A quantitative notion of redundancy for finite frames

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Definition of a Frame

1952 Duffin/Schaeffer

\((\varphi_i)_{i=1}^N\) is a **frame** for a \(n\)-dimensional Hilbert space \(\mathbb{H}_n\) if there exists (lower, upper frame bounds) \(A, B > 0\) so that

\[
A \|x\|^2 \leq \sum_{i=1}^{N} |\langle x, \varphi_i \rangle|^2 \leq B \|x\|^2 \quad \forall x \in H
\]
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If $A = B = 1$: We call this a Parseval frame and if $\| \varphi_i \| = c$ for all $i = 1, 2, \ldots, N$, this is an equal-norm frame.
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If \(A=B=1\): We call this a **Parseval frame** and if \(\|\varphi_i\| = c\) for all \(i = 1, 2, \ldots, N\), this is an **equal-norm frame**.

In this case we say the frame has **redundancy** \(\frac{N}{n}\).
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For example, the following frames all have redundancy 2 (where \((e_i)_{i=1}^n\) is an orthonormal basis for \(\mathbb{H}_n\)):
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For example, the following frames all have redundancy 2
(where \((e_i)_{i=1}^{n}\) is an orthonormal basis for \(\mathbb{H}_n\)):

\[(e_i)_{i=1}^{n} \cup (e_i)_{i=1}^{n}\]

\[(e_i)_{i=1}^{n} \cup \{e_1, e_1, \ldots, e_1\}\] where \(e_1\) occurs \(n\)-times
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For example, the following frames all have redundancy 2 (where \((e_i)_{i=1}^n\) is an orthonormal basis for \(\mathbb{H}_n\)):

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\[
(e_i)_{i=1}^n \cup \{e_1, e_1, \ldots, e_1\} \quad \text{where } e_1 \text{ occurs } n\text{-times}
\]

\[
(e_i)_{i=1}^n \cup \{0, 0, \ldots, 0\} \quad \text{where } 0 \text{ occurs } n\text{-times}
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Upper and Lower Redundancy

We would like a notion of **frame redundancy** to distinguish between our three examples above. To achieve this goal, for a frame $\Phi$ we will introduce:
Upper and Lower Redundancy

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**Upper Redundancy** $R_\Phi^+$

**Lower Redudancy** $R_\Phi^-$
Desiderata 1

Triviality. Redundancy should not count zero vectors.
Desiderata 2

**Generalization.** If $\Phi$ is an equal-norm Parseval frame, then in this special case the customary notion of redundancy shall be attained, i.e.,

$$R^-_\Phi = R^+_\Phi = \frac{N}{n}.$$
Nyquist Property. The condition $R_\Phi^--R_\Phi^+=R_\Phi^++R_\Phi^-=1$ shall characterize tightness of a normalized version of $\Phi$, thereby supporting the intuition that upper and lower redundancy being different implies ‘non-uniformity’ of the frame. In particular, $R_\Phi^- = R_\Phi^+ = 1$ shall be equivalent to orthogonality as the ‘limit-case’.
Upper and Lower Redundancy. Upper and lower redundancy shall be ‘naturally’ related by $0 < \mathcal{R}_\Phi^- \leq \mathcal{R}_\Phi^+ < \infty$. 
Desiderata 5

Additivity. Upper and lower redundancy shall be subadditive and superadditive, respectively, with respect to unions of frames. They shall be additive provided that the redundancy is uniform, i.e., $R^\Phi_\Phi = R^\Phi_\Phi$. 
Invariance. Redundancy shall be invariant under the action of a unitary operator on the frame vectors, under scaling of single frame vectors, as well as under permutation, since intuitively all these actions should have no effect on, for instance, robustness against erasures, - which is one property redundancy shall intuitively measure.
Spanning Sets. The lower redundancy shall measure the maximal number of spanning sets of which the frame consists. This immediately implies that the lower redundancy is a measure for robustness of the frame against erasures in the sense that any set of a particular number of vectors can be deleted yet leave a frame.
Desiderata 8

*Linearly Independent Sets.* The upper redundancy shall measure the minimal number of linearly independent sets the frame consists of.
Local Redundancy

Notation: If $y \in \mathcal{H}_n$, $\langle y \rangle$ will denote the span of $y$ and $P_{\langle y \rangle}$ will denote the orthogonal projection onto $\langle y \rangle$. The unit sphere of $\mathcal{H}_n$ will be denoted $S_n$. \

\[ R_\Phi(x) = \sum_{i=1}^{N} \| P_{\langle \varphi_i \rangle}(x) \|^2. \]
Local Redundancy

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Definition

Let \( \Phi = (\varphi_i)_{i=1}^N \) be a frame for a finite-dimensional real or complex Hilbert space \( \mathcal{H}_n \). For each \( x \in S_n \), the redundancy function \( R_\Phi : S_n \to \mathbb{R}^+ \) is defined by

\[
R_\Phi(x) = \sum_{i=1}^N \| P_{\langle \varphi_i \rangle}(x) \|^2.
\]
An equivalent formulation of $\mathcal{R}_\Phi(x)$ is:

$$\mathcal{R}_\Phi(x) = \sum_{i: \phi_i \neq 0} \left| \langle x, \frac{\phi_i}{\|\phi_i\|} \rangle \right|^2$$
A New Notion of Redundancy

**Definition**

Let \( \Phi = (\varphi_i)_{i=1}^{N} \) be a frame for a finite-dimensional real or complex Hilbert space \( \mathcal{H}_n \). Then the *upper redundancy of \( \Phi \) is defined by*

\[
R_\Phi^+ = \max_{x \in S_n} R_\Phi(x)
\]

Moreover, \( \Phi \) has a *uniform redundancy*, if

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R_\Phi^- = R_\Phi^+.
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Definition

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and the lower redundancy of \( \Phi \) by

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R^-_{\Phi} = \min_{x \in \mathbb{S}_n} R_{\Phi}(x).
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A New Notion of Redundancy

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Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for a finite-dimensional real or complex Hilbert space $\mathcal{H}_n$. Then the *upper redundancy of $\Phi$* is defined by

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and the *lower redundancy of $\Phi$* by

$$R^-_{\Phi} = \min_{x \in S_n} R_{\Phi}(x).$$

Moreover, $\Phi$ has a *uniform redundancy*, if

$$R^-_{\Phi} = R^+_\Phi.$$
Note that this definition yields:

\[ R^-_\Phi = A \quad R^+_\Phi = B \]

where \( A, B \) are the lower and upper frame bounds of

\[ \{ \frac{\varphi_i}{\|\varphi_i\|} : \varphi_i \neq 0 \} \]
This definition satisfies all eight of our Desiderata for redundancy of frames. In particular, let $\Phi = (\varphi_i)_{i=1}^{N}$ be a frame for an $n$-dimensional real or complex Hilbert space $\mathcal{H}_n$. Then we have:
Generalization

If $\Phi$ is an equal-norm Parseval frame, then

$$\mathcal{R}_{\Phi^-} = \mathcal{R}_{\Phi^+} = \frac{N}{n}.$$
Nyquist Property

The following conditions are equivalent:

(i) We have $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$.  
(ii) The normalized version of $\Phi$ is tight.
Nyquist Property

The following conditions are equivalent:

(i) We have $R_{\Phi}^-=R_{\Phi}^+$.  
(ii) The normalized version of $\Phi$ is tight.

Also the following conditions are equivalent:

(i’) We have $R_{\Phi}^-=R_{\Phi}^+=1$.  
(ii’) $\Phi$ is orthogonal.
Upper and Lower Redundancy

We have

\[ 0 < \mathcal{R}^-_\Phi \leq \mathcal{R}^+_\Phi < \infty. \]
Additivity

For each orthonormal basis \((e_i)_{i=1}^n\),

\[ \mathcal{R}_{\Phi \cup (e_i)_{i=1}^n}^\pm = \mathcal{R}_\Phi^\pm + 1. \]
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\]

Moreover, for each frame \(\Phi'\) in \(\mathcal{H}_n\),

\[
\mathcal{R}_{\Phi \cup \Phi'}^- \geq \mathcal{R}_\Phi^- + \mathcal{R}_{\Phi'}^- \quad \text{and} \quad \mathcal{R}_{\Phi \cup \Phi'}^+ \leq \mathcal{R}_\Phi^+ + \mathcal{R}_{\Phi'}^+.
\]
Additivity

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\mathcal{R}_{\Phi \cup \Phi'}^- \geq \mathcal{R}_\Phi^- + \mathcal{R}_{\Phi'}^- \quad \text{and} \quad \mathcal{R}_{\Phi \cup \Phi'}^+ \leq \mathcal{R}_\Phi^+ + \mathcal{R}_{\Phi'}^+.
\]

In particular, if \(\Phi\) and \(\Phi'\) have uniform redundancy, then

\[
\mathcal{R}_{\Phi \cup \Phi'}^- = \mathcal{R}_\Phi^- + \mathcal{R}_{\Phi'}^- = \mathcal{R}_{\Phi \cup \Phi'}^+.
\]
Invariance

Redundancy is invariant under application of a unitary operator $U$ on $\mathcal{H}_n$, i.e.,

$$\mathcal{R}_{U(\Phi)}^\pm = \mathcal{R}_\Phi^\pm,$$
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under scaling of the frame vectors, i.e.,

$$\mathcal{R}_{(c_i\varphi_i)_{i=1}^N}^\pm = \mathcal{R}_\Phi^\pm, \quad c_i \text{ scalars},$$
Invariance

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under scaling of the frame vectors, i.e.,

$$\mathcal{R}_{(c_i\varphi_i)_{i=1}^N} = \mathcal{R}_\Phi, \quad c_i \text{ scalars},$$

and under permutations, i.e.,

$$\mathcal{R}_{(\varphi_{\pi(i)})_{i=1}^N} = \mathcal{R}_\Phi, \quad \pi \in S_{\{1,\ldots,N\}},$$
Spanning Sets

Φ contains \( \lfloor R - \Phi \rfloor \) disjoint spanning sets.
Spanning Sets

\(\Phi\) contains \(\lfloor R^-_{\Phi} \rfloor\) disjoint spanning sets.

In particular, any set of \(\lceil R^-_{\Phi} \rceil - 1\) vectors can be deleted yet leave a frame.
If Φ does not contain any zero vectors, then it can be partitioned into $\lceil \mathcal{R}_\Phi^+ \rceil$ linearly independent sets.
Our Example 1

\[ \Phi_1 = (e_i)_{i=1}^n \cup (e_i)_{i=1}^n \]

Then \( R - \Phi_1 = 2 = R + \Phi_1 \), and this is an equal norm tight frame.
Our Example 1

\[
\Phi_1 = (e_i)^n_{i=1} \cup (e_i)^n_{i=1}
\]

Then

\[
\mathcal{R}_{\Phi_1}^- = 2 = \mathcal{R}_{\Phi_1}^+,
\]

and this is an equal norm tight frame.
Our Example 2

\[ \Phi_2 = (e_i)_{i=1}^n \cup \{ e_1, e_1, \ldots, e_1 \} \text{ where } e_1 \text{ occurs } n \text{-times} \]
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Then

\[ \mathcal{R}_{\Phi_2}^- = 1 \quad \mathcal{R}_{\Phi_2}^+ = n + 1 \]
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Moreover,

any partition of \( \Phi_2 \) contains at most one spanning set,
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Moreover,

any partition of \( \Phi_2 \) contains at most one spanning set,

any partition of \( \Phi_2 \) into linearly independent sets must contain at least \( (n + 1) \)-sets.
Our Example 3

\[ \Phi_3 = (e_i)_{i=1}^n \cup \{0,0,\ldots,0\} \text{ where } 0 \text{ occurs } n \text{-times} \]
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\[ \Phi_3 = (e_i)^n_i \cup \{0, 0, \ldots, 0\} \text{ where } 0 \text{ occurs } n\text{-times} \]

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Our Example 3

\[ \Phi_3 = (e_i)_{i=1}^n \cup \{0, 0, \ldots, 0\} \text{ where } 0 \text{ occurs } n\text{-times} \]

Then

\[ \mathcal{R}_{\Phi_3}^- = \mathcal{R}_{\Phi_3}^+ = 1 \]

Moreover,

\( \Phi_3 \) contains one spanning set and one linearly independent set.
Theorem

If the \( \{ \varphi_i \} \) are non-zero vectors, then the upper and lower redundancies of a frame \( \Phi = (\varphi_i)_{i=1}^{N} \) for a real or complex Hilbert space \( \mathcal{H}_n \) having dimension \( n \geq 2 \) satisfy the inequalities

\[
0 < \mathcal{R}_\Phi^- \leq \frac{N}{n} \leq \mathcal{R}_\Phi^+ < N,
\]

Moreover, if \( \mathcal{R}_\Phi^- = \frac{N}{n} \) or \( \mathcal{R}_\Phi^+ = \frac{N}{n} \), then the normalized version of \( \Phi \) is a tight frame.
Theorem

If the \( \{ \varphi_i \} \) are non-zero vectors, then the upper and lower redundancies of a frame \( \Phi = (\varphi_i)_{i=1}^{N} \) for a real or complex Hilbert space \( \mathcal{H}_n \) having dimension \( n \geq 2 \) satisfy the inequalities

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Moreover, if \( R^-_\Phi = \frac{N}{n} \) or \( R^+_\Phi = \frac{N}{n} \), then the normalized version of \( \Phi \) is a tight frame.
Theorem

Let \( n \leq N, \ r_1 \in (0, \frac{N}{n}], \) and \( r_2 \in \left[ \frac{N}{n}, N \right). \) Then the following conditions are equivalent.

(i) There exists a frame \( \Phi = (\varphi_i)_{i=1}^{n} \) for \( \mathcal{H}_n, \ n \geq 2, \) such that

\[
\mathcal{R}_\Phi^- = r_1 \quad \text{and} \quad \mathcal{R}_\Phi^+ = r_2.
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Theorem

Let $n \leq N$, $r_1 \in (0, \frac{N}{n}]$, and $r_2 \in \left[\frac{N}{n}, N\right)$. Then the following conditions are equivalent.

(i) There exists a frame $\Phi = (\varphi_i)_{i=1}^N$ for $\mathcal{H}_n$, $n \geq 2$, such that

$$\mathcal{R}_\Phi^- = r_1 \quad \text{and} \quad \mathcal{R}_\Phi^+ = r_2.$$

(ii) We have

$$(n - 1)r_1 + r_2 \leq N.$$
For every $r_1 \in (0, \frac{N}{n}]$ and every $r_2 \in [\frac{N}{n}, N)$, we can find unit-norm frames $\Phi = (\varphi_i)_{i=1}^N$ and $\Psi = (\psi_i)_{i=1}^N$ with

$$\mathcal{R}_\Phi^- = r_1 \quad \text{and} \quad \mathcal{R}_\Psi^+ = r_2.$$
Theorem

Let $f : \mathbb{S}_n \rightarrow \mathbb{R}_0^+$, $\mathcal{H}$ be an $n$-dimensional real or complex Hilbert space with $n \geq 3$, and let $q$ be the extension of $f$ to $\mathcal{H}$ given by $q(0) = 0$ and $q(x) = \|x\|^2 f(x/\|x\|)$ for any $x \neq 0$. Let $\omega$ denote the probability measure on the unit sphere which is invariant under all unitaries. Then the following conditions are equivalent:
Characterization of Redundancy Functions

**Theorem**

(i) *There exists a frame* $\Phi$ *for* $\mathcal{H}_n$ *such that*

\[ f(x) = R_\Phi(x) \quad \text{for all } x \in S_n. \]
Characterization of Redundancy Functions

Theorem

(i) There exists a frame $\Phi$ for $\mathcal{H}_n$ such that

$$f(x) = R_{\Phi}(x) \quad \text{for all } x \in \mathcal{S}_n.$$  

(ii) The function $f$ is strictly positive on $\mathcal{S}$, its extension $q$ satisfies the parallelogram identity

$$q(x + y) + q(x - y) = 2(q(x) + q(y)) \quad \text{for all } x, y \in \mathcal{S}_n$$

and $f$ integrates to

$$\int_{\mathcal{S}_n} f(x) d\omega(x) = N/n$$

with some integer $N \geq n$. 

The extreme cases

Let $0 < \varepsilon < 1$ and choose $\Phi_3 = (\varphi_i)_{i=1}^N$ as

$$\varphi_i = \begin{cases} e_1 & : i = 1, \\ \sqrt{1 - \varepsilon^2}e_1 + \varepsilon e_i & : i \neq 1. \end{cases}$$
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This frame is strongly concentrated around the vector $e_1$. 
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\end{cases}
$$

This frame is strongly concentrated around the vector $e_1$.

Also, this frame is a linearly independent set.
Redundancy for Finite Frames

\[ R_{\Phi_3}(e_1) = \sum_{i=1}^{N} \| P_{\langle \varphi_i \rangle}(e_1) \|^2 \]

\[ = 1 + \sum_{i=2}^{N} |\langle e_1, \sqrt{1 - \varepsilon^2} e_1 + \varepsilon e_i \rangle|^2 \]

\[ = 1 + (N - 1)(1 - \varepsilon^2). \]
Redundancy for this example

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Hence,

\[ 1 + (N - 1)(1 - \varepsilon^2) \leq \mathcal{R}_{\Phi_3}^+ < N. \]
It is not all Good News

We can also compute

$$\mathcal{R}_{\Phi_3}(e_2) = \sum_{i=1}^{N} \| P_{\langle \varphi_i \rangle}(e_2) \|^2 = \sum_{i=2}^{N} |\langle e_2, \sqrt{1 - \varepsilon^2} e_1 + \varepsilon e_i \rangle|^2 = \varepsilon^2,$$
It is not all Good News

We can also compute

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\mathcal{R}_{\Phi_3}(e_2) = \sum_{i=1}^{N} \|P_{\langle \varphi_i \rangle}(e_2)\|^2 = \sum_{i=2}^{N} |\langle e_2, \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i \rangle|^2 = \varepsilon^2,
\]

Hence,

\[
0 < \mathcal{R}_{\Phi_3} \leq \varepsilon^2.
\]
The frame $\Phi_3$ is not orthogonal, nor is it tight. Note that $\lfloor R_{\Phi_3} - 1 \rfloor = 0$ although there does exist a partition into one spanning set.

Again, we see that we can do better than this by merely taking the whole frame which happens to be linearly independent. However, our redundancy notion is giving pretty good information even in this extreme case. i.e. The very fact that $0 < R_{\Phi_3} \leq \varepsilon^2$, tells us that this frame is heavily piled up on itself.
The frame $\Phi_3$ is not orthogonal, nor is it tight. Note that $[\mathcal{R}_{\Phi_3}^-] = 0$ although there does exist a partition into one spanning set.

Our results show that this frame can be partitioned into $N - 1$ linearly independent sets. Again, we see that we can do better than this by merely taking the whole frame which happens to be linearly independent.
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