Quiz I MATH 355 January 22, 2010

1. Consider the differential equation

\[
\frac{dy}{dt} = y^3 - 3y^2 + 2y = y(y-1)(y-2)
\]

a) For what values of \( y \) is \( y(t) \) in equilibrium? For what values of \( y \) is \( y(t) \) increasing? For what values of \( y \) is \( y(t) \) decreasing?

\[
\begin{align*}
0 &< y < 1, \\
1 &< y < 2, \\
y &> 2
\end{align*}
\]

b) Make a rough sketch of the slope field.

c) Sketch the graphs of the solutions \( y(t) \) with the initial conditions \( y(0) = 1 \), \( y(0) = 3 \), \( y(0) = -1 \), \( y(1) = 1/2 \)

2. Give the general solution for the following differential equation as well as the solution to the initial value problem.

\[
\frac{dx}{dt} = \frac{t^2}{x + t^3x}
\]

\( x(0) = -2 \)

\[
\theta + \hat{c} + \checkmark
\]

\[
\begin{align*}
y - &+ \\
y - 1 &- + \\
y - 2 &- - + \\
&- + \\
\end{align*}
\]

\[
x = \frac{\sqrt{2 + 2t^3 + c}}{3}
\]

3. Solve the following mixture problem to find how much chocolate is in the vat at any time \( t \). A vat initially contains 1000 gallons of plain milk. A chocolate milk mixture containing 1/2 lb of chocolate per gallon is added at the rate of 50 gallons per minute. The mixture in the vat is constantly stirred and the result is poured out at the rate of 50 gallons per minute.

\[
\frac{dc}{dt} = 50 \cdot \frac{1}{20} - \frac{c}{500} \cdot \frac{1}{50} = 25 - \frac{c}{20} = \frac{500 - c}{20}
\]

\[
\frac{dc}{dt} = \frac{c - 500}{500 - c} = -\frac{1}{20} \frac{dt}{dt} + \frac{c}{500} \cdot \frac{1}{20} t
\]

\[
c(0) = 0, \quad c(t) = 500 - 500e^{-\frac{1}{20} t}
\]
A general type of question in mathematics:
Does a solution exist to a given problem?
Is that solution unique?

Example: \( x^2 + 1 = 0 \), zeros? none if restrict to real #s
If allow complex solutions, there are 2: \( \pm i \).

For a differential equation \( \frac{dy}{dt} = f(t,y) \), Note that a solution is a function \( y(t) \) and needs to be defined on an interval so that:

\[
\frac{dy}{dt} = f(t,y)
\]

* So existence:

Theorem: If \( f(t,y) \) is continuous on an open rectangle \( a < t < b, c < y < d \) and \( (t_0, y_0) \) is a point in that rectangle then there is an \( \epsilon > 0 \) and a function \( y(t) \) defined on \( (t_0 - \epsilon, t_0 + \epsilon) \) satisfying the initial value problem:

\[
\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0
\]

\[
y(a) = -\frac{1}{(y_0^2 - 1) \epsilon}
\]

The RHS is continuous, for example, on \( (1.2, 4.1) \) there is a solution \( y(t) \) defined in some neighborhood of \( 1.2 \) so that \( y'(1.2) = \frac{1}{(y(1.2)^2 - 1) \epsilon} \), \( y(1.2) = 4.1 \)

* This function is what? How large can an interval be defined for and satisfy the differential equation?
Uniqueness has somewhat more interesting consequences. As an example, if we know solutions for an initial value problem are unique and we know an equilibrium solution, we also know that the graphs of other solutions cannot cross over that line determined by the equilibrium solution.

\[ \text{We have already seen this with the logistic differential equation.} \]

**Uniqueness Theorem**

Given the differential equation \( \frac{dy}{dt} = f(t, y) \) with \( f(t, y) \) and \( \frac{df}{dy} \) continuous on the open rectangle \( a < t < b, \ c < y < d \). If \( (t_0, y_0) \) is in this rectangle and \( y_1(t), y_2(t) \) satisfy the initial value problem \( \frac{dy_i}{dt} = f(t, y_i), \ y_i(t_0) = y_0 \) for \( i = 1, 2 \) on some interval \( (t_0 - \varepsilon, t_0 + \varepsilon) \) then they agree on that interval.

\[ \frac{dy}{dt} = f(y) \]

\[
\begin{align*}
y_1(t) &= 4 & \text{for all } t \text{ is sol;} \\
y_2(t) &= 2 & \text{is is is is} \\
y_3(t) &= 0 & \text{is is is is}
\end{align*}
\]

Initial Cond: \( y(0) = 1 \)

\[ 0 < y(t) < 2 \text{ for all } t \text{ where } y \text{ is defined.} \]
#10 \( \frac{dy}{dt} = 2\sqrt{|y|} \quad \text{< autonomous} \)
\[ \sqrt{|y|} \text{ not differentiable at } y=0 \]

1) \( y = 0 \) is a sol:
\[ \frac{dy}{dt} = 0 = |y|, \checkmark \]

2) \( y > 0 \):
\[ \frac{dy}{\sqrt{y}} = 2dt, \quad 2\sqrt{y} = 2t + C. \quad \sqrt{y} = t + C. \]
Since \( \sqrt{y} > 0 \), \( t + C > 0 \):
\[ t > -C \quad y = (t + C)^2 \]

3) \( y < 0 \):
\[ \frac{dy}{\sqrt{-y}} = 2dt, \quad (-\sqrt{y}) = -t + k. \quad y = -(k-t)^2 \]
Since \( (-\sqrt{y}) > -t + k > 0 \):
\[ k > t, \quad y = -(k-t)^2 \]

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Example of a solution
\[ y_1(t) = \frac{1}{t-1} \quad y_2(t) = \frac{1}{t-2} \quad \text{are solutions of} \quad \frac{dy}{dt} = -y^2. \]

Similarly for \( y_2 \).

\[ \frac{dy}{dt} = -\frac{1}{(t-1)^2} = -y_1^2. \]

Know \( y_1(t) < y(t) < y_2(t) \).

Because \( y_1 \rightarrow 0, t \rightarrow \infty \) as \( t \rightarrow -\infty \) y_2.

Also \( y(t) \) has a vertical asymptote between 1 and 2 because of the vertical asymptotes of \( y_1 \) and \( y_2 \).