NONLINEAR APPROXIMATION WITH FRAMES

Abstract. Borrowing a few important facts from [Young], these notes describe elementary results regarding the N-term nonlinear approximation problem for frames.

1. Introduction

Let $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be an unconditional basis for a Banach space, $\mathbb{B}$. A fundamental problem in approximation theory concerns the determination of the best approximate of $x \in \mathbb{B}$ (in terms of the norm) which can be written as a linear combination of at most N basis elements. Let Σ_N denote the set of such linear combinations, i.e.,

$$\Sigma_N = \left\{ y \in \mathbb{B} : y = \sum_{k=1}^{N} c_k e_{n_k},\ \text{where } c_k \in \mathbb{C} \text{ and } n_k \in \mathbb{N} \right\}.$$

The *N-term error of approximation* is defined for each $x \in \mathbb{B}$ by

$$\sigma_N(x) = \inf\{\|x - y\| : y \in \Sigma_N\}.$$

This quantity can be difficult to estimate and an optimal approximate $y \in \Sigma_N$ may not even exist. One approach to near-optimal N-term approximation is the so called greedy algorithm, which chooses summands by decreasing norm. In other words, if $x \in \mathbb{B}$ is given by

$$x = \sum_{n\in\mathbb{N}} c_n e_n$$

and $\{n_k\}_{k\in\mathbb{N}}$ is a rearrangement of the natural numbers such that

$$\|c_{n_1} e_{n_1}\| \geq \|c_{n_2} e_{n_2}\| \geq \|c_{n_3} e_{n_3}\| \geq \cdots,$$

then the *greedy algorithm of step N* maps x to $G_N(x)$ given by

$$G_N(x) = \sum_{k=1}^{N} c_{n_k} e_{n_k}.$$

It is self-evident that $\sigma_N(x) \leq \|x - G_N(x)\|$ for all $N \in \mathbb{N}$ and $x \in \mathbb{B}$. If, conversely, there exists $C \geq 1$ such that

$$\|x - G_N(x)\| \leq C\sigma_N(x), \quad \forall\, x \in \mathbb{B},\ N \in \mathbb{N}, \tag{1}$$

then the basis $\mathcal{B}$ is said to be *greedy* for $\mathbb{B}$.

2. The orthonormal case

Orthonormal bases for a Hilbert space represent the paradigm for greedy algorithms.

Proposition 1. *Orthonormal bases are greedy.*

Proof. Let $x \in \mathbb{H}$ (a Hilbert space) and let $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathbb{H}$. Let $\{n_k\}_{k\in\mathbb{N}}$ be a rearrangement of the natural numbers such that $\{\|c_{n_k} e_{n_k}\|\}_{k\in\mathbb{N}}$ is decreasing. Fix $N \in \mathbb{N}$ and choose $y \in \Sigma_N$ with representation

$$y = \sum_{k=1}^{N} d_{m_k} e_{m_k}.$$

Then, due to the orthonormality, $\|x-y\|$ is given by

$$\|x-y\|^2 = \sum_{n\in\mathcal{J}_1} |c_n - d_n|^2 + \sum_{n\in\mathcal{J}_2} |c_n|^2,$$

where $\mathcal{J}_1 = \{m_k : 1 \le k \le N\}$ and $\mathcal{J}_2 = \mathbb{N}\setminus\mathcal{J}_1$. Meanwhile,

$$\|x - G_N(x)\|^2 = \sum_{k>N} |c_{n_k}|^2 \le \sum_{n\in\mathcal{J}_2} |c_n|^2,$$

by the definition of the greedy algorithm. It follows that $\|x - G_N(x)\| = \sigma_N(x)$ for all $x\in\mathbb{H}$ and $N\in\mathbb{N}$. □

3. Frame theory

A *frame* for a separable Hilbert space $\mathbb{H}$ is a collection $\{e_n\}_{n\in\mathbb{N}}$ together with constants $0 < A \le B < \infty$ such that for each $x \in \mathbb{H}$

$$A\|x\|^2 \le \sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 \le B\|x\|^2.$$

Associated to a frame is the *frame operator*, T, given by

$$Tx = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle e_n.$$

Proposition 2 ([Young])**.** *The frame operator T satisfies*

i) *T is bounded and self-adjoint;*
ii) *T is bounded below;*
iii) *T is one-to-one;*
iv) *T is onto.*

Proof. Each property will be proven independently.

i) Consider the mapping $\{c_n\}_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} c_n e_n$. Let $y = \sum_{n\in\mathbb{N}} c_n e_n$ and observe that

$$\|y\|^4 = |\langle y, y\rangle|^2 = \left|\sum_{n\in\mathbb{N}} \overline{c_n}\,\langle y, e_n\rangle\right|^2 \le \sum_{n\in\mathbb{N}} |c_n|^2 \sum_{n\in\mathbb{N}} |\langle y, e_n\rangle|^2 \le B\|y\|^2 \sum_{n\in\mathbb{N}} |c_n|^2.$$

This estimate shows that $\|y\|^2 \le B\sum_{n\in\mathbb{N}} |c_n|^2$, so setting $c_n = \langle x, e_n\rangle$, it follows that

$$\|Tx\|^2 \le B^2\|x\|^2.$$

An elementary calculation shows that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y\in\mathbb{H}$, i.e., T is self-adjoint.

ii) Given $x\in\mathbb{H}$,

$$\langle Tx, x\rangle = \sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2,$$

so

$$A\|x\|^2 \le \langle Tx, x\rangle \le \|Tx\|\,\|x\|$$

for each $x\in\mathbb{H}$. Hence, $\|Tx\| \ge A\|x\|$ showing that T is bounded below.

iii) If $Tx = Ty$ then $\|T(x-y)\| = 0$, forcing $x = y$.

iv) First it will be shown that the range of T is closed. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in the range of T that converges to $y\in\mathbb{H}$. Let $\{x_n\}_{n\in\mathbb{N}}$ such that $Tx_n = y_n$. Because T is bounded below it follows that $\|x_k - x_\ell\| \le A^{-1}\|y_k - y_\ell\|$ and thus $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x\in\mathbb{H}$. But T is bounded, so another sequence argument shows that $Tx = y$ and, hence, the range of T is closed.

Now suppose that T is not onto and let $y\in\mathbb{H}$ such that $y \ne 0$ and $\langle Tx, y\rangle = 0$ for all $x\in\mathbb{H}$. Therefore, $\langle x, Ty\rangle = 0$ for each $x\in\mathbb{H}$, forcing $Ty = 0$. This implies $y = 0$, which is a contradiction.

□

The previous proposition shows that T has a bounded inverse, allowing for the definition of a *dual frame*, $\{\tilde{e}_n\}_{n\in\mathbb{N}}$, given by

$$\tilde{e}_n = T^{-1}e_n.$$

Proposition 3 ([Young])**.** *The collection $\{\tilde{e}_n\}_{n\in\mathbb{N}}$ is a frame with bounds B^{-1}, A^{-1} and for each $x \in \mathbb{H}$*

$$x = \sum_{n\in\mathbb{N}} \langle x, \tilde{e}_n\rangle e_n. \tag{2}$$

Moreover, if $a_n = \langle x, \tilde{e}_n\rangle$ and $x = \sum_{n\in\mathbb{N}} b_n e_n$ then

$$\sum_{n\in\mathbb{N}} |b_n|^2 = \sum_{n\in\mathbb{N}} |a_n|^2 + \sum_{n\in\mathbb{N}} |a_n - b_n|^2. \tag{3}$$

Proof. The identity (2) will be established first. Indeed, let $x \in \mathbb{H}$ and notice that T^{-1} is self-adjoint so that

$$\sum_{n\in\mathbb{N}} \langle x, \tilde{e}_n\rangle e_n = \sum_{n\in\mathbb{N}} \langle x, T^{-1}e_n\rangle e_n = \sum_{n\in\mathbb{N}} \langle T^{-1}x, e_n\rangle e_n = TT^{-1}x = x.$$

Now, calculate

$$\|x\|^2 = \langle x, x\rangle = \sum_{n\in\mathbb{N}} \langle x, \tilde{e}_n\rangle\langle e_n, x\rangle \leq \left(\sum_{n\in\mathbb{N}} |\langle x, \tilde{e}_n\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2\right)^{\frac{1}{2}} \leq \sqrt{B}\|x\| \left(\sum_{n\in\mathbb{N}} |\langle x, \tilde{e}_n\rangle|^2\right)^{\frac{1}{2}},$$

which verifies the claimed lower bound for the dual frame. To see the upper bound, consider the following sequence of estimates

$$\sum_{n\in\mathbb{N}} |\langle x, \tilde{e}_n\rangle|^2 = \sum_{n\in\mathbb{N}} \langle x, \tilde{e}_n\rangle \langle \tilde{e_n}, x\rangle = \langle x, T^{-1}x\rangle \leq \frac{1}{A}\|x\|^2,$$

since $\|T^{-1}x\| \leq \frac{1}{A}\|x\|$.

Finally, to verify (3) let a_n and b_n as in the statement of the proposition, i.e.,

$$\sum_{n\in\mathbb{N}} a_n e_n = \sum_{n\in\mathbb{N}} b_n e_n$$

with $a_n = \langle x, \tilde{e}_n\rangle$. Consider the inner product of each side of the above equality with $T^{-1}x$,

$$\sum_{n\in\mathbb{N}} a_n\langle e_n, T^{-1}x\rangle = \sum_{n\in\mathbb{N}} b_n\langle e_n, T^{-1}x\rangle,$$

which is equivalent to

$$\sum_{n\in\mathbb{N}} |a_n|^2 = \sum_{n\in\mathbb{N}} b_n\overline{a_n}.$$

Hence,

$$\begin{aligned}\sum_{n\in\mathbb{N}} |a_n - b_n|^2 &= \sum_{n\in\mathbb{N}} |a_n|^2 - 2\text{Real}\left\{\sum_{n\in\mathbb{N}} a_n\overline{b_n}\right\} + \sum_{n\in\mathbb{N}} |b_n|^2 \\ &= \sum_{n\in\mathbb{N}} |b_n|^2 - \sum_{n\in\mathbb{N}} |a_n|^2.\end{aligned}$$

□

4. Nonlinear approximation with frames

In this section an estimate is derived for the greedy algorithm applied to the N-term approximation of $x \in \mathbb{H}$ given a frame expansion. Let $x \in \mathbb{H}$ and suppose that x is expressible as an unconditionally convergent series,

$$x = \sum_{n\in\mathbb{N}} c_n e_n,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a frame with bounds A, B. Let $\{n_k\}_{k\in\mathbb{N}}$ be a rearrangement of the natural numbers such that

$$\|c_{n_1} e_{n_1}\| \geq \|c_{n_2} e_{n_2}\| \geq \|c_{n_3} e_{n_3}\| \geq \cdots,$$

and define $G_N(x) = \sum_{k=1}^N c_{n_k} e_{n_k}$, as in Section 1.

Proposition 4. *Let $x \in \mathbb{H}$ and $N \in \mathbb{N}$, then*

$$\|x - G_N(x)\|^2 \leq B \sum_{k>N} |c_{n_k}|^2.$$

Proof. Let $\{\tilde{e}_n\}_{n\in\mathbb{N}}$ denote the dual frame. The vector $x - G_N(x)$ may thus be expressed as both

$$x - G_N(x) = \sum_{k>N} c_{n_k} e_{n_k} \quad \text{and} \quad x - G_N(x) = \sum_{n\in\mathbb{N}} \langle x - G_N(x), \tilde{e}_n \rangle e_n.$$

By Proposition 3,

$$B^{-1}\|x - G_N(x)\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x - G_N(x), \tilde{e}_n\rangle|^2 \leq \sum_{k>N} |c_{n_k}|^2,$$

which leads to the desired estimate. □

Remark 1. *If the frame $\{e_n\}_{n\in\mathbb{N}}$ is actually an orthonormal basis, then Proposition 4 reduces precisely to the estimate shown in the proof of Proposition 1.*

References

[Young] R. Young, *An introduction to nonharmonic Fourier series*, revised edition, Academic Press, San Diego, (2001).