Quincunx wavelets on $\mathbb{T}^2$

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Abstract

- This work examines finite-dimensional wavelet systems in $L^2(\mathbb{T})$ and $L^2(\mathbb{T}^2)$ in which dilation is achieved by a dyadic downsampling of the Fourier transform. At scale $j > 0$ these systems will have dimension 2^j.

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Part I: The Circle

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- Dilation on the circle performs a *downsampling* of the Fourier coefficients.
- Dilation on the circle is not invertible, hence MRAs will be one-sided.
- Dilation of a trigonometric polynomial will eventually result in a constant function, i.e., if f is a trigonometric polynomial then $D^j f = \hat{f}(0)$ for sufficiently large $j \in \mathbb{N}$.

Shift-Invariant Spaces

Definition 2

The *principal shift-invariant space of order* 2^j generated by $\phi \in L^2(\mathbb{T})$ is the finite-dimensional space $V_j(\phi) = \text{span} X_j(\phi)$, where

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Definition 3

The *bracket product of order* 2^j of two functions $f, g \in L^2(\mathbb{T})$ is the vector $[\hat{f}, \hat{g}]_j \in \ell(\mathbb{Z}_{2^j})$ defined by

$$[\hat{f}, \hat{g}]_j(n) = 2^j \sum_{k \in \mathbb{Z}} \hat{f}(n + k2^j)\, \overline{\hat{g}(n + k2^j)}, \quad 0 \leq n \leq 2^j - 1.$$

Here, $\ell(\mathbb{Z}_{2^j})$ is the finite-dimensional space of functions defined on $\mathbb{Z}/2^j\mathbb{Z}$.

Shift-Invariant Spaces

Proposition 1

For all $f, g \in L^2(\mathbb{T})$,

$$\mathcal{F}_{2^j}\left(\{\langle f, T^n_{2^{-j}} g\rangle\}_{n=0}^{2^j-1}\right) = 2^{-\frac{j}{2}}[\hat{f}, \hat{g}]_j,$$

where $\mathcal{F}_{2^j}$ *is the Fourier transform on* $\ell(\mathbb{Z}_{2^j})$.

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Proposition 2

The collection $X_j(\phi)$ *forms an orthonormal basis for* $V_j(\Phi)$ *if and only if*

$$[\hat{\phi}, \hat{\phi}]_j(n) = 1, \quad n \in \mathbb{Z}_{2^j}.$$

Refinable Functions

Definition 4

A function $\phi \in L^2(\mathbb{T})$ is said to be *refinable of order* 2^j if there exists a *mask* $c \in \ell(\mathbb{Z}_{2^j})$ such that

$$D\phi = \sum_{n \in \mathbb{Z}_{2^j}} c(n) T^n_{2^{-j}} \phi. \tag{1}$$

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Lemma 1

Suppose that $\phi \in L^2(\mathbb{T})$ *is refinable of order* 2^j*, then there exists* $m \in \ell(\mathbb{Z}_{2^j})$ *such that*

$$\hat{\phi}(2k) = m(k)\hat{\phi}(k), \quad k \in \mathbb{Z}. \tag{2}$$

Refinability of Dilates

Remark 2

If $\phi \in L^2(\mathbb{T})$ is refinable of order 2^j with filter $m \in \ell(\mathbb{Z}_{2^j})$ then

$$D^2\phi = \sum_{n \in \mathbb{Z}_{2^{j-1}}} \left(\sum_{\ell \in \{0,1\}} c(n + \ell 2^{j-1}) \right) T^n_{2^{j-1}} D\phi,$$

i.e., $D\phi$ is refinable of order 2^{j-1}.

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Remark 3

Notice that $\widehat{D\varphi}(0) = m(0)\hat{\varphi}(0)$. Hence, if φ is refinable with $\hat{\varphi}(0) \neq 0$, it follows that $m(0) = 1$.

Multiresolution Analysis

Definition 5

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Remark 4

Notice that MRA properties i, ii, and iv imply that a scaling function φ is necessarily refinable of order 2^j. Moreover, it follows from MRA properties iii and iv that $D^j\varphi$ must be constant and nonzero, implying that $\hat{\varphi}(0) \neq 0$.

Characterization of Scaling Functions

Theorem 1

Suppose $\varphi \in L^2(\mathbb{T})$ *is a refinable function of order* 2^j *with* $\hat{\varphi}(0) \neq 0$. *Then* φ *is the scaling function of an MRA of order* 2^j *if and only if*

$$|m_0(n)|^2 + |m_0(n + 2^{j-1})|^2 = 1, \quad n \in \mathbb{Z}_{2^j}, \tag{3}$$

and

$$[\hat{\varphi}, \hat{\varphi}]_j(n) = 1, \quad n \in \mathbb{Z}_{2^j}. \tag{4}$$

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Remark 5

Equation (3) will be referred to as the *Smith-Barnwell equation* for the filter.

Existence of Scaling Functions

Theorem 2

Suppose $m_0 \in \ell(\mathbb{Z}_{2^j})$ satisfies (3) with $m_0(0) = 1$. Then m_0 is the low-pass filter of a trigonometric polynomial scaling function of order 2^j.

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3. For $-2^{j-2} \leq k \leq 2^{j-2} - 1$ and $1 \leq n \leq j-1$, define $\hat{\varphi}(2^n(2k+1))$ according to (1), i.e.,

$$\hat{\varphi}(2^n(2k+1)) = m_0(2^{n-1}(2k+1))\, \hat{\varphi}(2^{n-1}(2k+1)).$$

Orthonormal Wavelets

Definition 6

Let $\{V_k\}_{k=0}^{j}$ be an MRA of order 2^j. A function $\psi \in V_j$ is a *wavelet* for the MRA if the collection

$$\{2^{\frac{j-k}{2}} T^n_{2^{-k}} D^{j-(k+1)} \psi : 0 \leq k \leq j-1,\ n \in \mathbb{Z}_{2^k}\}$$

is an orthonormal basis for $V_j \ominus V_0$.

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This construction rests on a decomposition $V_k = V_{k-1} \oplus W_{k-1}$, $1 \leq k \leq j$, where W_k is of the form

$$W_k = V_k(D^{j-(k+1)}\psi).$$

The High-Pass Filter

Theorem 3

Suppose that φ *is the scaling function of an MRA of order* 2^j *and define* $\psi \in V_j$ *by*

$$\hat{\psi}(k) = m_1(k)\hat{\varphi}(k), \quad k \in \mathbb{Z},$$

where $m_1 \in \ell(\mathbb{Z}_{2^j})$ *is chosen as*

$$m_1(n) = \overline{m_0(n + 2^{j-1})}\, e^{-2\pi i 2^{-j} n}, \quad n \in \mathbb{Z}_{2^j}. \tag{5}$$

Then ψ *is a wavelet for the MRA.*

Borrowing from the Line

Proposition 3

Suppose $c \in \ell^2(\mathbb{Z})$ is an absolutely summable sequence whose Fourier transform $m = \hat{c}$ satisfies

$$|m(\xi)|^2 + |m(\xi + 2^{-1})|^2 = 1, \quad \xi \in \mathbb{T},$$

If $c_0 \in \ell(\mathbb{Z}_{2^j})$ is defined by

$$c_0(n) = 2^{\frac{j}{2}} \sum_{k \in \mathbb{Z}} c(n + k2^j), \quad n \in \mathbb{Z}_{2^j},$$

then $m_0 = 2^{\frac{j}{2}} \widehat{c_0}$ satisfies the Smith-Barnwell equation (3).

The Haar Scaling Function

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Example 1 (Haar Filter)

Fix $j = 3$ and let $c \in \ell(\mathbb{Z}_8)$ be given by $c(0) = c(1) = \frac{1}{2}$ with $c(n) = 0$ for $n \neq 0, 1$. The low-pass filter $m_0 \in \ell(\mathbb{Z}_8)$ is given by

$$m_0(n) = e^{-\pi i n/8} \cos\left(n\pi/8\right), \quad n \in \mathbb{Z}_8.$$

It is easy to verify the Smith-Barnwell equation (3).

The Haar Scaling Function

$$
\begin{aligned}
\hat{\varphi}(-3) &= \frac{1}{\sqrt{8}} \longrightarrow \hat{\varphi}(-6) = \hat{\varphi}(-3)m_0(5) \longrightarrow \hat{\varphi}(-12) = \hat{\varphi}(-6)m_0(5)m_0(2), \\
\hat{\varphi}(-1) &= \frac{1}{\sqrt{8}} \longrightarrow \hat{\varphi}(-2) = \hat{\varphi}(-1)m_0(7) \longrightarrow \hat{\varphi}(-4) = \hat{\varphi}(-2)m_0(7)m_0(6), \\
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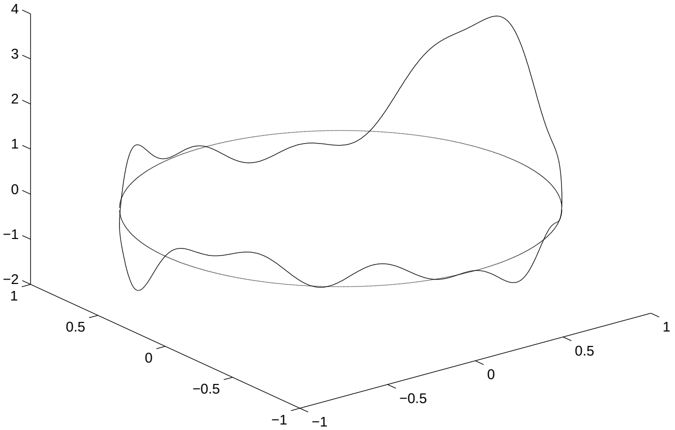
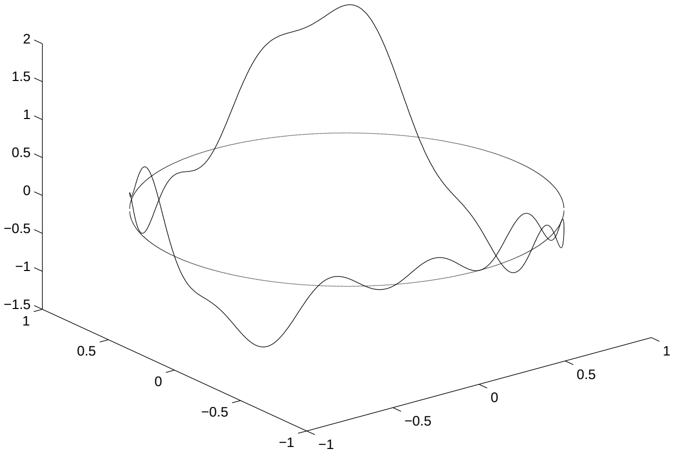
\end{aligned}
$$

Each “strand” terminates because the next computation would include $m_0(4) = 0$.

The Haar Scaling Function ($j = 3$)



The Haar Wavelet ($j = 3$)



Approximation Error

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Definition 7

The error of approximation, denoted $E_{2^j}(k)$, is defined as

$$E_{2^j}(k) = \left[1 - 2^j|\hat{\varphi}(k)|^2\right]^{\frac{1}{2}}, \quad k \in \mathbb{Z}.$$

An elementary calculation shows that $E_{2^j}(k)$ is the approximation error $\|Pf - f\|$ where $f = e^{2\pi ikx}$ and P is the orthogonal projection onto $V_j(\varphi)$.

An Approximation Result

If $m(\xi)$ is a continuous function on the circle with $m(0) = 1$ and satisfying the Smith-Barnwell equation (3), one can define a scaling function associated to m of order 2^j.

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Fix $r \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $j > 0$ such that $E_{2^j}(k) < \varepsilon$ for $|k| < r$, where φ is constructed as in Theorem 2.

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Remark 6

Notice that this discussion does not apply to the classical Shannon filter or scaling function. The next example seeks to remedy this situation.

The Shannon Scaling Function

Example 2 (Shannon Filter)

Let $m_0 \in \ell(\mathbb{Z}_{2^j})$ be defined for $j > 2$ by

$$m_0(n) = \begin{cases} 1, & n < \frac{1}{4}2^j \text{ or } n > \frac{3}{4}2^j, \\ \frac{1}{\sqrt{2}}, & n = \frac{1}{2}2^j \text{ or } n = \frac{3}{4}2^j, \\ 0, & \text{otherwise}, \end{cases} \qquad n \in \mathbb{Z}_{2^j},$$

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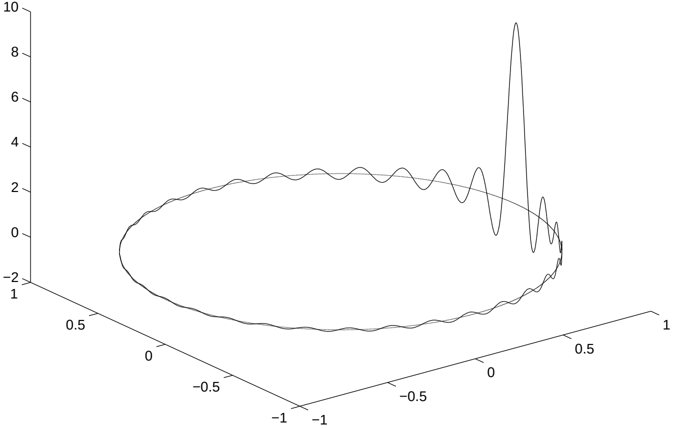
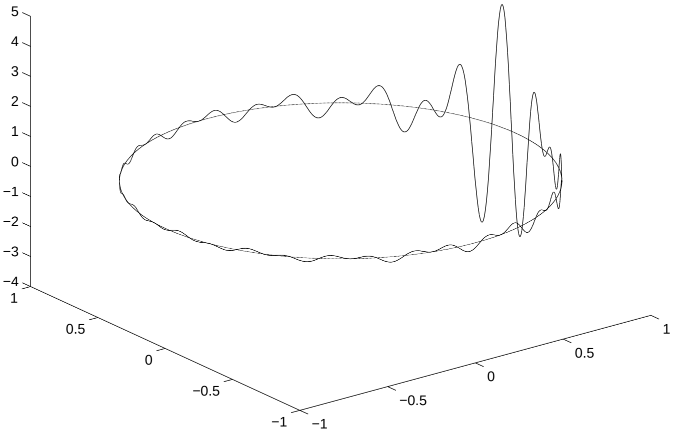
If φ is constructed as in Theorem 2, then $\hat{\varphi}(k) = 2^{-\frac{j}{2}}$ whenever $|k| < 2^{j-1}$.

The Shannon Scaling Function ($j = 6$)



The Shannon Wavelet ($j = 6$)



Part II: The Torus

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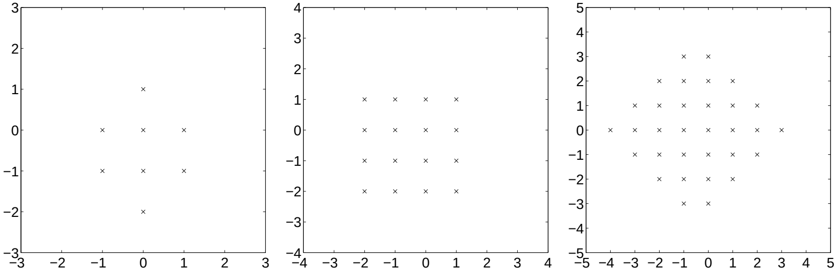
  - The *dual lattice of order 2^j generated by A*:

    $$\Gamma_j^* = \mathbb{Z}^2/B^j\mathbb{Z}^2$$

    Convention: $\Gamma_j^* = B^jR \cap \mathbb{Z}^2$, where $R = (-\frac{1}{2}, \frac{1}{2}] \times (-\frac{1}{2}, \frac{1}{2}]$.

The Dual Lattices Γ_j^*



Γ_3^* Γ_4^* Γ_5^*

Dilation & Translation

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- Translation, $T_\alpha : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$, $\alpha \in \Gamma_j$, is defined by

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Dilation & Translation

Recall:

- Dilation, $D : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$, is defined by

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- Interplay of D and T:

$$DT_\alpha f = T_{B\alpha} Df, \quad \alpha \in \Gamma_j.$$

Note that $B\alpha \in \Gamma_j$ for all $\alpha \in \Gamma_j$.

Shift-Invariant Spaces

Definition 8

Let $\phi \in L^2(\mathbb{T}^2)$. The *principal A-shift-invariant space of order* 2^j *generated by* ϕ, denoted $V_j(\phi)$, is the finite-dimensional subspace of $L^2(\mathbb{T}^2)$ spanned by the collection

$$X_j(\phi) = \{T_\alpha \phi : \alpha \in \Gamma_j\}. \tag{6}$$

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Definition 9

Let $f, g \in L^2(\mathbb{T}^2)$. The *A-bracket product of f and g of order* 2^j is the element of $\ell(\Gamma_j^*)$ defined by

$$[\hat{f}, \hat{g}]_{A^j}(\beta) = 2^j \sum_{k \in B^j \mathbb{Z}^2} \hat{f}(\beta + k)\overline{\hat{g}(\beta + k)}, \quad \beta \in \Gamma_j^*.$$

Here, $\ell(\Gamma_j^*)$ is the space of $\mathbb{C}$-valued functions on Γ_j^*.

Shift-Invariant Spaces

Lemma 2

Define $e_{j,\alpha} \in \ell(\Gamma_j^*)$, $j > 0$, $\alpha \in \Gamma_j$, *by*

$$e_{j,\alpha}(\beta) = \exp\left(2\pi i\langle \alpha, \beta\rangle\right), \quad \beta \in \Gamma_j^*.$$

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Proposition 5

The collection $X_j(\phi)$ *forms an orthonormal basis for* $V_j(\phi)$ *if and only if*

$$[\hat{\phi}, \hat{\phi}]_{A^j}(\beta) = 1, \quad \beta \in \Gamma_j^*.$$

Refinable Functions

Definition 10

A function $\phi \in L^2(\mathbb{T}^2)$ is *A-refinable of order* 2^j if there exists a mask $c \in \ell(\Gamma_j)$ such that

$$D\phi = \sum_{\alpha \in \Gamma_j} c(\alpha)\, T_\alpha \phi. \tag{7}$$

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Lemma 3

If ϕ is refinable of order 2^j, then

$$\hat{\phi}(Ak) = m(k)\hat{\phi}(k), \quad k \in \mathbb{Z}^2, \tag{8}$$

where $m \in \ell(\Gamma_j^*)$ is given by

$$m(\beta) = \sum_{\alpha \in \Gamma_j} c(\alpha)\overline{e_{j,\alpha}(\cdot)}, \quad \beta \in \Gamma_j^*.$$

Refinability of Dilates

Lemma 4

If ϕ is refinable of order 2^j with filter $m \in \ell(\Gamma_j^*)$, then $D\phi$ is refinable of order 2^{j-1} with filter $m(A\cdot) \in \ell(\Gamma_{j-1}^*)$.

Multiresolution Analysis

Definition 11

A *multiresolution analysis (MRA) of order* 2^j ($j \in \mathbb{N}$) is a collection of closed subspaces of $L^2(\mathbb{T}^2)$, $\{V_k\}_{k=0}^{j}$, satisfying

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ii) For $1 \leq k \leq j$, $f \in V_k$ if and only if $Df \in V_{k-1}$;

iii) V_0 is the subspace of constant functions;

iv) There exists a *scaling function* $\varphi \in V_j$ such that $X_k(2^{\frac{j-k}{2}} D^{j-k}\varphi)$ is an orthonormal basis for V_k, $0 \leq k \leq j$.

Main Results

Theorem 4

Suppose that $\varphi \in L^2(\mathbb{T}^2)$ is refinable of order 2^j ($j \in \mathbb{N}$) with $\hat{\varphi}(0) \neq 0$. Then φ is the scaling function of an MRA of order 2^j if and only if

$$|m_0(\beta)|^2 + |m_0(\beta + B^{j-1}\beta_1)|^2 = 1, \quad \beta \in \Gamma^*_{j-1}, \tag{9}$$

and

$$[\hat{\varphi}, \hat{\varphi}]_{A^j}(\beta) = 1, \quad \beta \in \Gamma^*_j, \tag{10}$$

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Theorem 5

Fix $j > 0$ and let $m_0 \in \ell(\Gamma_j^*)$ be a candidate low-pass filter satisfying (3) and $m_0(0) = 1$. Then m_0 is the low-pass filter of a trigonometric polynomial scaling function of order 2^j.

Orthonormal MRA Wavelets

Definition 12

Let $\{V_k\}_{k=0}^{j}$ be an MRA of order 2^j. A function $\psi \in V_j$ is a *wavelet* for the MRA if the collection

$$\left\{2^{\frac{j-k}{2}} T_\alpha D^{j-(k+1)} \psi : 0 \leq k \leq j-1,\ \alpha \in \Gamma_k\right\}$$

is an orthonormal basis for $V_j \ominus V_0$.

The High-Pass Filter

Theorem 6

Let φ be the scaling function of an MRA of order 2^j. Define ψ by

$$\hat{\psi}(k) = m_1(k)\hat{\varphi}(k), \quad k \in \mathbb{Z}^2,$$

where $m_1 \in \ell(\Gamma_j^*)$ is defined by

$$m_1(\beta) = \overline{m_0(\beta + B^{j-1}\beta_1)}\exp\left(2\pi i\langle A^{-(j-1)}\alpha_1, \beta\rangle\right). \tag{11}$$

Then, ψ is an orthonormal wavelet for the MRA.

Real-Valued Scaling Functions

Proposition 6

Let $m_0 \in \ell(\Gamma_j^*)$ *be a low-pass filter satisfying* (3) *and such that* $m_0(0) = 1$ *and* $m_0(-\beta) = \overline{m_0(\beta)}$, $\beta \in \Gamma_j^*$. *Then there is a real-valued scaling function* φ *which is refinable with respect to* m_0 *giving rise to an MRA of order* 2^j.

Construction of Real-Valued Scaling Functions

Let $\mathcal{B} = \Gamma_j^* \setminus A\mathbb{Z}^2$. Define φ by

1) Let $\hat{\varphi}(0) = 2^{-\frac{j}{2}}$.
2) If $\beta, -\beta \in \mathcal{B}$, let $\hat{\varphi}(\beta) = 2^{-\frac{j}{2}}$ and define

$$\hat{\varphi}(A^k\beta) = \hat{\varphi}(\beta) \prod_{\ell=0}^{k-1} m_0(A^\ell \beta), \quad 1 \leq k \leq j-1.$$

3) If $\beta \in \mathcal{B}$, but $-\beta \notin \mathcal{B}$, let $\hat{\varphi}(\pm\beta) = 2^{-\frac{j+1}{2}}$ and define

$$\hat{\varphi}(\pm A^k\beta) = \hat{\varphi}(\pm\beta) \prod_{\ell=0}^{k-1} m_0(\pm A^\ell \beta), \quad 1 \leq k \leq j-1.$$

4) The remaining Fourier coefficients will be zero.

Shannon Filter

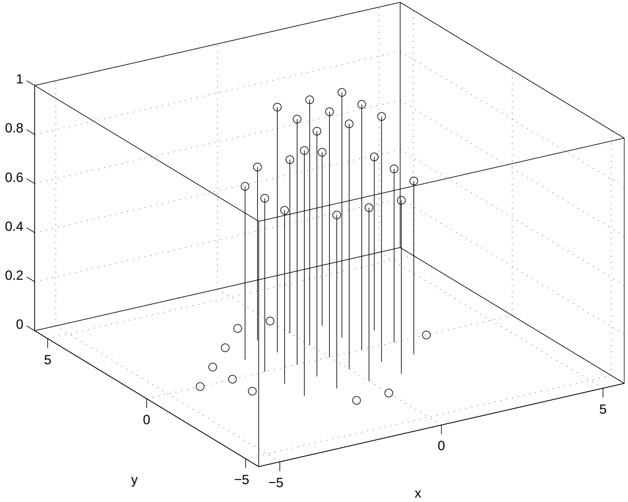
Proposition 7 (Shannon Filter)

Fix $j \geq 2$ *and let* $S_j = \{\beta \in \Gamma_j^* : \beta, -\beta \in \Gamma_{j-1}^*\}$. *The low-pass filter* $m_0 \in \ell(\Gamma_j^*)$ *defined by*

$$m_0(\beta) = \begin{cases} 1 & \beta \in S_j \\ \frac{1}{\sqrt{2}} & \beta \in \Gamma_{j-1}^* \setminus S_j \\ \sqrt{1 - |m_0(\beta - B^{j-1}\beta_1)|^2} & \text{otherwise}, \end{cases} \quad \beta \in \Gamma_j^*,$$

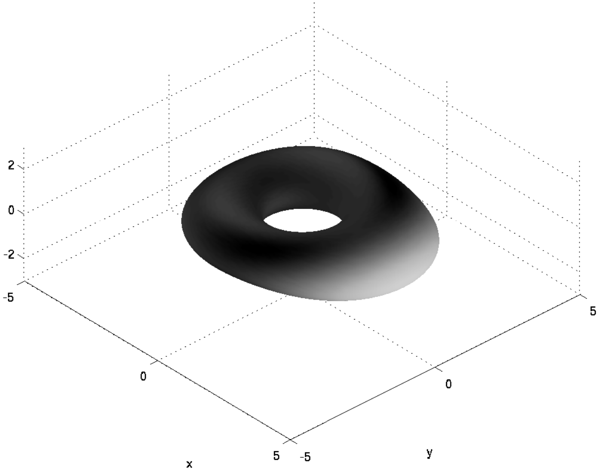
satisfies (3) *and is symmetric in the sense that* $m_0(-\beta) = m_0(\beta)$, $\beta \in \Gamma_j^*$.

Shannon Filter



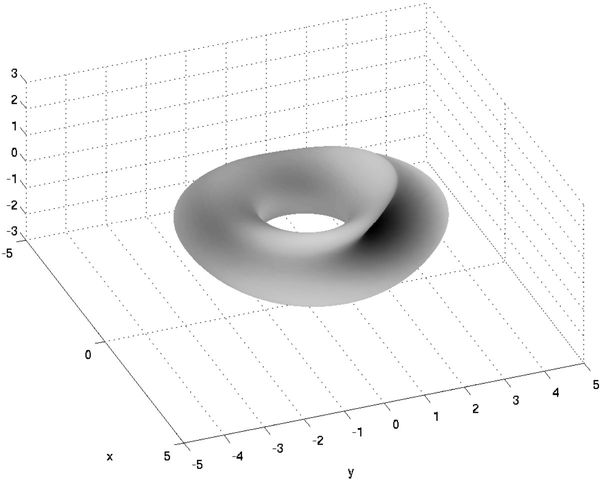
Shannon filter m_0 for $j = 5$

Shannon Scaling Function



Shannon Scaling Function φ for $j = 5$

Shannon Wavelet



Shannon Wavelet ψ for $j = 5$

Approximation with the Shannon Wavelet

Proposition 8

Let φ be the scaling function corresponding to the low-pass filter of Proposition 7 given by Proposition 6. If $j \geq 6 + \log_2 r^2$, then $E_j(k) = 0$ for all $k \in \{k = (k_1, k_2) : \max\{|k_1|, |k_2|\} \leq r\}$.

Haar Filter

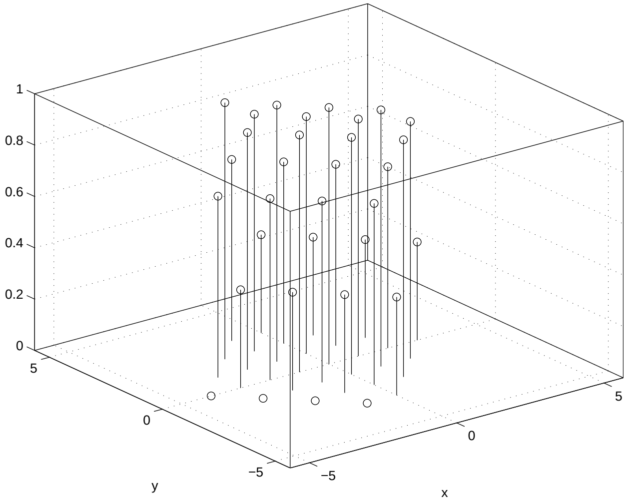
Proposition 9 (Haar Filter)

Fix $j \geq 2$. *Define* $m_0 \in \ell(\Gamma_j^*)$ *by*

$$m_0(\beta) = \frac{1}{2}\left(1 + \exp\left(-2\pi i \langle A^{-(j-1)}\alpha_1, \beta \rangle\right)\right),$$

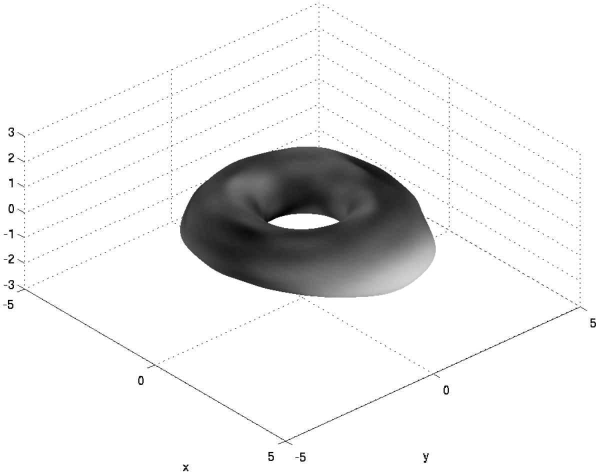
where α_1 *is the nonzero element of* Γ_1. *Then* m_0 *satisfies* (3) *with* $m_0(0) = 1$ *and is conjugate-symmetric, i.e.,* $m_0(-\beta) = \overline{m_0(\beta)}$, $\beta \in \Gamma_j^*$.

Haar Filter



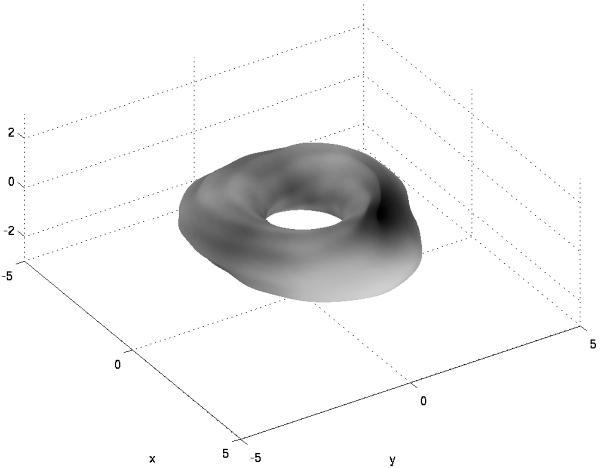
Haar filter m_0 for $j = 5$

Haar Scaling Function



Haar Scaling Function φ for $j = 5$

Haar Wavelet



Haar Wavelet ψ for $j = 5$

Approximation with the Haar Wavelet

Proposition 10

Let φ be the scaling function corresponding to the low-pass filter of Proposition 9 given by Proposition 6. Then for any $r \in \mathbb{Z}^2$,

$$\lim_{j \to \infty} E_j(r) = 0.$$

The Last Slide

The End.