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Frame potential and finite abelian groups

Brody D. Johnson and Kasso A. Okoudjou

Abstract. This article continues a prior investigation of the authors with the goal of extending characterization results of convolutional tight frames from the context of cyclic groups to general finite abelian groups. The collections studied are formed by translating a number of *generators* by elements of a fixed subgroup and it is shown, under certain norm conditions, that tight frames with this structure are characterized as local minimizers of the frame potential. Natural analogs to the downsampling and upsampling operators of finite cyclic groups are studied for arbitrary subgroups of finite abelian groups. Directions of further study are also proposed.

1. Introduction

In recent years, there has been much activity in the study of frames for finite-dimensional Hilbert spaces. Topics of interest include the characterization and construction of frames of various kinds, e.g., equiangular frames [**Renes**], harmonic frames [**VW2**], [compound] geometrically uniform frames [**BE**], tight frames [**BF, CFKLT**], frames with symmetries [**VW1**], frames resistant to erasures [**HP**], etc. Throughout this section, let $\mathbb{H}$ denote a finite dimensional real or complex Hilbert space. Recall that a *frame* for $\mathbb{H}$ is a collection $X = \{f_k\}_{k=1}^n \subseteq \mathbb{H}$ for which there exist real numbers $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{n} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathbb{H}.$$

If it is possible to choose $A = B$ then X is called a *tight frame*.

One recent line of research began with the work of Benedetto and Fickus in [**BF**], where a characterization of tight frames composed of unit-norm vectors was given in terms of the frame potential.

Definition 1.1. Let $\mathbb{H}$ be a finite dimensional real or complex Hilbert space. Let $X = \{f_k\}_{k=1}^N \subseteq \mathbb{H}$, then the *frame potential* of the collection X is defined as

$$\mathrm{FP}(X) = \sum_{j=1}^{N} \sum_{k=1}^{N} |\langle f_j, f_k\rangle|^2 .$$

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Following the work of Benedetto and Fickus, Casazza et al. [**CFKLT**] considered frames composed of vectors with arbitrary norms and arrived at the following description of tight frames.

THEOREM 1.2 (Theorem 10 of [**CFKLT**]). *Let $\mathbb{H}$ be a d-dimensional real or complex Hilbert space and fix $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$, $M \geq d$. Denote by m_0 the smallest index $0 \leq m \leq M-1$ for which*

$$(d-m)a_0^2 \leq \sum_{j=m}^{M-1} a_j^2 \tag{1.1}$$

holds. If $X = \{f_j\}_{j=0}^{M-1} \subset \mathbb{H}$ is a local minimizer of the frame potential subject to the constraint $\|f_j\| = a_j$, $0 \leq j \leq M-1$, then X may be divided into two mutually orthogonal subcollections: $\{f_j\}_{j=0}^{m_0-1}$, which consists of mutually orthogonal nonzero vectors, and $\{f_j\}_{j=m_0}^{M-1}$, which is a tight frame for its $(d-m_0)$-dimensional span. In the event that $m_0 = 0$, X is a tight frame for $\mathbb{H}$.

When $m = 0$, (1.1) is referred to as the *fundamental frame inequality*,

$$da_0^2 \leq \sum_{j=0}^{M-1} a_j^2. \tag{1.2}$$

In light of Theorem 1.2, it is clear that this inequality provides a sufficient condition for the existence of a tight-frame having a specified nonincreasing sequence of norms.

REMARK 1.3. The following additional facts from [**CFKLT**] related to Theorem 1.2 and the inequality (1.2) will also be relevant to this work.

(i) The fundamental frame inequality (1.2) is a necessary condition on the norms associated to any tight frame after rearrangement into decreasing order.
(ii) Any local minimizer of the frame potential (associated with a fixed sequence of norms) must also be a global minimizer.

Another line of research, originated by Vale and Waldron [**VW1**], deals with an examination of certain symmetries possessed by tight frames for finite dimensional Hilbert spaces. Following [**VW1**], define the *symmetry group* of a frame $X = \{f_k\}_{k=1}^n$ for $\mathbb{H}$ to be the group

$$\mathrm{Sym}(X) = \{U \in \mathcal{U}(\mathbb{H}) : U(X) = X\}.$$

Here, $\mathcal{U}(\mathbb{H})$ denotes the group of unitary linear transformations on $\mathbb{H}$ under composition. This motivates a natural question: *Under what conditions, if any, can tight frames with specified symmetries be characterized as local minimizers of the frame potential?*

EXAMPLE 1.4. Let $\omega = e^{2\pi i/3}$ and define A to be the 2×2 matrix

$$A = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}.$$

Given $u \in \mathbb{C}^2$ with $\|u\| = 1$, consider the collection $X = \{A^j u\}_{j=0}^2$. It will be shown that the local minimizers of the frame potential of X (under the constraint

that $\|u\| = 1$) are precisely the tight frames of this form. By symmetry,

$$\mathrm{FP}(X) = 3\sum_{k=0}^{2} \left|\langle A^k u, u\rangle\right|^2,$$

so letting $u = (u_1, u_2)$ one seeks to minimize (after an elementary computation)

$$f(u_1, u_2) = 3\left(1 + 2|u_1|^4 - 2|u_1|^2|u_2|^2 + 2|u_2|^4\right)$$

subject to the constraint that $g(u_1, u_2) = |u_1|^2 + |u_2|^2 = 1$. The method of Lagrange multipliers reveals that the minima occur when $|u_1|^2 = |u_2|^2$, i.e., u is of the form $(\alpha/\sqrt{2}, \beta/\sqrt{2})$, where $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha| = |\beta| = 1$. In any case, this leads to a $\frac{3}{2}$-tight frame for $\mathbb{C}^2$. Hence, each local minimizer of $\mathrm{FP}(X)$ leads to a tight frame. Moreover, if there were another choice for u which led to a tight frame it would achieve the same minimum value for f and, therefore, would be among the solutions found through Lagrange multipliers. This shows that the tight frames of the form $\{A^j u\}_{j=0}^2$ are precisely the local minimizers of the frame potential.

The present work seeks an answer to the preceding question for certain symmetry groups in association with the group algebra $\ell(G)$, the real or complex Hilbert space of functions defined on a finite abelian group G. Notice that each element $g \in G$ leads to a natural *translation* operator on $\ell(G)$,

$$T_g : \ell(G) \to \ell(G), \quad (T_g f)(g') = f(g'g^{-1}).$$

Therefore, given a subgroup H of G, any frame of the form

$$X_H = \{T_h f_k : h \in H,\, f_k \in \ell(G),\, 0 \le k \le n-1\} \tag{1.3}$$

satisfies $H \le \mathrm{Sym}(X_H)$. Moreover, such frames bear a close relationship with the convolution structure of $\ell(G)$, which leads to efficient implementation in applications by way of the fast Fourier transform. In this sense, this work continues an investigation of the authors' from [**FJKO**] in which tight frames generated by translations in $\ell(\mathbb{Z}/d\mathbb{Z})$ were characterized as the local minimizers of the frame potential under certain norm conditions related to (1.1).

2. Preliminaries

Throughout this section G will denote a finite abelian group. Recall that the inner product on $\ell(G)$ is given by

$$\langle f_1, f_2\rangle = \sum_{g\in G} f_1(g)\overline{f_2(g)}, \quad f_1, f_2 \in \ell(G).$$

The convolution of $f_1, f_2 \in \ell(G)$, denoted $f_1 * f_2 \in \ell(G)$, is given by

$$f_1 * f_2(g) = \sum_{x\in G} f_1(x) f_2(g^{-1}x), \quad g \in G.$$

This work is concerned with the study of collections of *filters*, $X = \{f_m\}_{m=0}^{n-1} \in \ell(G)$, which will be used to analyze and synthesize general elements of $\ell(G)$ via convolution. It is natural to *sample* such convolutions over a subgroup H, which may be loosely interpreted as a downsampling operation with respect to the quotient group G/H. With this motivation, define the *sampling operator over* H, $\mathcal{S}_H : \ell(G) \to \ell(H)$, by

$$(\mathcal{S}_H f)(h) = f(h), \quad h \in H.$$

Similarly, define the *upsampling operator over* H, $\mathcal{S}_H^* : \ell(H) \to \ell(G)$, by

$$(\mathcal{S}_H^* f)(g) = \begin{cases} f(g), & g \in H \\ 0, & g \notin H, \end{cases} \quad g \in G.$$

Figure 1 depicts a typical filterbank, composed of analysis and synthesis stages that make use of convolution (represented by rectangular elements) as well as the sampling and upsampling operators (represented by circular elements). The analysis stage implements the *involution* of each filter, an operation on $\ell(G)$ given by $\tilde{f}(g) = \overline{f(g^{-1})}$.

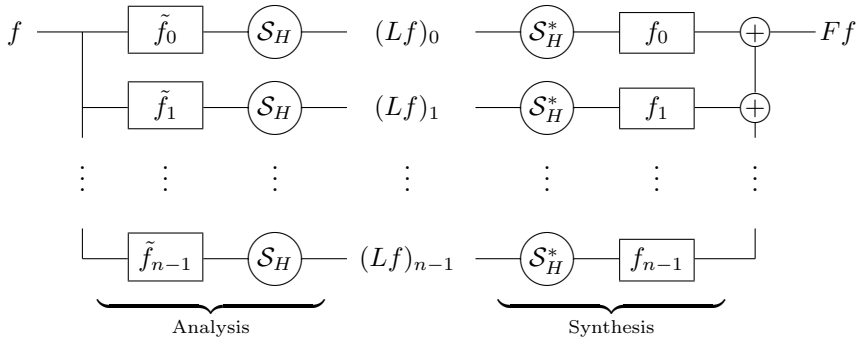

FIGURE 1. Block diagram of an n-channel filterbank on $\ell(G)$.

Associated with the filterbank of Figure 1 are several important operators. The *filterbank analysis operator*, $L : \ell(G) \to \bigoplus_{m=0}^{n-1} \ell(H)$, is given by

$$Lf = (Lf)_0 \oplus \cdots \oplus (Lf)_{n-1},$$

where $(Lf)_m = \mathcal{S}_H(f * \tilde{f}_m)$. The adjoint of the filterbank analysis operator, $L^* : \oplus_{m=0}^{n-1} \ell(H) \to \ell(G)$, is called the *filterbank synthesis operator* and is described by

$$L^* \left(\oplus_{m=0}^{n-1} y_m \right) = \sum_{m=0}^{n-1} (\mathcal{S}_H^* y_m) * f_m.$$

The composition of the filterbank synthesis and analysis operators, L^*L, is called the *filterbank frame operator* for the filterbank and is denoted by $F : \ell(G) \to \ell(G)$. The frame operator is described explicitly by

$$Ff = \sum_{m=0}^{n-1} \left[\mathcal{S}_H^* \mathcal{S}_H (f * \tilde{f}_m) \right] * f_m.$$

The reader should note that the filterbank frame operator associated with X is precisely the ordinary frame operator associated with the collection X_H, as defined by (1.3). Of particular interest are collections of filters for which the associated filterbank frame operator is a scalar multiple of the identity, i.e., filters which give rise to tight frames for $\ell(G)$. With this in mind, the main goal of this work is to provide a characterization of filterbanks which give rise to tight frames in terms of the frame potential. Although neither of the works [**BF, CFKLT**] includes the use of filterbanks, one could interpret their results in terms of a filterbank for $\ell(G)$ with sampling over the trivial subgroup. (In this case, the group structure of G plays no

role in the analysis.) As noted in Section 1, the results of [**FJKO**] correspond to a non-trivial filterbank with $G = \mathbb{Z}_d$ and H a cyclic subgroup of G.

The methods of [**FJKO**] rely on a block diagonalization of L^* called the *modulated filter representation*, from which it follows that the frame characteristics of a filterbank for $\ell(\mathbb{Z}_d)$ are equivalent to the combined frame characteristics of a series of d/N trivial filterbanks (downsampled over the trivial subgroup) for $\ell(\mathbb{Z}_N)$. The characterization of tight frames for the latter case is described by the following theorem.

THEOREM 2.1 (Theorem 12 of [**FJKO**]). *Let $\{a_m\}_{m=0}^{n-1} \subset \mathbb{R}$ be such that $a_0 \geq a_1 \geq \cdots \geq a_{n-1} > 0$. Let d and N be positive integers such that $N \mid d$ and $N \leq n$. Denote by m_0 the smallest index $0 \leq m \leq N-1$ such that*

$$(N-m)a_m^2 \leq \sum_{j=m}^{n-1} a_j^2. \tag{2.1}$$

If the collections $Y_j := \{y_{m,j}\}_{m=0}^{n-1} \subset \ell(\mathbb{Z}_N)$ form a local minimizer of the combined frame potential, $\sum_{j=0}^{\frac{d}{N}-1} \mathrm{FP}(Y_j)$, under the constraint that

$$\sum_{j=0}^{\frac{d}{N}-1} \|y_{m,j}\|^2 = \frac{d}{N} a_m^2, \quad 0 \leq m \leq n-1,$$

then each collection Y_j may be divided into two mutually orthogonal subcollections of $\ell(\mathbb{Z}_N)$: $\{y_{m,j}\}_{m=0}^{m_0-1}$, which consists of mutually orthogonal, nonzero vectors, and $\{y_{m,j}\}_{m=m_0}^{n-1}$, which is a tight frame for its $(N-m_0)$-dimensional span. Moreover, for each j the norms of the vectors of Y_j must satisfy $\|y_{m,j}\| = a_m$ for $0 \leq m \leq m_0-1$ and $\sum_{m=m_0}^{n-1} \|y_{m,j}\|^2 = \sum_{m=m_0}^{n-1} a_m^2$. In the event that $m_0 = 0$ each collection Y_j, $0 \leq j \leq \frac{d}{N}-1$, is a tight frame for $\ell(\mathbb{Z}_N)$ with a common frame bound.

The reader should note the similarity of (2.1) to (1.1). It is apparent from the statement of Theorem 2.1 that this result is somewhat technical, yet it will play an important role in the analysis of filterbanks with sampling over arbitrary subgroups H. Before such an analysis can be made, however, the behavior of $\mathcal{S}_H$ and $\mathcal{S}_H^*$ under the discrete Fourier transform will be examined.

3. The discrete Fourier transform

Throughout this section G will represent a finite abelian group. Recall that a *character* of G is a group homomorphism $\chi : \ell(G) \to \mathbb{T}$, where $\mathbb{T}$ represents the multiplicative group of unimodular complex numbers. The *dual group* to G is denoted by $\widehat{G}$ and consists of all characters of G under pointwise multiplication. By the duality theorem of Pontryagin $\widehat{G}$ is, in fact, isomorphic to G, a result which can be obtained here as an easy consequence of the fundamental theorem of finite abelian groups.

Indeed, G is isomorphic to a direct sum of cyclic groups, i.e.,

$$G \simeq (\mathbb{Z}/m_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/m_r\mathbb{Z}),$$

with $m_j \mid m_{j+1}$, $1 \leq j \leq r-1$. To each $a = (a_1, \ldots, a_r) \in G$ there is an associated character, χ_a, given by

$$\chi_a(x) = \prod_{j=1}^{r} \chi_{a_j}(x_j), \quad x = (x_1, \ldots, x_r) \in G, \tag{3.1}$$

where $\chi_{a_j}(x_j) = \exp(2\pi i a_j x_j / m_j)$. For $a, b \in G$, therefore, it is evident that $\chi_a \chi_b = \chi_{ab}$. The fact that the characters given by (3.1) exhaust $\widehat{G}$ is a consequence of the following lemma.

LEMMA 3.1 ([**Terras**]). *Let $\chi, \psi \in \widehat{G}$, then*

$$\langle \chi, \psi \rangle = \begin{cases} |G|, & \chi = \psi, \\ 0, & \text{otherwise.} \end{cases}$$

The preceding discussion illustrates the fact that the characters of G form an orthogonal basis of $\ell(G)$ and, consequently, that any $f \in \ell(G)$ is uniquely determined by its inner products with the characters. This notion is the foundation for the discrete Fourier transform on G.

DEFINITION 3.2. The *discrete Fourier transform* (DFT) of $f \in \ell(G)$ is defined by

$$\mathcal{F}f(\chi) = \hat{f}(\chi) = \sum_{x \in G} f(x)\overline{\chi(x)}, \quad \chi \in \widehat{G}.$$

The following lemma summarizes the basic properties of the DFT on a finite abelian group, G.

LEMMA 3.3 ([**Terras**]). *Basic Properties of the DFT.*

(i) *$\mathcal{F} : \ell(G) \to \ell(\widehat{G})$ is a bijective linear map.*

(ii) *For $f_1, f_2 \in \ell(G)$, the convolution,*

$$(f_1 * f_2)(x) = \sum_{y \in G} f_1(y) f_2(x - y), \quad x \in G,$$

satisfies

$$\mathcal{F}(f_1 * f_2)(\chi) = \mathcal{F}f_1(\chi)\mathcal{F}f_2(\chi), \quad \chi \in \widehat{G}.$$

(iii) *For $f \in \ell(G)$,*

$$f(x) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \mathcal{F}f(\chi)\chi(x), \quad x \in G.$$

(iv) *For $f_1, f_2 \in \ell(G)$,*

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \langle \mathcal{F}f_1, \mathcal{F}f_2 \rangle.$$

Another important aspect of Fourier analysis on finite abelian groups, particularly for the problems considered in this work, lies in the relationship between the dual groups of G and a subgroup $H \leq G$. The following proposition is adapted from Proposition 6.1 of [**Serre**].

PROPOSITION 3.4. *Suppose $H \leq G$, let $x \in G \setminus H$, and denote by H^x the subgroup of G generated by H and x. Let $m_x = \min\{n \in \mathbb{N} : x^n \in H\}$. Then each $\chi \in \widehat{H}$ extends to m_x orthogonal characters in $\widehat{H^x}$, $\{\chi_j\}_{j=0}^{m_x-1}$, and*

$$\widehat{H^x} = \{\chi_j : \chi_j|_H = \chi, \chi \in \widehat{H},\ 0 \leq j \leq m_x - 1\}.$$

PROOF. Fix $x \in G \setminus H$ and let $m := m_x$. Observe that H^x is given by

$$H^x = \{x^k h : 0 \leq k \leq m-1,\ h \in H\}.$$

If $x^k h_1 = x^\ell h_2$ for some $0 \leq k, \ell < m$ and $h_1, h_2 \in H$ (without loss, assume $k \geq \ell$), then $x^{k-\ell} \in H$ with $0 \leq k - \ell < m$, a contradiction. It follows that $[H^x : H] = m$.

Fix $\chi \in \widehat{H}$ and let $\omega = \chi(x^m)$. Define α_j, $0 \leq j \leq m-1$, as the distinct solutions of

$$\alpha_j^m = \omega.$$

Note that the collection $\{\alpha_j\}_{j=0}^{m-1}$ consists of a constant multiple of the mth roots of unity. Now define χ_j, $0 \leq j \leq m-1$, by $\chi_j(x^k h) = \alpha_j^k \chi(h)$. It is routine to verify that these elements of $\ell(H^x)$ are characters. Moreover, it follows that

$$\langle \chi_{j_1}, \chi_{j_2} \rangle_{H^x} = \sum_{h \in H} |\chi(h)|^2 \sum_{k=0}^{m-1} \alpha_{j_1}^k \, \overline{\alpha_{j_2}^k}.$$

Hence, the orthogonality of $\{\chi_j\}_{j=0}^{m-1}$ follows from the orthogonality of the characters of $\mathbb{Z}/m\mathbb{Z}$. Thus, each character of $\widehat{H}$ extends to m orthogonal characters of $\widehat{H^x}$.

If $\chi, \psi \in \widehat{H}$ then

$$\langle \chi_{j_1}, \psi_{j_2} \rangle_{H^x} = \sum_{h \in H} \chi(h)\, \overline{\psi(h)} \sum_{k=0}^{m-1} \alpha_{j_1}^k \, \overline{\tilde{\alpha}_{j_2}^k}$$

and the orthogonality follows from the orthogonality of characters in $\widehat{H}$. Dimensional considerations reveal that $\widehat{H^x}$ consists precisely of the character extensions claimed in the statement of the proposition. □

Let $H \leq G$. Given $\chi \in \widehat{H}$, let $\widehat{G}_\chi$ consist of all the characters in $\widehat{G}$ whose restrictions to H coincide with χ, i.e.,

$$\widehat{G}_\chi = \{\psi \in \widehat{G} : \psi|_H = \chi\}.$$

COROLLARY 3.5. *Let $H \leq G$ and $\chi \in \widehat{H}$. Then $|\widehat{G}_\chi| = [G : H]$.*

PROOF. Let $x_1, x_2, \ldots, x_n \in G$ such that $x_1 \notin H$ and $x_k \notin H_{k-1}$, $1 \leq k \leq n$, where $H_0 = H$, $H_n = G$, and H_k is the subgroup generated by H_{k-1} and x_k. Such elements exist because G is a finite group.

The result will be demonstrated by induction. The proof of Proposition 3.4 shows for $\chi \in \widehat{H}$ that $|(\widehat{H_1})_\chi| = [H_1 : H]$. Assume that $|(\widehat{H_k})_\chi| = [H_k : H]$, with $1 \leq k \leq n-1$. Appealing again to the proof of Proposition 3.4 it is clear that for any $\psi \in (\widehat{H_k})_\chi$,

$$|\widehat{(H_{k+1})_\psi}| = [H_{k+1} : H_k].$$

But each element of $\widehat{(H_{k+1})_\psi}$, $\psi \in (\widehat{H_k})_\chi$, is an element of $\widehat{(H_{k+1})_\chi}$, so

$$|\widehat{(H_{k+1})_\chi}| = [H_{k+1} : H_k]\,[H_k : H] = [H_{k+1} : H].$$

The $k = n$ instance of the induction statement is the conclusion of the proposition. □

COROLLARY 3.6. *Let $H \leq G$ and $\chi \in \widehat{H}$. Then,*

$$\sum_{\psi \in \widehat{G}_\chi} \psi(g) = \begin{cases} [G:H]\chi(g), & g \in H, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. By the definition of $\widehat{G}_\chi$, $\psi(g) = \chi(g)$ whenever $g \in H$. Hence, the claimed formula for $g \in H$ follows from Corollary 3.5. It is, therefore, sufficient to prove that if $g \notin H$, then $\sum_{\psi \in \widehat{G}_\chi} \psi(g) = 0$. Let $\{x_k\}_{k=1}^n$ and H_k, $0 \leq k \leq n$, be as in the proof of Corollary 3.5. Recall from the proof of Proposition 3.4 that each extension of $\varphi \in \widehat{H_k}$ to an element ψ_j of H_{k+1} is defined by

$$\psi_j(x_{k+1}^\ell h) = \alpha_j^\ell \varphi(h),$$

where $h \in H_k$ and, letting $m = [H_{k+1} : H_k]$, α_j is an mth root of $\varphi(x_{k+1}^m)$. (Note that $x_{k+1}^m \in H_k$.) It follows that

$$\sum_{\psi \in (\widehat{H_{k+1}})_\varphi} \psi(g) = \varphi(h) \sum_{j=0}^{m-1} \alpha_j^\ell,$$

where $g = x^\ell h$, for some $h \in H_k$ and $0 \leq \ell < m$. If $\ell \neq 0$, i.e., $g \notin H_k$, the sum is zero because $\{\alpha_j\}_0^{m-1}$ consists of a constant multiple of the mth roots of unity. The result follows by an induction argument similar to that of Corollary 3.5. □

The preceding corollaries set the stage for descriptions of sampling and upsampling over a subgroup $H \leq G$ in terms of the DFT.

PROPOSITION 3.7. *Let G be a finite abelian group with subgroup H. Then*

(i) *For $f \in \ell(H)$,*

$$\widehat{\mathcal{S}_H^* f}(\chi) = \hat{f}(\chi|_H), \quad \chi \in \widehat{G}.$$

(ii) *For $f \in \ell(G)$,*

$$\widehat{\mathcal{S}_H f}(\chi) = \frac{1}{[G:H]} \sum_{\psi \in \widehat{G}_\chi} \hat{f}(\psi), \quad \chi \in \widehat{H}.$$

PROOF. Let $f \in \ell(H)$ and observe that

$$\widehat{\mathcal{S}_H^* f}(\chi) = \sum_{g \in G} (\mathcal{S}_H^* f)(g) \overline{\chi(g)} = \sum_{h \in H} f(h) \overline{\chi(h)} = \hat{f}(\chi|_H).$$

Given $f \in \ell(G)$, then

$$\begin{aligned} \widehat{\mathcal{S}_H f}(\chi) &= \sum_{h \in H} f(h) \overline{\chi(h)} \\ &= \sum_{g \in G} f(g) \frac{1}{[G:H]} \sum_{\psi \in \widehat{G}_\chi} \overline{\psi(g)} \\ &= \frac{1}{[G:H]} \sum_{\psi \in \widehat{G}_\chi} \hat{f}(\psi), \end{aligned}$$

where the second equality follows from Corollary 3.6. □

4. Convolutional systems for $\ell(G)$

The results of the previous section make it possible to apply the approach of [**FJKO**] for the study of convolutional systems in $\ell(\mathbb{Z}/d\mathbb{Z})$ to convolutional systems in $\ell(G)$, where G is an arbitrary finite abelian group. The major tools required are the modulated filter representation (which rests on the sampling formulas of Proposition 3.7) and Theorem 2.1. The formal definition of a convolutional system for $\ell(G)$ follows.

DEFINITION 4.1. Let $\{f_m\}_{m=0}^{n-1} \subset \ell(G)$, where G is a finite abelian group. Given a subgroup H of G, the collection

$$X_H\left(\{f_m\}_{m=0}^{n-1}\right) = \{T_h f_m : h \in H,\ 0 \le m \le n-1\}$$

will be referred to as the *convolutional system generated by* $\{f_m\}_{m=0}^{n-1}$ *with sampling over* H.

Note that the frame operator of such a system coincides with the filterbank frame operator described in Section 2 and depicted in Figure 1.

4.1. The modulated filter representation. The purpose of the modulated filter representation is to generate a factorization of the filterbank synthesis operator, L^*, which will lead to a new system with equivalent frame properties whose synthesis operator possesses a block diagonal representation.

Towards this end, consider $y = y_0 \oplus \cdots \oplus y_{n-1} \in \bigoplus_{m=0}^{n-1} \ell(H)$ under the action of L^*. Applying $\mathcal{F}_G$ to L^*y yields

$$(\mathcal{F}_G L^* y)(\psi) = \sum_{m=0}^{n-1} \hat{f}_m(\psi)\, \hat{y}_m(\psi|_H), \quad \psi \in \widehat{G},$$

by Proposition 3.7. Alternatively, given $\chi \in \widehat{H}$,

$$(\mathcal{F}_G L^* y)(\psi) = \sum_{m=0}^{n-1} \hat{f}_m(\psi)\, \hat{y}_m(\chi), \quad \psi \in \widehat{G}_\chi. \tag{4.1}$$

Notice that the value of $\mathcal{F}_G L^* y$ for each $\psi \in \widehat{G}_\chi$, $\chi \in \widehat{H}$, depends only on the Fourier transform of the components of y at χ. There is no implicit ordering for the elements of $\widehat{G}$ and $\widehat{H}$, but, for the sake of a more explicit matrix represenation of the filterbank analysis operator, let the elements of $\widehat{H}$ be enumerated as $\widehat{H} = \{\chi_\ell : 1 \le \ell \le |H|\}$ and those of $\widehat{G}$ in terms of $\widehat{G}_{\chi_\ell} = \{\psi_{\ell,j} : 1 \le j \le [G:H]\}$. (Notice that the latter enumeration is justified by Corollary 3.5.) Under these conventions, (4.1) may be formulated as the matrix product

$$\frac{1}{\sqrt{|G|}} \begin{bmatrix} \mathcal{F}_G L^* y(\psi_{\ell,1}) \\ \vdots \\ \mathcal{F}_G L^* y(\psi_{\ell,[G:H]}) \end{bmatrix} = H^*_{\mathrm{mod}}(\ell)\, \frac{1}{\sqrt{|H|}} \begin{bmatrix} \hat{y}_0(\chi_\ell) \\ \vdots \\ \hat{y}_{n-1}(\chi_\ell) \end{bmatrix}, \quad 1 \le \ell \le |H|, \tag{4.2}$$

where $H^*_{\mathrm{mod}}(\ell)$ is a $[G:H] \times n$ matrix whose (j,m) entry is given by

$$[H^*_{\mathrm{mod}}(\ell)](j,m) = \frac{1}{\sqrt{[G:H]}} \hat{f}_m(\psi_{\ell,j}).$$

The additional constants present in (4.2) account for the fact that $\mathcal{F}_G$ and $\mathcal{F}_H$, as defined in Section 3, are not unitary. Collecting the $|H|$ equations represented

by (4.2) into a single matrix equation reveals the block diagonal nature of the modulated filter representation,

$$\frac{1}{\sqrt{|G|}}\left[\begin{array}{c} \mathcal{F}_G L^* y(\varphi_{1,1}) \\ \vdots \\ \mathcal{F}_G L^* y(\varphi_{1,[G:H]\}}) \\ \hline \mathcal{F}_G L^* y(\varphi_{2,1}) \\ \vdots \\ \mathcal{F}_G L^* y(\varphi_{2,[G:H]\}}) \\ \hline \vdots \\ \hline \mathcal{F}_G L^* y(\varphi_{|H|,1}) \\ \vdots \\ \mathcal{F}_G L^* y(\varphi_{|H|,[G:H]\}}) \end{array}\right] = \frac{1}{\sqrt{|H|}}\left[\begin{array}{ccc} H^*_{\text{mod}}(\chi_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & H^*_{\text{mod}}(\chi_{|H|}) \end{array}\right]\left[\begin{array}{c} \mathcal{F}_H y_0(\chi_1) \\ \vdots \\ \mathcal{F}_H y_{n-1}(\chi_1) \\ \hline \mathcal{F}_H y_0(\chi_2) \\ \vdots \\ \mathcal{F}_H y_{n-1}(\chi_2) \\ \hline \vdots \\ \hline \mathcal{F}_H y_0(\chi_{|H|}) \\ \vdots \\ \mathcal{F}_H y_{n-1}(\chi_{|H|}) \end{array}\right].$$

The block matrix on the right-hand side of the last equality is called the *block adjoint modulated filter matrix* and will be denoted by H^*_{mod}. Notice that $U_1 = |G|^{-1/2}\mathcal{F}_G \in \mathcal{U}(\ell(G))$ and $U_2 = |H|^{-1/2}\mathcal{F}_H \in \mathcal{U}(\ell(H))$, which implies

$$L^* = U_1^{-1} H^*_{\text{mod}}(\oplus_\ell U_2). \tag{4.3}$$

It should be noted that given another ordering of the elements comprising $\widehat{G}$ or $\widehat{H}$ one could introduce additional permutation matrices to achieve the above block diagonalization. Equation (4.3) motivates the definition of the $[G:H]$ collections,

$$Y_\ell = \{y_{m,\ell}\}_{m=0}^{n-1} \subset \ell(\mathbb{Z}/[G:H]\mathbb{Z}), \quad 1 \le \ell \le |H|,$$

where $y_{m,\ell}(j) = [G:H]^{-1/2}\hat{f}_m(\psi_{\ell,j})$. As a result of this definition, $H^*_{\text{mod}}(\ell)$ is the synthesis operator corresponding to Y_ℓ. Moreover, since L^*L and $H^*_{\text{mod}}H_{\text{mod}}$ are unitarily equivalent, it follows that:

(1) the frame bounds for $X_H(\{f_m\}_{m=0}^{n-1})$ are the minimum of the lower frame bounds and the maximum of the frame bounds for the collections Y_ℓ;
(2) the frame potential of $X_H(\{f_m\}_{m=0}^{n-1})$ is equal to the sum of the frame potential of the collections Y_ℓ, $1 \le \ell \le |H|$.

The arguments used to justify these facts can be found in [**FJKO**] and, therefore, will not be repeated here.

4.2. The main result.

THEOREM 4.2. *Let $\{a_m\}_{m=0}^{n-1} \subset \mathbb{R}$ be such that $a_0 \ge a_1 \ge \cdots \ge a_{n-1} > 0$. Let G be a finite abelian group and H a subgroup of G with $n \ge [G:H]$. Denote by m_0 the smallest index $0 \le m \le [G:H]-1$ such that*

$$([G:H]-m)a_m^2 \le \sum_{j=m}^{n-1} a_j^2. \tag{4.4}$$

If $X_H(\{f_m\}_{m=0}^{n-1}) \subset \ell(G)$ is a local minimizer of the frame potential under the constraint that $\|f_m\| = a_m$, $0 \le m \le n-1$, then $X_H(\{f_m\}_{m=0}^{n-1})$ may be divided into two mutually orthogonal subcollections of $\ell(G)$: $X_H(\{f_m\}_{m=0}^{m_0-1})$, which consists of mutually orthogonal, nonzero vectors, and $X_H(\{x_m\}_{m=m_0}^{n-1})$, which is a tight

frame for its $|H|([G:H]-m_0)$-dimensional span. In particular, if $m_0 = 0$ then $X_H(\{f_m\}_{m=0}^{n-1})$ is a tight frame for $\ell(G)$.

PROOF. Consider the collections Y_ℓ, $1 \leq \ell \leq |H|$, defined in the previous section and observe that, up to a constant multiple, the Fourier coefficients of a given filter f_m are distributed as the values of the elements $y_{m,\ell}(j)$, $1 \leq \ell \leq |H|$, $1 \leq j \leq [G:H]$. Hence, the constraint $\|f_m\| = a_m$ can be restated as

$$\sum_{\ell=1}^{|H|} \|y_{m,\ell}\|^2 = |H|a_m^2, \quad 0 \leq m \leq n-1.$$

Referring to Theorem 2.1 with $N = [G:H]$ it is not difficult to see that if the hypotheses of Theorem 4.2 are satisfied, then those of Theorem 2.1 will be satisfied with the same m_0. Theorem 2.1 thus provides a decomposition of each Y_ℓ, $1 \leq \ell \leq |H|$, into mutually orthogonal subcollections $\{y_{m,\ell}\}_{m=0}^{m_0-1}$ and $\{y_{m,\ell}\}_{m=m_0}^{n-1}$, the former collection consisting of mutually orthogonal vectors and the latter comprising a tight frame for its span. In particular, for each $1 \leq \ell \leq |H|$, $\|y_{m,\ell}\|^2 = a_m^2$ for $m < m_0$ and

$$\sum_{m=m_0}^{n-1} \|y_{m,\ell}\|^2 = \sum_{m=m_0}^{n-1} a_m^2, \quad m_0 \leq m \leq n-1. \tag{4.5}$$

Let F_{Y_ℓ} denote the frame operator of Y_ℓ on $\ell(\mathbb{Z}/[G:H]\mathbb{Z})$ and $F_Y = H_{\text{mod}}^* H_{\text{mod}}$ the frame operator induced by H_{mod}^* on $\ell(G)$. It follows from the above decomposition that each $y_{m,\ell}$ is an eigenvector of F_{Y_ℓ} and that the eigenvalue is independent of ℓ. If $m < m_0$, then the eigenvalue is a_m^2, while if $m \geq m_0$, the eigenvalue is given by (4.5). As a result, $\oplus_\ell y_{m,\ell}$ is an eigenvector of F_Y with the corresponding eigenvalue. By construction, $\oplus_\ell y_{m,\ell}$ is equal to $\sqrt{|H|}U_1 f_m$ and

$$F_Y = H_{\text{mod}}^* H_{\text{mod}} = U_1 L^* L U_1^{-1} = U_1 F U_1^{-1},$$

where F is the frame operator of the collection X_H. Since $U_1 f_m$ is an eigenvector of F_Y, it follows that f_m is an eigenvector of F. Moreover, the eigenvalue of f_m for F must equal that of $\oplus_\ell y_{m,\ell}$ for F_Y, while the symmetry of the collection X_H forces each translate $T_h f_m$, $h \in H$, to be an eigenvector with the same eigenvalue. Finally, note that the claimed orthogonality of the subcollections $X_H(\{f_m\}_{m=0}^{m_0-1})$ and $X_H(\{x_m\}_{m=m_0}^{n-1})$ (along with the mutual orthogonality of the vectors in the former subcollection) is inherited from the above decomposition of Y_ℓ, since the frame operators F_Y and F are unitarily equivalent. This completes the proof. □

The final result of this section examines underdetermined systems and follows from an argument analogous to that used in [**FJKO**].

COROLLARY 4.3. *Let $\{a_m\}_{m=0}^{n-1} \subset \mathbb{R}$ be such that $a_0 \geq a_1 \geq \cdots \geq a_{n-1} > 0$. Let G be a finite abelian group and H a subgroup of G with $n \leq [G:H]$. If $X_H(\{f_m\}_{m=0}^{n-1}) \subset \ell(G)$ is a local minimizer of the frame potential under the constraint that $\|f_m\| = a_m$, $0 \leq m \leq n-1$, then $X_H(\{f_m\}_{m=0}^{n-1})$ is an orthogonal sequence in $\ell(G)$.*

5. Directions for further study

Recall that one consequence of the characterization results of Theorems 1.2 and 4.2 is the existence of tight frames with a certain structure. In the case of

Theorem 1.2 the structure imposed is limited to the norms of the frame elements, while in Theorem 4.2 the frames studied are required to consist of the translates of elements of prescribed norms over a subgroup. In each case, the fundamental frame inequality describes when the imposed structure permits tight frames. It is natural to wonder what other structures lead to similar descriptions of tight frames and whether some version of the fundamental frame inequality will always appear in the characterization. The following specific questions are motivated by this idea.

(1) *Is it possible to extend the characterization of tight frames in terms of the frame potential to convolutional systems for $\ell(G)$, where G is an arbitrary finite group?*
(2) *What other symmetries or structures of systems in a finite dimensional Hilbert space lead to similar characterizations of tight frames in terms of the frame potential?*

In a certain sense, the frame potential measures the orthogonality of a given collection of vectors and, as such, minimizers of the frame potential can be thought to represent maximally orthogonal collections. Recent interest in equiangular frames seems to take a step away from this concept; however, it is reasonable to ask whether potential methods can be adapted sufficiently to produce existence results for equiangular frames.

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Department of Mathematics and Computer Science, Saint Louis University, Saint Louis, MO 63103
E-mail address: brody@slu.edu

Department of Mathematics, University of Maryland, College Park, MD 20742
E-mail address: kasso@math.umd.edu