Geometric analysis on Cantor sets and trees

Anders Björn ¹  Jana Björn ¹  James T. Gill ²  Nageswari Shanmugalingam ³

¹Linköpings Universitet ²Saint Louis University
³University of Cincinnati

October 18, 2013
Trees and Cantor sets

Poincare Inequality

Trace and Extension

Quasisymmetry and Rough Quasiisometry
A tree is a connected graph $X$ without cycles. A rooted tree is a tree with a distinguished vertex called the root denoted $0_X$. We discuss two different metrics on $X$. The simple graph distance denoted $|x - y|$ and . . .
...a uniformizing distance given by

$$d_\epsilon(x, y) = \int_{[x,y]} e^{-\epsilon|z|} \, dz$$

(1)

where $\epsilon > 0$ is chosen, $[x, y]$ is the unique path between $x$ and $y$, $|z| = |z - 0_X|$, and $d|z|$ is the measure that gives each edge Lebesgue measure 1. The diameter of the tree with respect to this metric is $2/\epsilon$. 
We assume that each vertex of the tree has at least one child and at most $K$ children. We equip $X$ with the doubling weighted measure

$$d\mu(x) = e^{-\beta|x|}d|x|$$

where $\beta > K$ (necessary for doubling). In fact if $0 < r' \leq r$,

$$\frac{\mu(B(x, r'))}{\mu(B(x, r))} \geq C \left(\frac{r'}{r}\right)^s$$

where $s = \max\{1, \beta/\epsilon\}$. 

A. Björn
Geometric analysis on Cantor sets and trees
We now look at $\partial X$, the boundary. The boundary will be identified with geodesics starting at the root. We write

$$\zeta = 0x_1x_2x_3 \ldots$$

Where $x_i$ is a vertex at $|\cdot|$ distance $i$ from the root and is connected to $x_{i-1}$. (We use Roman letters for $X$ and Greek letters for $\partial X$.)

Let $\zeta, \xi \in \partial X$ with $x_k$ the largest vertex they have in common with the above identification. Then we set

$$d_{\partial X}(\zeta, \xi) = 2 \int_k^\infty e^{-\epsilon t} \, dt = \frac{2}{\epsilon} e^{-\epsilon k}$$
This metric is an ultrametric,

\[ d_{\partial X}(\xi, \zeta) \leq \max\{d_{\partial X}(\xi, \eta), d_{\partial X}(\eta, \zeta)\} \]

So \( B(\zeta, r) = B(\xi, r) \) if \( \zeta \in B(\xi, r) \), every point in the ball can be the center.

It turns out that when \( X \) is a \( K - \text{regular} \) tree, \( \partial X \) is an Ahlfors \( Q \)-regular space with Hausdorff dimension

\[
\log K \\
\epsilon
\]
One can show $\partial X$ is a totally disconnected, perfect, compact metric space, hence it is a Cantor set.

If $K = 2$ and $\epsilon = \log 3$ makes $\partial X$ isometric to the Cantor ternary set. Some other important Cantor sets can be achieved in this way (Garnett-Ivanov set with positive length but zero analytic capacity).
Let $u$ be a locally summable function on the vertices of the tree $X$. We say that $g : X \to [0, \infty]$ (g is valued on the edges too) is an upper gradient if

$$|u(z) - u(y)| \leq \int_{\gamma} g d_\epsilon s$$

where $\gamma$ is the geodesic from $x$ to $y$ and $d_\epsilon s$ denote the arc length measure with respect to the metric $d_\epsilon$. 
The Newtonian space $N^{1,p}(X)$ is the collection of functions on $X$ such that the norm

$$\|u\|_{N^{1,p}(X)} := \left( \int_X u^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p} < \infty$$

where the infimum is taken over all upper gradients.
For $1 \leq p < \infty$ we show that $(X, d_\varepsilon, \mu)$ supports at $(p, p)$-Poincaré inequality:

$$
\frac{1}{\mu(B)} \int_B |u - u_B|^p d\mu \leq C \frac{r}{\mu(B)} \int_B g^p d\mu
$$

for all balls $B(x, r)$ and $u$ in $N^{1,p}(X)$ with $g$ any upper gradient.
Question: If a function $u$ on $X$ a regular $K$-ary tree is well-behaved, can it be extended to a function on $\partial X$? Also, can a function on $\partial X$ be traced to a function on $X$, perhaps by averaging?
**Question:** If a function $u$ on $X$ a regular $K$-ary tree is well-behaved, can it be extended to a function on $\partial X$? Also, can a function on $\partial X$ be traced to a function on $X$, perhaps by averaging?

**Answer:** Yes!
**Question:** If a function $u$ on $X$ a regular $K$-ary tree is well-behaved, can it be extended to a function on $\partial X$? Also, can a function on $\partial X$ be traced to a function on $X$, perhaps by averaging?

**Answer:** Yes! But we have to use Besov spaces.
Let $f : \partial X \to \mathbb{R}$. Let $\nu$ denote the normalized $Q$-dimensional Hausdorff measure on $\partial X$. For $t > 0$ and $p \geq 1$ we set

$$E_p(f, t) := \left( \int_{\partial X} \int_{B(\zeta, t)} |f(\zeta) - f(\xi)|^p \, d\nu(\xi) \, d\nu(\zeta) \right)^{1/p},$$

and for $\theta > 0$ and $q \geq 1$,

$$\|f\|_{B^\theta_{p,q}(\partial X)} := \left( \int_0^{2\text{diam}X} \left( \frac{E_p(f, t)}{t^\theta} \right)^q \, dt \right)^{1/q}. \quad (2)$$

The Besov space $B^\theta_{p,q}(\partial X)$ consists of all measurable functions $f$ on $\partial X$ for which this seminorm is finite. In this paper we only deal with the Besov spaces for which $q = p$, that is, the spaces $B^\theta_{p,p}(\partial X)$. The expression

$$\|f\|_{B^\theta_{p,p}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{B^\theta_{p,p}(\partial X)}$$

is a norm on $B^\theta_{p,p}(\partial X)$. 
An Example: Let $f(\zeta) = d_{\partial X}(\zeta, \zeta_0)^\alpha$ for some $\zeta_0 \in \partial X$ and $\alpha > 0$. This function is in $B^\theta_{p,p}(\partial X)$ when

$$\alpha > \theta - \frac{Q}{p} > 0$$

where $Q = \frac{\log K}{\epsilon}$ is the Hausdorff dimension. This shows that the Besov spaces for different $\theta$ are distinct. In fact, we show that

$$0 < \theta_1 < \theta_2 \implies B^\theta_{p,p}(\partial X) \cup B^\theta_{p,p}(\partial X)$$

so the spaces are nested.
Theorem Let $X$ be a regular $K$-ary tree. Let $\epsilon$ be the exponential weight for $d_X$ and $\beta$ the exponential weight for $\mu$. Let $\beta > \log K$ and $p \geq 1$. Then the trace space of the Newtonian space $N^{1,p}(X)$ is the Besov space $B^{\theta}_{p,p}(\partial X)$, where

$$\theta = 1 - \frac{\beta/\epsilon - \log(K)/\epsilon}{p}$$
**Theorem** Let $X$ be a regular $K$-ary tree. Let $\epsilon$ be the exponential weight for $d_X$ and $\beta$ the exponential weight for $\mu$. Let $\beta > \log K$ and $p \geq 1$. Then the trace space of the Newtonian space $N^{1,p}(X)$ is the Besov space $B^\theta_{p,p}(\partial X)$, where

$$\theta = 1 - \frac{\beta/\epsilon - \log(K)/\epsilon}{p}$$

Note the dimensional constants. They are very similar to results of Jonsson and Wallin in $\mathbb{R}^n$: For an Ahlfors $d$ regular compact subset $K$, the trace space of $W^{1,p}$ is $B^\tau_{p,p}(K)$ where $\tau = 1 - (n - d)/p$. 
We achieve both a sharp trace and a sharp extension theorem.
We achieve both a sharp trace and a sharp extension theorem.

The proof of the extension part of the theorem is mostly a calculation. One defines the extension on $X$ via averaging and carefully adds things up.
We achieve both a sharp trace and a sharp extension theorem.

The proof of the extension part of the theorem is mostly a calculation. One defines the extension on $X$ via averaging and carefully adds things up.

The trace part of the theorem is more involved.
A homeomorphism $f : Z \to W$ between two metric spaces is **quasisymmetric** if there is an increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that whenever $x, y, z \in Z$ and $x \neq z$, then

$$\frac{d_W(f(x), f(y))}{d_W(f(x), f(z))} \leq \eta \left( \frac{d_Z(x, y)}{d_Z(x, z)} \right)$$

It is known that if $Z$ and $W$ are uniformly perfect that $\eta$ may be chosen of the form

$$\eta(t) = \begin{cases} 
At^\alpha, & \text{if } t \leq 1, \\
A t^{1/\alpha}, & \text{if } t \geq 1,
\end{cases}$$
A mapping $F : X \to Y$ between two metric spaces is an $(L, \Lambda)$ rough quasiisometry if whenever $x, y \in X$ we have (using Polish notation for both metrics)

$$\frac{1}{L} |x - y| - \Lambda \leq |F(x) - F(y)| \leq L |x - y| + \Lambda$$

and for each $y$ there is a point $x \in X$ such that $|F(x) - y| \leq L + \Lambda$. 
Quasisymmetries extend to Rough Quasiisometries:

**Theorem** Let $X$ and $Y$ be two rooted trees, such that each vertex has at least two children and with respective exponential weights $\epsilon_X$ and $\epsilon_Y$. If $f : \partial X \to \partial Y$ is an $\eta$-quasisymmetry then there exists an $(L, \Lambda)$-rough quasiisometry $F : X \to Y$ which extends along geodesics in $X$ to $f$. There exist sharp quantitative formulae for $L$ and $\Lambda$ depending on the $\epsilon$'s and the quasisymmetry powers.
Quasisymmetries extend to Rough Quasiisometries:

**Theorem** Let $X$ and $Y$ be two rooted trees, such that each vertex has at least two children and with respective exponential weights $\epsilon_X$ and $\epsilon_Y$. If $f : \partial X \to \partial Y$ is an $\eta$-quasisymmetry then there exists an $(L, \Lambda)$-rough quasiisometry $F : X \to Y$ which extends along geodesics in $X$ to $f$. There exist sharp quantitative formulae for $L$ and $\Lambda$ depending on the $\epsilon$'s and the quasisymmetry powers.

Note that the graph metric is used in the rough quasiisometry, not the uniformizing metric.
Rough Quasiisometries extend to Quasisymmetries:

**Theorem** If $F : X \rightarrow Y$ is a rough quasiisometry with constant $L_1$ on the left inequality and $L_2$ on the right inequality, then there exists a quasisymmetric extension with

$$
\eta(t) = \begin{cases} 
At^\alpha, & \text{if } t \leq 1, \\
At^{1/\gamma}, & \text{if } t \geq 1,
\end{cases}
$$

where $\alpha = \frac{L_1 \epsilon_Y}{\epsilon_X}$ and $\gamma = \frac{L_2 \epsilon_Y}{\epsilon_X}$. 
Thanks to the audience for your time and Leonid and Jeremy for organizing the special section and the invitation.
Thanks to the audience for your time and Leonid and Jeremy for organizing the special section and the invitation.

*Advertisement* Matthew Badger, Joan Lind, and G. are organizing a section at the 2014 Spring Southeastern Sectional Meeting entitled “Complex Analysis, Probability and Metric Geometry” we hope you consider coming.