2k_1 + \sqrt{2}(k_2 - k_1)) & z + \pi(k_2 - k_1) & 1 & 1 & 1 \\ \pi(2k_1 + \sqrt{2}(k_2 - k_1)) & z + \pi(k_2 - k_1) & 1 & 1 & 1 \\ \pi(2k_1 + \sqrt{2}(k_2 - k_1)) & z + \pi(k_2 - k_1) & 1 & 1 & 1 \end{bmatrix} : \ z \in \mathbb{C}, \ k_1 \in \mathbb{Z}, \ k_2 \in \mathbb{Z} \right\}.
\]

This is a closed Lie subgroup of \( G \), but as subgroup of \( N = \left\{ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} : \ s \in \mathbb{R}, \ z \in \mathbb{C} \right\} \), it is not locally closed.

On the other hand, in the following example \( G_1 \) is closed in \( GL(V) \), and yet we shall see in Example 4.12 that its orbits are almost everywhere non-regular.

**Example 4.7.** Take now \( V = \mathbb{C}^2 \oplus \mathbb{R}^3 \), and let \( \mathfrak{g} \) be the commutative algebra \((A_1, A_2)_{\mathbb{R}}\) where:

\[
A_1 = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\]

Then we have, up to complex conjugation, three purely imaginary roots \( \lambda_j = i\beta_j \), with

\[
\beta_1(A_1) = 1, \quad \beta_1(A_2) = 0,
\]

\[
\beta_2(A_1) = 0, \quad \beta_2(A_2) = 1,
\]

\[
\beta_3(A_1) = 0, \quad \beta_3(A_2) = 0.
\]

Here \( \mathfrak{t} \) is trivial, Lemma 4.4 gives \( B_j = A_j \), \((j = 1, 2)\), and \( G_1 \) is the subgroup of \( GL(5, \mathbb{C}) \) defined by:

\[
G_1 = \left\{ \begin{bmatrix} 1 & 1 & 1 & \pi k_1 & 1 \\ 1 & 1 & 1 & \pi k_1 & 1 \\ 1 & 1 & 1 & \pi k_1 & 1 \\ \pi k_1 & 1 & 1 & \pi k_1 & 1 \\ \pi k_1 & 1 & 1 & \pi k_1 & 1 \end{bmatrix} : k_1 \in \mathbb{Z}, \ k_2 \in \mathbb{Z} \right\}.
\]
This is a closed Lie subgroup of $G$, and a closed subgroup of
\[
N = \left\{ \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ s & t & \cdots & 1 \end{pmatrix} : s \in \mathbb{R}, \ t \in \mathbb{R} \right\}.
\]

Standard facts about lattices yield the following.

**Lemma 4.8.** We have $G_1$ is closed in $GL(V)$ if and only if $\Gamma$ is closed, if and only if
\[
dim(X_1 + \mathfrak{k}, X_2 + \mathfrak{k}, \ldots, X_m + \mathfrak{k})_{\mathbb{R}} = dim(X_1 + \mathfrak{k}, X_2 + \mathfrak{k}, \ldots, X_m + \mathfrak{k})_{\mathbb{Q}}.
\]

**Proof.** Since the exponential is a global homeomorphism from $n$ onto $N$ and $N$ is closed in $GL(V)$, then the first equivalence is clear. Now denote by $p : n \longrightarrow n/\mathfrak{k}$ the canonical projection. Then $\Gamma = p^{-1}(p(\Gamma))$ is closed if and only if $p(\Gamma)$ is discrete, if and only if $p(\Gamma)$ is a lattice in $n/\mathfrak{k}$.

But $p(\Gamma) = 2\pi \sum_s \mathbb{Z} p(X_s)$, and this is a lattice if and only if
\[
dim(p(X_1), p(X_2), \ldots, p(X_m))_{\mathbb{R}} = dim(p(X_1), p(X_2), \ldots, p(X_m))_{\mathbb{Q}}.
\]

□

Even though we do not suppose that $G_1$ is closed in $N$, we shall nevertheless study the action of $N$ to get a criterion for the regularity of $G$. First, we define a Zariski open subset of $V$ in which the action of $N$ is easily described.

Fix a basis $\{Y_1, \ldots, Y_r\}$ for $\mathfrak{k}$. It is clear that the set $\{Y_1, \ldots, Y_r, X_1, X_2, \ldots, X_m\}$ spans $n$, though $\{X_1, X_2, \ldots, X_m\}$ is not necessarily linearly independent. For the moment put $Y_{r+s} = X_s, 1 \leq s \leq m$. We linearly order all finite sequences of positive integers as follows. If $j$ and $j'$ are any finite sequences of positive integers, we say that $j < j'$ if

1. $|j| > |j'|$, or
2. if $|j| = |j'|$, then for some $r$, $j_s = j'_s, 1 \leq s < r$ and $j_r < j'_r$.

For each $v \in V$ the Lie algebra of the stabilizer $N(v)$ of $v$ in $N$ is $n(v) = \{Y \in n : Y \cdot v = 0\}$. Now for each $v \in V$ put
\[
j(v) = \{1 \leq j \leq q : n_j \not\subset n_{j-1} + n(v)\}
\]
and write $j(v)$ as an increasing sequence. Set $J = \{j(v) : v \in V\}$ and $j = \min J$. It is well-known that $\Omega_0 = \{v \in V : j(v) = j\}$ is a Zariski open subset of $V$. Put
\[
m = (Y_j : j \in j)_{\mathbb{R}}.
\]
For each $v \in \Omega_0$, $m$ is a supplementary space for $n(v)$ in $n$. We denote by $\rho(Y, v)$ the projection of $Y$ into $n(v)$ parallel to $m$ and $\eta(Y, v) = Y - \rho(Y, v)$, so that $\eta(Y, v)$ is the projection of $Y$ into $m$ parallel to $n(v)$. It is easily seen that the maps
\[
\rho : n \times \Omega_0 \longrightarrow n(v) \quad \text{and} \quad \eta : n \times \Omega_0 \longrightarrow m
\]
are non-singular and rational. Observe that for each $1 \leq j \leq q$, if $j \notin j$ then $Y_j \in n(v) \cap n_j + n_{j-1}$. It follows inductively that $\rho(Y_j, v)$ belongs to $n(v) \cap n_j$, and in particular, that $\rho(\ell, v) = \ell \cap n(v)$ and hence $\eta(\ell, v) = \ell \cap m$.

Write $s = j - r$; then $\{X_s : s \in s\}$ is linearly independent and for each $s \in s$, $\eta(X_s, v) = X_s$. Fix $u \notin s, 1 \leq u \leq m$. Then we have a non-singular rational function $Z_u : \Omega_0 \longrightarrow \ell \cap m$, and for each $s \in s$, a non-singular rational, real-valued function $c_{s, u}$ on $\Omega_0$, such that
\[
\eta(X_u, v) = \sum_{s \in s} c_{s, u}(v)X_s + Z_u(v).
\]

**Remark 4.9.** Since $G$ is abelian, $n(s \cdot v) = n(v)$, for any $s \in G$. Therefore, we also have $j(s \cdot v) = j(v)$ for any $v$, and $\Omega_0$ is a $G$-invariant subset of $V$. Similarly, the projection functions $\rho(\cdot, v)$ and $\eta(\cdot, v)$ are $G$-invariant: for any $s \in G$, $\eta(\cdot, s \cdot v) = \eta(\cdot, v)$. Hence the functions $c_{s, u}$ are $G$-invariant.

We can now characterize regularity of a $G_1$-orbit in terms of the explicitly computable objects above.

**Lemma 4.10.** Let $v \in \Omega$. Then the following are equivalent:

1. $G_1 \cdot v$ is regular.
2. $G_1 \cdot v$ is closed.
3. $\eta(\Gamma) \cdot v$ is closed in $gl(V)$.
4. $\eta(\Gamma, v)$ is locally closed in $gl(V)$.
5. $n(v) + \Gamma$ is closed in $gl(V)$.

**Proof.** The map $X \mapsto \exp X \cdot v$ is a homeomorphism of $m$ with $N \cdot v$, and $G_1 \cdot v$ is the image of $\eta(\Gamma, v)$ under this map. Since $m$ is closed in $gl(V)$ and $N \cdot v$ is closed in $V$, then $\eta(\Gamma, v)$ is closed (resp. locally closed) if and only if $G_1 \cdot v$ is closed (resp. regular) in $V$. In particular, 2. and 3. (resp. 1. and 4.) are equivalent.

By definition of $\eta$, we have $n(v) + \Gamma = \eta^{-1}(\eta(\Gamma, v), v)$. It follows that $n(v) + \Gamma$ is closed if and only if $\eta(\Gamma, v)$ is closed in $gl(V)$, and thus 3. and 5. are equivalent.

Now we can write
\[
\eta(\Gamma, v) = \eta(\ell, v) + \sum_{s \in s} 2\pi\mathbb{Z} \eta(X_s, v) + \sum_{u \notin s} 2\pi\mathbb{Z} \eta(X_u, v)
\]

\[
= \ell \cap m + \sum_{s \in s} 2\pi\mathbb{Z} X_s + \sum_{u \notin s} 2\pi\mathbb{Z} \left( \sum_{s \in s} c_{s, u}(v)X_s \right).
\]
Put $p : m \rightarrow m/(\frak{k} \cap m)$; then $\eta(\Gamma, v) = p^{-1}(p(\eta(\Gamma, v)))$ is closed if and only if $p(\eta(\Gamma, v))$ is closed, if and only if $p(\eta(\Gamma, v))$ is a lattice, if and only if $p(\eta(\Gamma, v))$ is locally closed, if and only if $\eta(\Gamma, v)$ is locally closed. Thus 3. and 4. are equivalent. □

For each $v \in \Omega_0$, put $\Delta(v) = \Delta_G(v) = n(v) + \Gamma$; it is clear that $\Delta(v)$ is explicitly computable and depends only upon $G$ and $v$. However, in terms of the constructions above (which depend upon the choice of $Y_1, \ldots, Y_q$), we have

$$\Delta(v) = n(v) + \eta(\Gamma, v) = n(v) + \frak{k} \cap m + \sum_{s \in S} 2\pi Z X_s + \sum_{u \notin S} 2\pi Z \left( \sum_{s \in S} c_{s,u}(v) X_s \right).$$

The preceding shows that $G_1 \cdot v$ is regular if and only if $\Delta(v)$ is closed. We return to our previously computed examples to illustrate the above.

**Example 4.11.** Let us first come back to Example 4.6. In this example $g = (A_1, \ldots, A_4)_\mathbb{R}$ and

$$G_1 = \exp(\mathbb{R} A_3 + \mathbb{R} A_4 + 2\pi Z X_1 + 2\pi Z X_2),$$

so

$$n = (Y_1, Y_2, Y_3, Y_4)_\mathbb{R}$$

where $Y_1 = A_3, Y_2 = A_4, Y_3 = X_1$, and $Y_4 = X_2$. Any $N$-orbit has dimension at most two and if $v_3 \neq 0$, then $Y_1$ and $Y_2$ act non-trivially and independently on $v$. Hence the minimum index set is $j = \{1, 2\}$, and the Zariski-open layer is

$$\Omega_0 = \{v \in V : j(v) = j\} = \{[v_1, \ldots, v_4]^t \in \mathbb{C}^4 : v_3 \neq 0\}.$$

We have $m = \frak{k} = (A_3, A_4)_\mathbb{R}$, and we then get

$$\rho(\psi_3, v) = \rho(X_1, v) = X_1 - \left( -\frac{1}{2} \right) A_3 - \left( 1 - \frac{\sqrt{2}}{2} \right) \left( \Re \left( \frac{v_2}{v_3} \right) A_3 + \Im \left( \frac{v_2}{v_3} \right) A_4 \right),$$

$$\rho(\psi_4, v) = \rho(X_2, v) = X_2 - \frac{1}{2} A_3 - \frac{\sqrt{2}}{2} \left( \Re \left( \frac{v_2}{v_3} \right) A_3 + \Im \left( \frac{v_2}{v_3} \right) A_4 \right).$$

Thus for all $v \in \Omega_0$, $\eta(X_s, v) = X_s - \rho(X_s, v)$ belongs to $\frak{k}$. Observe that both $\eta(\Gamma, v) = \frak{k}$ and $\Delta(v) = n$ are closed, and hence the $G_1$-orbit of $v$ is regular (even though $\Gamma$ is not closed.)

**Example 4.12.** Let us come back to Example 4.7. In this example, $\Gamma = 2\pi Z X_1 + 2\pi Z X_2$ and

$$G_1 = \exp(2\pi Z X_1 + 2\pi Z X_2)$$
so \( n = (Y_1, Y_2) \mathbb{R} \) where \( Y_1 = X_1, Y_2 = X_2 \). Here an \( N \)-orbit has dimension at most one, the minimum index set is \( j = \{1\} \) and

\[
\Omega_0 = \{ v \in V : j(v) = j \} = \{ [v_1, \ldots, v_5] : v_3 \neq 0 \}.
\]

For \( v \in \Omega_0 \) we have

\[
\rho(X_2, v) = X_2 - \frac{v_4}{v_3} X_1;
\]

hence \( \eta(\Gamma, v) = \{ 2\pi(k_1 + \frac{u}{v_3} k_2)X_1 : k_1, k_2 \in \mathbb{Z} \} \) and

\[
\Delta(v) = \mathbb{R}(X_2 - \frac{v_4}{v_3} X_1) + 2\pi(\mathbb{Z} + \frac{v_4}{v_3} \mathbb{Z}) X_1.
\]

Both are not closed for any \( v \) in the dense, conull set \( \mathcal{U}_0 = \{ v \in V : v_3 \neq 0, \frac{u}{v_3} \notin \mathbb{Q} \} \), (even though \( \Gamma \) is closed) and every orbit in \( \mathcal{U}_0 \) is not regular.

In the preceding examples, either \( \Omega_0 \) is \( G_1 \)-regular (Example 4.6), or almost all orbits in \( \Omega_0 \) are not regular (Example 4.7). The following shows that these are the only possibilities for \( \Omega_0 \). Moreover, we see that if \( \Omega_0 \) is regular, then the function \( \Delta : \Omega_0 \rightarrow P(\mathbb{n}) \) is constant.

**Lemma 4.13.** The following are equivalent.

1. \( \Omega_0 \) is \( G_1 \)-regular.
2. \( \Delta(v) \) is independent of \( v \in \Omega_0 \) and is a closed subset of \( gl(V) \).
3. Each of the rational functions \( c_{s,u}, s \in \mathfrak{s}, u \notin \mathfrak{s} \), is constant and takes a rational value.

Moreover, if \( \Omega_0 \) is not \( G_1 \)-regular, then there is a conull \( G \)-invariant \( \mathfrak{g}_s \) subset of \( \Omega_0 \) in which all \( G_1 \)-orbits are not regular.

**Proof.** Suppose that \( \Omega_0 \) is \( G_1 \)-regular. By Lemma 4.10, \( \eta(\Gamma, v) \) is closed for all \( v \in \Omega_0 \). Referring to the expression (4.6) for \( \eta(\Gamma, v) \), we see that each of the values \( c_{s,u}(v) \) in this expression must be a rational number. Since \( c_{s,u} \) is a real-valued, rational function on an open set that takes only rational values, it must be constant. Hence for each \( u \notin \mathfrak{s} \) we have \( b_{s,u} \in \mathbb{Q}, s \in \mathfrak{s} \) such that \( c_{s,u}(v) = b_{s,u} \) holds for all \( v \in \Omega_0 \) and we can write

\[
\eta(X_u, v) = \sum_{s \in \mathfrak{s}} b_{s,u} X_s + Z_u(v)
\]
where \( Z_u(v) \in \mathfrak{k} \cap \mathfrak{m} \). Now for \( s \in \mathfrak{s} \) we have \( \rho(X_s, v) = 0 \), and so

\[
\mathfrak{n}(v) = \rho(\mathfrak{n}, v) = \rho(\mathfrak{k}, v) + \sum_{s=1}^{m} \mathbb{R}_s \rho(X_s, v)
\]

\[
= \mathfrak{k} \cap \mathfrak{n}(v) + \sum_{u \not\in \mathfrak{s}} \mathbb{R}(X_u) + \sum_{u \not\in \mathfrak{s}} (X_u - \sum_{s \in \mathfrak{s}} b_{s,u} X_s).
\]

Hence

\[
\mathfrak{h} + \mathfrak{n}(v) = \mathfrak{k} + \sum_{u \not\in \mathfrak{s}} \mathbb{R} \left( X_u - \sum_{s \in \mathfrak{s}} b_{s,u} X_s \right)
\]

is independent of \( v \in \Omega_0 \). If we put \( \mathfrak{h} = \mathfrak{k} + \mathfrak{n}(v) \) then

\[
\Delta(v) = \mathfrak{h} + \sum_{s \in \mathfrak{s}} 2\pi \mathbb{Z} X_s + \sum_{u \not\in \mathfrak{s}} 2\pi \mathbb{Z} \left( \sum_{s \in \mathfrak{s}} b_{s,u} X_s \right).
\]

Thus \( \Delta(v) \) is a closed subset of \( \mathfrak{gl}(V) \), independent of \( v \in \Omega_0 \).

Now suppose that \( \Omega_0 \) is not \( G_1 \)-regular, so for some \( v_0 \in \Omega_0 \), \( \eta(\Gamma, v_0) \) is not closed. This means that for some \( s \in \mathfrak{s} \) and some \( u \not\in \mathfrak{s} \), the value \( c_{s,u}(v_0) \) of the rational function \( c_{s,u} \) is an irrational number. It follows that \( \Delta(v_0) \) is not closed, and that the non-empty set

\[
\mathcal{U}_0 = \{ v \in \Omega : c_{s,u}(v) \not\in \mathbb{Q} \} = \bigcap_{r \in \mathbb{Q}} \{ v \in \Omega : c_{s,u}(v) \neq r \}
\]

is a conull \( \mathfrak{G}_d \)-set. Since \( c_{s,u} \) is \( G \)-invariant (Remark 4.9), then \( \mathcal{U}_0 \) is \( G \)-invariant, and for each \( v \in \mathcal{U}_0 \), the expression (4.6) shows that \( \eta(\Gamma, v) \) is not closed, hence \( G_1 \cdot v \) is not regular.

\[\square\]

Define the Zariski open \( G \)-invariant subset \( \Omega \) of \( V \) by

\[
\Omega = \Omega_0 \cap \Omega_1
\]

A combination of Proposition 3.1, Lemma 4.3, and Lemma 4.13 yields the following regularity criterion.

**Theorem 4.14.** (a) If \( \Omega \) is \( G \)-regular, then \( \Omega_0 \) is \( G_1 \)-regular while if \( \Omega_0 \) is not \( G_1 \)-regular, then there is a \( G \)-invariant, conull, \( \mathfrak{G}_d \) subset \( \mathcal{U} \) in \( V \) such that any \( G \)-orbit \( \mathcal{O} \) in \( \mathcal{U} \) is not regular.

(b) Suppose that \( G \) satisfies the rationality condition; then the following are equivalent:

1. \( \Omega \) is \( G \)-regular,
2. \( \Omega_0 \) is \( G_1 \)-regular,
3. the set-valued function $v \mapsto \Delta(v)$ is constant and its value is a closed subset of $gl(V)$.

Proof. Suppose that $\Omega_0$ is not $G_1$ regular. Then by Lemma 4.13, we have a conull, $G$-invariant $\mathfrak{S}_\delta$-set $U_0$ in $\Omega_0$ such that every $G_1$-orbit in $U_0$ is not regular. Now $U = U_0 \cap \Omega$ is a $G$-invariant, conull $\mathfrak{S}_\delta$ subset in which every orbit is not regular. This proves the first part of the theorem.

As for part 2, recall that by Lemma 4.13, $G_1$-regularity of $\Omega_0$ is equivalent with the condition that $\Delta(v)$ does not depend upon $v \in \Omega$ and is closed. Suppose $\Omega_0$ is $G_1$-regular and that $G$ satisfies the rationality condition. Let $v = (w, z) \in \Omega$ with $O = Gv$, $O_0 = b(G)w$ and $\omega = \phi_w(G_1)z$. By Lemma 4.3, $O_0$ is regular, and since $G_1v$ is regular then $\omega$ is regular. Hence by Proposition 3.1, $O$ is regular. This completes the proof. □

References

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