ON THE OVERSAMPLING OF AFFINE WAVELET FRAMES

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Abstract. The properties of oversampled affine frames are considered here with two main goals in mind. The first goal is to generalize the approach of Chui and Shi to the matrix oversampling setting for expanding, lattice-preserving dilations, whereby we obtain a new proof of the Second Oversampling Theorem for affine frames. The Second Oversampling Theorem, proven originally by Ron and Shen via Gramian analysis, states that oversampling an affine frame with dilation M by a matrix P will result in a frame with the same bounds (after renormalization) provided that P and M satisfy a certain relative primality condition. In this case, the matrix P is said to be admissible for M. The second goal of this work is to examine the compatibility of admissible oversampling with the refinable affine frames arising from a certain class of scaling functions. In this setting we show that oversampling dual affine systems by an admissible P preserves the multiresolution structure and, from this fact, that the oversampled systems remain dual. We then show that the admissibility of P is also sufficient to endow the dual oversampled systems with a discrete wavelet transform. The novelty of this work lies both in our approach to the Second Oversampling Theorem as well as our consideration of oversampling in the context of multiresolution analysis.

Key words. affine system, oversampling, wavelet, multiresolution analysis

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1. Introduction. Unless otherwise stated, M will denote a fixed $n \times n$ dilation matrix with integer entries such that each eigenvalue λ of M satisfies $|\lambda| > 1$. We will refer to $M \in GL_n(\mathbb{R})$ as *expanding* if its eigenvalues satisfy this latter condition. Thus M is a $\mathbb{Z}^n$-lattice preserving, expanding dilation. The unitary dilation operator on $L^2(\mathbb{R}^n)$ induced by M will be denoted D and is defined by $Df(x) := |\det M|^{\frac{1}{2}} f(Mx)$ for $f \in L^2(\mathbb{R}^n)$. For $u \in \mathbb{R}^n$, let T_u denote the usual translation operator, i.e. $T_u f(x) := f(x-u)$. With these basic ingredients we may now recall the definition of an affine system.

DEFINITION 1.1. *Let $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$. The affine system generated by Ψ, denoted $X(\Psi)$, is the collection*

$$X(\Psi) = \{\psi_{\ell;j,k} : 1 \leq \ell \leq L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where $\psi_{\ell;j,k} := D^j T_k \psi_\ell$.

Our interest lies in those affine systems that constitute frames for $L^2(\mathbb{R}^n)$.

DEFINITION 1.2. *Let $\mathbb{H}$ be a Hilbert space. The collection $\{h_j\}_{j\in J} \subset \mathbb{H}$ is a frame for $\mathbb{H}$ if there exist constants $A, B > 0$ such that for all $f \in \mathbb{H}$*

$$A\|f\|_{\mathbb{H}}^2 \leq \sum_{j\in J} |\langle f, h_j\rangle_{\mathbb{H}}|^2 \leq B\|f\|_{\mathbb{H}}^2. \tag{1.1}$$

The constants A and B are referred to as the lower and upper frame bounds, respectively. In the case that $A = B$ the frame is said to be *tight.* If only the right inequality holds, the system is called a *Bessel system* and in this case B is referred to as the

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Bessel bound. We say two frames for $\mathbb{H}$, $\{h_j\}_{j\in J}$ and $\{\tilde{h}_j\}_{j\in J}$, are *dual frames* if for each $f \in \mathbb{H}$ we have

$$f = \sum_{j\in J} \langle f, \tilde{h}_j \rangle h_j. \tag{1.2}$$

Let $GL_n(\mathbb{Z})$ denote the set of all $n \times n$ matrices with integer entries having nonzero determinant. Given $P \in GL_n(\mathbb{Z})$, we now define the oversampled affine system generated by a family Ψ relative to P.

Definition 1.3. *Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$. The oversampled affine system generated by Ψ relative to $P \in GL_n(\mathbb{Z})$, denoted $X^P(\Psi)$, is the collection*

$$X^P(\Psi) := \{\psi^P_{\ell;j,k} : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where $\psi^P_{\ell;j,k} := \frac{1}{\sqrt{p}} D^j T_{P^{-1}k} \psi_\ell$ and $p := |\det P|$.

The factor $\frac{1}{\sqrt{p}}$ in Definition 1.3 compensates for the increase in the density of the lattice of translations caused by the oversampling. This allows us to compare the frame bounds of the oversampled and non-oversampled systems.

This notion of oversampling was introduced by Chui and Shi in [2] when they proved that oversampling a dyadic affine frame ($M = 2$) in one dimension by p odd preserves the frame bounds. In [3], Chui and Shi later extended this result to the multivariate case in which the dilation $M \in GL_n(\mathbb{Z})$ is expanding and $P = pI$ with $\gcd(p, |\det M|) = 1$. The result is referred to there as the Second Oversampling Theorem. Since the one-dimensional result appeared, several other researchers have investigated the problem of bound-preserving oversampling for affine frames. In the case that $M, P \in GL_n(\mathbb{Z})$ with M expanding, Ron and Shen have used their Gramian analysis to show that a relative primality condition on the lattices $M^T\mathbb{Z}^n$ and $P^T\mathbb{Z}^n$ is sufficient for bound-preserving oversampling [7]. More recently, the work of Laugesen [6] provides another approach to the Second Oversampling Theorem which employs the concept of almost periodicity. In [6] it is observed that the conditions on M and P for bound-preserving oversampling described in [7] and [6] are equivalent. We should also mention that Chui, Czaja, Maggioni, and Weiss have developed a notion of tightness-preserving oversampling based on the characterizion of affine tight-frames [1]. In their work the dilation matrix M is not required to have integer entries; however, the result only applies to tight-frames.

During the revision of this paper the author has become aware of an interesting work by Hernández, Labate, Weiss, and Wilson [4] in which the various embodiments of oversampling are unified into a single theory, including quasi-affine systems as well as oversampled affine systems.

The techniques used by Ron and Shen in [7] and Laugesen in [6] are quite different from those used originally by Chui and Shi in [2]. In each case a notion of relative primality between the dilation matrix and the oversampling matrix has proven essential in the proof of the Second Oversampling Theorem. One of the goals of our work is to extend the original ideas of Chui and Shi to the matrix oversampling case with a careful development of the relative primality conditions. These conditions are introduced in section 2, where we define a notion of admissible oversampling and develop related elementary results. In section 3 we present our version of the Second Oversampling Theorem.

The second goal of this work is to explore the compatibility of admissible oversampling with multiresolution analysis. In section 4 we restrict attention to dual refinable affine systems associated with a certain class of scaling functions. We introduce multiresolution operators for the oversampled systems and show that they behave much like those associated with the original affine systems. This allows us to prove that the duality of refinable affine frames is preserved under admissible oversampling. Finally, we show that admissible oversampling endows the dual oversampled systems with a discrete wavelet transform (DWT).

To close the section let us note that we will adopt the following definition for the Fourier transform, $\hat{f}$, of $f \in L^2(\mathbb{R}^n)$,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x\rangle} dx.$$

2. Admissible Oversampling Matrices. Given a candidate oversampling matrix, $P \in GL_n(\mathbb{Z})$, we are concerned with the quotient group $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. Let $\{\theta_r\}_{r=0}^{p-1}$ be a complete set of distinct coset representatives of the quotient group $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$, where again $p = |\det P|$. In the following proposition we consider conditions on P such that the action of M on $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ is nice.

Proposition 2.1. *Suppose $M, P \in GL_n(\mathbb{Z})$ with $m := |\det M|$ and $p := |\det P|$. Let $\{\theta_r\}_{r=0}^{p-1}$ be a complete set of distinct coset representatives of $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ with $\theta_0 = 0$. Suppose $PMP^{-1} \in GL_n(\mathbb{Z})$. Then $\{M\theta_r\}_{r=0}^{p-1}$ is a complete set of representatives of $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ if and only if M and P satisfy $M^{-1}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n = \mathbb{Z}^n$.*

Proof: The statement is trivial if $p = 1$. We proceed to prove the result in the case that $p \geq 2$.

($\Rightarrow$) By way of contradiction assume that $M^{-1}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n \supsetneq \mathbb{Z}^n$. Then there exists θ_{r_0}, $1 \leq r_0 \leq p-1$, such that $\theta_{r_0} \in \left(P^{-1}\mathbb{Z}^n \bigcap M^{-1}\mathbb{Z}^n\right) \setminus \mathbb{Z}^n$. This implies that $M\theta_{r_0} \equiv 0 \pmod{\mathbb{Z}^n}$ which means $\{M\theta_r\}_{r=0}^{p-1}$ can not be a complete set of representatives of $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. This is a contradiction.

($\Leftarrow$) The condition $PMP^{-1} \in GL_n(\mathbb{Z})$ implies that $Mx \in P^{-1}\mathbb{Z}^n$ if $x \in P^{-1}\mathbb{Z}^n$, i.e. the multiplication map induced by M maps $P^{-1}\mathbb{Z}^n$ into itself. Suppose $\{M\theta_r\}_{r=0}^{p-1}$ is not a complete set of coset representatives of $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. Then there is some θ_{r_0}, $1 \leq r_0 \leq p-1$, such that $M\theta_{r_0} \in \mathbb{Z}^n$ which implies that $\theta_{r_0} \in M^{-1}\mathbb{Z}^n$. Since $r_0 \neq 0$, $\theta_{r_0} \notin \mathbb{Z}^n$ implying $\theta_{r_0} \in \left(P^{-1}\mathbb{Z}^n \bigcap M^{-1}\mathbb{Z}^n\right) \setminus \mathbb{Z}^n$ and we have $M^{-1}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n \supsetneq \mathbb{Z}^n$, a contradiction. □

Definition 2.2. *Let $P \in GL_n(\mathbb{Z})$. P is an admissible oversampling matrix for M if $PMP^{-1} \in GL_n(\mathbb{Z})$ and $M^{-1}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n = \mathbb{Z}^n$. If the matrix M is clear from the context we will simply say that P is admissible.*

In the terminology of the preceding proposition, notice that if $\gcd(m, p) = 1$ then $M^{-1}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n = \mathbb{Z}^n$. Indeed, suppose $\theta \in \left(M^{-1}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n\right)$. Since the order of θ divides both m and p we conclude that $\theta \in \mathbb{Z}^n$.

Example 1. *Consider the following examples of admissible oversampling matrices.*

(a) *Let* $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$*, the Quincunx dilation matrix, and let* $P = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$*. It is easy to check that* PMP^{-1} *has integer entries and, in light of the previous remark,* P *is admissible.*

(b) *Let* $M = mI_n$*, where* $m \geq 2$ *is an integer and* I_n *is the* $n \times n$ *identity matrix. Clearly,* $PMP^{-1} \in GL_n(\mathbb{Z})$ *for all* $P \in GL_n(\mathbb{Z})$*, which means a sufficient condition for* P *to be admissible is that* $\gcd(m, |\det P|) = 1$.

Given that P is admissible, Proposition 2.1 tells us that the mapping $\theta_r \mapsto M\theta_r$, $0 \leq r \leq p-1$, acts to permute the coset representatives of $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. Let σ be the permutation of $\{0, \ldots, p-1\}$ such that $\theta_{\sigma(r)} \equiv M\theta_r \pmod{\mathbb{Z}^n}$ in $P^{-1}\mathbb{Z}^n$. Let σ^{-1} be the associated inverse permutation. The following result, which replaces Lemma 2 of [2] in this setting, describes a basic property of the permutation σ.

LEMMA 2.3. *Let* $j_0 \in \mathbb{Z}$*. If* $P \in GL_n(\mathbb{Z})$ *is admissible, then for* $j \geq j_0$ *and* $0 \leq r \leq p-1$, $\theta_{\sigma^j(r)} \equiv M^{j-j_0}\theta_{\sigma^{j_0}(r)} \ (mod\, \mathbb{Z}^n)$ *in* $P^{-1}\mathbb{Z}^n$.

Proof: The statement holds trivially for $j = j_0$. By induction, assume the formula holds for j and we will derive the formula for $j+1$. Proposition 2.1 and the definition of σ imply

$$\theta_{\sigma^{j+1}(r)} \equiv M\theta_{\sigma^j(r)} \equiv M^{j+1-j_0}\theta_{\sigma^{j_0}(r)} \pmod{\mathbb{Z}^n}. \quad \square$$

COROLLARY 2.4. *Let* $j, j_0 \in \mathbb{Z}$ *with* $j \geq j_0$ *and suppose* $P \in GL_n(\mathbb{Z})$ *is admissible. For* $0 \leq r \leq p-1$,

$$\{D^j T_{\theta_{\sigma^j(r)}+k}\psi_\ell : 1 \leq \ell \leq L, k \in \mathbb{Z}^n\} = \{T_{M^{-j_0}\theta_{\sigma^{j_0}(r)}} D^j T_k \psi_\ell : 1 \leq \ell \leq L, k \in \mathbb{Z}^n\}.$$

3. The Second Oversampling Theorem.

We now seek to describe our version of the Second Oversampling Theorem, generalizing the approach of Chui and Shi introduced in [2]. We begin by demonstrating the preservation of Bessel bounds for admissible oversampling, which follows essentially from Proposition 2.1.

LEMMA 3.1. *Suppose* $X(\Psi)$ *is a Bessel system with bound* $B > 0$ *relative to an expanding dilation matrix* $M \in GL_n(\mathbb{Z})$*. If* $P \in GL_n(\mathbb{Z})$ *is an admissible oversampling matrix, then* $X^P(\Psi)$ *is a Bessel system with the same bound.*

Proof: The fact that $X(\Psi)$ is Bessel with bound $B > 0$ implies for each $f \in L^2(\mathbb{R}^n)$ that

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} \sum_{r=0}^{p-1} \frac{1}{p} \left| \langle T_{-\theta_r} f, \psi_{\ell;j,k} \rangle \right|^2 \leq B\|f\|^2.$$

We now relate this sum to the inner products of the oversampled system, $X^P(\Psi)$,

$$\begin{aligned} \sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} \sum_{r=0}^{p-1} \frac{1}{p} \left| \langle T_{-\theta_r} f, \psi_{\ell;j,k} \rangle \right|^2 &= \sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} \sum_{r=0}^{p-1} \frac{1}{p} \left| \langle f, D^j T_{M^j\theta_r+k}\psi_\ell \rangle \right|^2 \\ \text{(by Proposition 2.1)} \quad &= \sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} \frac{1}{p} \left| \langle f, D^j T_{P^{-1}k}\psi_\ell \rangle \right|^2 \end{aligned}$$

$$= \sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} \left| \langle f, \psi^P_{\ell;j,k} \rangle \right|^2.$$

Letting $J \geq 0$ and observing that $\|D^J f\|^2 = \|f\|^2$ it is easily shown that for each $f \in L^2(\mathbb{R}^n)$

$$\sum_{\ell=1}^{L} \sum_{j \geq -J} \sum_{k \in \mathbb{Z}^n} \left| \langle f, \psi^P_{\ell;j,k} \rangle \right|^2 \leq B\|f\|^2.$$

Letting $J \to \infty$ we see that $X^P(\Psi)$ is a Bessel system with upper bound B. $\square$

Given an admissible oversampling matrix we can use the permutation σ guaranteed by Proposition 2.1 to rewrite the oversampled system as the union of appropriate affine-like systems, one for each coset of $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. Namely,

$$X^P(\Psi) = \bigcup_{r=0}^{p-1} \frac{1}{\sqrt{p}} S_r \quad \text{(disjointly)},$$

where $S_r := \{D^j T_{\theta_{\sigma^j(r)}+k} \psi_\ell : 1 \leq \ell \leq L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. S_0 is precisely $X(\Psi)$ while the remaining collections S_r, $1 \leq r \leq p-1$, are slightly more complicated. This decomposition plays a key role in our proof of the Second Oversampling Theorem.

Theorem 3.2 (Second Oversampling Theorem). *Suppose $X(\Psi)$ is a frame with lower and upper bounds $A, B > 0$, respectively, relative to an expanding dilation matrix $M \in GL_n(\mathbb{Z})$. If $P \in GL_n(\mathbb{Z})$ is an admissible oversampling matrix, then $X^P(\Psi)$ is a frame with the same bounds.*

Remark: In [6], Laugesen shows that if $X(\Psi)$ and $X(\tilde{\Psi})$ are dual frames and P is admissible, then $X^P(\Psi)$ and $X^P(\tilde{\Psi})$ are also dual frames. This statement will be proven in the next section (see Theorem 4.4 below) for a certain class of refinable functions.

Proof: The preservation of the upper bound was discussed above; hence, it suffices to demonstrate the lower bound. Since $S_0 = X(\Psi)$, S_0 is a frame with lower bound A. It is, therefore, sufficient to prove that each of the collections S_r, $1 \leq r \leq p-1$, is a frame with lower bound A.

Fix r, $1 \leq r \leq p-1$, and let $f \in L^\infty_c(\mathbb{R}^n)$, the dense subset of $L^2(\mathbb{R}^n)$ consisting of essentially bounded functions of compact support. It is sufficient to demonstrate the lower bound for such an f. Suppose that $\operatorname{supp} f \subset K$, where K is a compact subset of $\mathbb{R}^n$ containing 0. Let $R := \operatorname{diam} K$. Lastly, let $\lambda_- > 1$ and λ_+ be the strict lower and upper bounds, respectively, for the moduli of the eigenvalues of M.

1. For $j_0 \in \mathbb{Z}$, let $f^r_{j_0} = T_{-M^{-j_0}\theta_{\sigma^{j_0}(r)}} f$. By defining $K^r_{j_0} := K - M^{-j_0}\theta_{\sigma^{j_0}(r)}$ we see that $\operatorname{supp} f^r_{j_0} \subset K^r_{j_0}$. Observe that by Corollary 2.4 we have

$$\sum_{g \in S_r} |\langle f, g \rangle|^2 = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \left| \langle f, D^j T_{\theta_{\sigma^j(r)}+k} \psi_\ell \rangle \right|^2$$
$$= \sum_{\ell=1}^{L} \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}^n} \left| \langle f^r_{j_0}, D^j T_k \psi_\ell \rangle \right|^2 + \sum_{\ell=1}^{L} \sum_{j < j_0} \sum_{k \in \mathbb{Z}^n} \left| \langle f, D^j T_{\theta_{\sigma^j(r)}+k} \psi_\ell \rangle \right|^2$$

$$\geq \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\left|\langle f_{j_0}^r,\psi_{\ell;j,k}\rangle\right|^2-\sum_{\ell=1}^{L}\sum_{j<j_0}\sum_{k\in\mathbb{Z}^n}\left|\langle f_{j_0}^r,\psi_{\ell;j,k}\rangle\right|^2$$
$$\geq A\|f\|^2-\sum_{\ell=1}^{L}\sum_{j<j_0}\sum_{k\in\mathbb{Z}^n}\left|\langle f_{j_0}^r,\psi_{\ell;j,k}\rangle\right|^2.$$

We are left to prove that the latter sum tends to 0 as $j_0\to-\infty$.

2. Let $\varepsilon>0$. Let us adopt the notation

$$S_{j_0}^r(f)=\sum_{\ell=1}^{L}\sum_{j<j_0}\sum_{k\in\mathbb{Z}^n}\left|\langle f_{j_0}^r,\psi_{\ell;j,k}\rangle\right|^2.$$

Estimating the inner product of $f_{j_0}^r$ with $\psi_{\ell;j,k}$ by

$$\left|\langle f_{j_0}^r,\psi_{\ell;j,k}\rangle\right|^2\leq\|f_{j_0}^r\|^2\,\|\psi_{\ell;j,k}\chi_{K_{j_0}^r}\|^2\leq\|f\|^2\int_{M^jK_{j_0}^r}|\psi_\ell(x-k)|^2dx,$$

we obtain a bound on $S_{j_0}^r(f)$:

$$S_{j_0}^r(f)\leq\|f\|^2\sum_{\ell=1}^{L}\sum_{j<j_0}\sum_{k\in\mathbb{Z}^n}\int_{M^jK_{j_0}^r-k}|\psi_\ell(x)|^2dx.$$

We will break up the sum over j into two pieces, corresponding to $j<j_0-J$ and $j_0-J\leq j<j_0$ where $J>0$ will be fixed below (independent of j_0). Since M is expanding it is well known that there exists $\beta\geq1$ such that for $x\in\mathbb{R}^n$ and $j>0$, we have the estimates

$$\beta^{-1}\lambda_-^j\|x\|\leq\|M^jx\|\leq\beta\lambda_+^j\|x\|,$$

and

$$\beta^{-1}\lambda_+^{-j}\|x\|\leq\|M^{-j}x\|\leq\beta\lambda_-^{-j}\|x\|,$$

where $\beta\geq1$ depends on λ_-, λ_+, and M.
We now make a pair of technical assumptions that will be used below, each of which relies on the expanding property of M.

(a) We may assume j_0 is negative and sufficiently less than 0 that

$$R<\frac{1}{2}\|M^{-j_0}\theta_{\sigma^{j_0}(r)}\|.$$

(b) We will assume $J>0$ is such that $\lambda_-^J>2\beta^{-1}(R+\|\theta_{\sigma^{j_0}(r)}\|)$.

3. Let us first handle the terms for which $j<j_0-J$. Consider the set

$$E:=\bigcup_{j<j_0-J}M^jK_{j_0}^r=\bigcup_{j<j_0-J}\big(M^j(K-M^{-j_0}\theta_{\sigma^{j_0}(r)})\big).$$

Our first estimate involves replacing the sum over j by integration over E, which requires the that the sets $\{M^jK_{j_0}^r\}_{j<j_0-J}$ have finite overlaps which

can be bounded independently of j_0. Supposing for the moment that this is the case, we have

$$I_1 := \|f\|^2 \sum_{\ell,k} \sum_{j<j_0-J} \int_{M^j K^r_{j_0}-k} |\psi_\ell(x)|^2 dx \le C\|f\|^2 \sum_{\ell,k} \int_{E-k} |\psi_\ell(x)|^2 dx.$$

We now investigate the disjointness of $\{M^j K^r_{j_0}\}_{j<j_0-J}$. Suppose that

$$M^{j_1} K^r_{j_0} \bigcap M^{j_2} K^r_{j_0} \neq \emptyset,$$

with $j_1 > j_2$, then $M^{j_1-j_2} K^r_{j_0} \bigcap K^r_{j_0} \neq \emptyset$. Thus it suffices to prove that there exists $j_1 > 0$ (independent of j_0) such that $M^j K^r_{j_0} \bigcap K^r_{j_0} = \emptyset$ for all $j \ge j_1$. For $x \in K^r_{j_0}$ we have by assumption (a) that

$$\|x\| \le R + \|M^{-j_0}\theta_{\sigma^{j_0}(r)}\| \le \frac{3}{2}\|M^{-j_0}\theta_{\sigma^{j_0}(r)}\|,$$

and

$$\|x\| \ge \|M^{-j_0}\theta_{\sigma^{j_0}(r)}\| - R \ge \frac{1}{2}\|M^{-j_0}\theta_{\sigma^{j_0}(r)}\|.$$

Again using the expanding property of M, we have for $x \in K^r_{j_0}$ and $j > 0$

$$\|M^j x\| \ge \frac{1}{\beta}\lambda_-^j \frac{1}{2}\|M^{-j_0}\theta_{\sigma^{j_0}(r)}\|.$$

Hence, a sufficient condition for the disjointness of the sets $M^j K^r_{j_0}$ and $K^r_{j_0}$ is

$$\frac{1}{\beta}\lambda_-^j \frac{1}{2}\|M^{-j_0}\theta_{\sigma^{j_0}(r)}\| \ge \frac{3}{2}\|M^{-j_0}\theta_{\sigma^{j_0}(r)}\|,$$

or, equivalently,

$$\lambda_-^j \ge 3\beta.$$

Again, since M is expanding we have $\lambda_- > 1$, so we may choose $j_1 > 0$ to be the smallest j for which this last inequality holds.

4. Returning to the estimate from 3, we will next fix J large enough to control the term I_1. Let us examine a typical j in the sum defining I_1, which is of the form $j = j_0 - J - j_1$ with $j_1 \ge 1$. If $x \in M^j K^r_{j_0}$, then

$$\|x\| \le \beta\lambda_-^{j_0-J-j_1} R + \beta\lambda_-^{-J-j_1}\|\theta_{\sigma^{j_0}(r)}\| \le \beta\lambda_-^{-J-j_1}\big(R + \|\theta_{\sigma^{j_0}(r)}\|\big),$$

where we have used the assumption that $j_0 < 0$ and β is as above. We conclude that if $x \in E$, then $\|x\| \le \lambda_-^{-J-1}\big(R + \|\theta_{\sigma^{j_0}(r)}\|\big) \le \frac{1}{2}$ by the assumption (b) above regarding J. This means that $\operatorname{diam} E \le 1$, implying that any distinct integer translates of E are disjoint. While the definition of E above does depend on both j_0 and J, we just showed that the measure of E can be made arbitrarily small independent of j_0, by choosing J sufficiently large. Since each $\psi_\ell \in L^2(\mathbb{R}^n)$ the dominated convergence theorem allows us to fix $J > 0$ so large that $I_1 < \varepsilon$ independent of j_0.

5. We now estimate the terms in $S^r_{j_0}(f)$ for $j_0 - J \leq j < j_0$ with J fixed as in the last step. It suffices to consider an arbitrary term of this sort, which can be written as $j = j_0 - j_1$ with $1 \leq j_1 \leq J$. By definition, $M^j K^r_{j_0} = M^{j_0 - j_1} K - M^{-j_1}\theta_{\sigma^{j_0}(r)}$, where $M^{-j_1}\theta_{\sigma^{j_0}(r)}$ is one of $p-1$ constants depending on j_0. Hence,

$$I_{2,j} := \sum_{\ell,k} \int_{M^j K^r_{j_0} - k} |\psi_\ell(x)|^2 dx = \sum_{\ell,k} \int_{M^{j_0-j_1}K - M^{-j_1}\theta_{\sigma^{j_0}(r)} - k} |\psi_\ell(x)|^2 dx.$$

Applying the dominated convergence theorem we may choose j_0 sufficiently less than zero that $I_{2,j} < \varepsilon$.

By definition we have

$$S^r_{j_0} = I_1 + \sum_{j=j_0-J}^{j_0-1} I_{2,j},$$

so, combining all the estimates, we have shown that $S^r_{j_0} \to 0$ as $j_0 \to -\infty$. Thus, S_r is a frame with lower bound A for each r, $0 \leq r \leq p-1$, completing the proof. □

4. Oversampling and Multiresolution Analysis. Consider two families of generating functions, $\Psi := \{\psi_1, \dots, \psi_L\}$ and $\tilde{\Psi} := \{\tilde{\psi}_1, \dots, \tilde{\psi}_L\} \subset L^2(\mathbb{R}^n)$. Let us assume that the families are produced by refinement with scaling functions $\varphi, \tilde{\varphi} \in \mathbb{E}$, respectively, where $\mathbb{E} := \{f \in L^2(\mathbb{R}^n) : [\hat{f}, \hat{f}] \in L^\infty(\mathbb{T}^n)\}$. Recall that $[f, g]$, the *bracket product of* $f, g \in L^2(\mathbb{R}^n)$, is defined by

$$[f, g] = \sum_{k \in \mathbb{Z}^n} T_{2\pi k} f \overline{T_{2\pi k} g}.$$

Adopting the convention that $\psi_0 := \varphi$ and $\tilde{\psi}_0 := \tilde{\varphi}$ for notational convenience, we have the refinement identities

$$\hat{\psi}_\ell(M^T\xi) = m_\ell(\xi)\hat{\varphi}(\xi) \text{ and } \hat{\tilde{\psi}}_\ell(M^T\xi) = \tilde{m}_\ell(\xi)\hat{\tilde{\varphi}}(\xi) \tag{4.1}$$

for $0 \leq \ell \leq L$ and a.e. $\xi \in \mathbb{R}^n$, where $m_\ell, \tilde{m}_\ell \in L^\infty(\mathbb{T}^n)$ for $0 \leq \ell \leq L$. We assume here that $M \in GL_n(\mathbb{Z})$ is expansive. Finally, we assume that the filters satisfy the generalized Smith-Barnwell equations for the dilation M, namely for $0 \leq s \leq m-1$ we have

$$\sum_{\ell=0}^{L} \overline{m_\ell(\xi)} \tilde{m}_\ell(\xi + 2\pi(M^T)^{-1}\vartheta_s) = \delta_{0,s} \quad \text{a.e. } \xi \in \mathbb{T}^n, \tag{4.2}$$

where $\{\vartheta_s\}_{p=0}^{m-1}$ is a complete set of distinct coset representatives of $\mathbb{Z}^n/M^T\mathbb{Z}^n$, $m := |\det M|$, and $\delta_{0,s}$ is the Kronecker delta. Implicitly assumed here is the fact that $\vartheta_0 = 0$. Together, the scaling functions and filters specify the generating families Ψ and $\tilde{\Psi}$ that define the affine systems $X(\Psi)$ and $X(\tilde{\Psi})$. We will rely on some basic properties of this class of refinable affine systems as found in [5].

Given that these two systems are dual frames for $L^2(\mathbb{R}^n)$ we are interested in two properties of the resulting oversampled systems. First, we will investigate when the oversampled affine systems $X^P(\Psi)$ and $X^P(\tilde{\Psi})$ relative to a matrix $P \in GL_n(\mathbb{Z})$ are again dual frames. Secondly, we will examine the scaling equations associated with the oversampled system and determine conditions on the oversampling matrix that endow the oversampled system with a bonafide DWT. We conclude the section by reconciling the conditions required for these two properties.

4.1. Multiresolution Operators and Duality. Our analysis will involve multiresolution operators that arise naturally as generalizations of the orthogonal projections found in the orthonormal MRA case. The affine approximation and detail operators at the scale $j \in \mathbb{Z}$, $\mathcal{P}_j$ and $\mathcal{Q}_j$, respectively, act on $f \in L^2(\mathbb{R}^n)$ by

$$\mathcal{P}_j f := \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\varphi}_{j,k} \rangle \varphi_{j,k} \text{ and } \mathcal{Q}_j f := \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{\ell;j,k} \rangle \psi_{\ell;j,k}, \tag{4.3}$$

whereas the oversampled affine approximation and detail operators at the scale j, $\mathcal{P}_j^P$ and $\mathcal{Q}_j^P$, respectively, are defined similarly by

$$\mathcal{P}_j^P f := \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\varphi}_{j,k}^P \rangle \varphi_{j,k}^P \text{ and } \mathcal{Q}_j^P f := \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{\ell;j,k}^P \rangle \psi_{\ell;j,k}^P. \tag{4.4}$$

We have the following basic properties for $\mathcal{P}_j$ and $\mathcal{Q}_j$.

LEMMA 4.1 ([5]). *Suppose $\varphi, \tilde{\varphi} \in \mathbb{E}$ and $\Psi, \tilde{\Psi} \subset L^2(\mathbb{R}^n)$ are such that (4.1) and (4.2) hold.*

(a) $\mathcal{P}_j$ and $\mathcal{Q}_j$ are bounded operators on $L^2(\mathbb{R}^n)$ for each $j \in \mathbb{Z}$.

(b) $\mathcal{P}_j + \mathcal{Q}_j = \mathcal{P}_{j+1}$ for each $j \in \mathbb{Z}$.

(c) For each $f \in L^2(\mathbb{R}^n)$, $\lim_{j \to -\infty} \|\mathcal{P}_j f\| = 0$.

(d) If $X(\Psi)$ and $X(\tilde{\Psi})$ are dual frames for $L^2(\mathbb{R}^n)$ then for each $f \in L^2(\mathbb{R}^n)$, we have

$$f = \lim_{j \to \infty} \mathcal{P}_j f = \sum_{j \in \mathbb{Z}} \mathcal{Q}_j f. \tag{4.5}$$

Our objective is to establish similar properties for the oversampled multiresolution operators armed with this information. We begin by expressing the oversampled multiresolution operators in terms of the original affine counterparts.

PROPOSITION 4.2. *Let $\varphi, \tilde{\varphi} \in \mathbb{E}$. Let $P \in GL_n(\mathbb{Z})$. For each $j \in \mathbb{Z}$, $\mathcal{P}_j^P$ and $\mathcal{Q}_j^P$ are bounded operators on $L^2(\mathbb{R}^n)$ and we have*

(a) $\mathcal{P}_j^P = \dfrac{1}{p} \displaystyle\sum_{r=0}^{p-1} T_{M^{-j}\theta_r} \mathcal{P}_j T_{-M^{-j}\theta_r}$,

(b) $\mathcal{Q}_j^P = \dfrac{1}{p} \displaystyle\sum_{r=0}^{p-1} T_{M^{-j}\theta_r} \mathcal{Q}_j T_{-M^{-j}\theta_r}$.

Proof: It is sufficient to derive (a). We have for each $f \in L^2(\mathbb{R}^n)$,

$$\mathcal{P}_j^P f = \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\varphi}_{j,k}^P \rangle \varphi_{j,k}^P$$

$$= \frac{1}{p}\sum_{r=0}^{p-1}\sum_{k\in\mathbb{Z}^n}\langle f, D^jT_{\theta_r+k}\tilde{\varphi}\rangle D^jT_{\theta_r+k}\varphi$$
$$= \frac{1}{p}\sum_{r=0}^{p-1}\sum_{k\in\mathbb{Z}^n}\langle T_{-M^{-j}\theta_r}f, D^jT_k\tilde{\varphi}\rangle T_{M^j\theta_r}D^jT_k\varphi$$
$$= \frac{1}{p}\sum_{r=0}^{p-1}T_{M^{-j}\theta_r}\mathcal{P}_jT_{-M^{-j}\theta_r}f. \quad \square$$

It is important to realize that the representations of $\mathcal{P}_j^P$ and $\mathcal{Q}_j^P$ above are independent of the choice of coset representatives. This is because $\mathcal{P}_j$ and $\mathcal{Q}_j$ are invariant under conjugation by translation operators over $M^{-j}\mathbb{Z}^n$. Indeed, for each $f \in L^2(\mathbb{R}^n)$ and $k_0 \in \mathbb{Z}^n$ we have

$$\begin{aligned} T_{M^{-j}k_0}\mathcal{P}_jT_{-M^{-j}k_0}f &= \sum_{k\in\mathbb{Z}^n}\langle T_{-M^{-j}k_0}f, D^jT_k\tilde{\varphi}\rangle T_{M^{-j}k_0}D^jT_k\varphi \\ &= \sum_{k\in\mathbb{Z}^n}\langle f, T_{M^{-j}k_0}D^jT_k\tilde{\varphi}\rangle T_{M^{-j}k_0}D^jT_k\varphi \\ &= \sum_{k\in\mathbb{Z}^n}\langle f, D^jT_{k+k_0}\tilde{\varphi}\rangle D^jT_{k+k_0}\varphi \\ &= \mathcal{P}_jf. \end{aligned}$$

Since any two representatives of the same coset in $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ differ by an element of $\mathbb{Z}^n$ the claim follows. This independence is particularly important for the following proposition.

Proposition 4.3. *Let $\varphi, \tilde{\varphi} \in \mathbb{E}$. If $P \in GL_n(\mathbb{Z})$ is an admissible oversampling matrix then for each $j \in \mathbb{Z}$*

$$\mathcal{P}_j^P + \mathcal{Q}_j^P = \mathcal{P}_{j+1}^P.$$

Proof: The condition on the oversampling matrix P guarantees that $\{M\theta_r\}_{r=0}^{p-1}$ is a complete set of coset representatives for $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. This fact together with the previous proposition imply for $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \mathcal{P}_j^Pf + \mathcal{Q}_j^Pf &= \frac{1}{p}\sum_{r=0}^{p-1}\Big(T_{M^{-j}\theta_r}\mathcal{P}_jT_{-M^{-j}\theta_r}f + T_{M^{-j}\theta_r}\mathcal{Q}_jT_{-M^{-j}\theta_r}f\Big) \\ &= \frac{1}{p}\sum_{r=0}^{p-1}T_{M^{-j}\theta_r}\mathcal{P}_{j+1}T_{-M^{-j}\theta_r}f \\ &= \frac{1}{p}\sum_{r=0}^{p-1}T_{M^{-(j+1)}M\theta_r}\mathcal{P}_{j+1}T_{-M^{-(j+1)}M\theta_r}f \\ &= \frac{1}{p}\sum_{r=0}^{p-1}T_{M^{-(j+1)}\theta_r}\mathcal{P}_{j+1}T_{-M^{-(j+1)}\theta_r}f \end{aligned}$$

$$= \mathcal{P}_{j+1}^P f. \quad \square$$

We are now in the position to examine the duality of the oversampled systems.

THEOREM 4.4. *Suppose $\varphi, \tilde{\varphi} \in \mathbb{E}$ and $\Psi, \tilde{\Psi} \subset L^2(\mathbb{R}^n)$ are such that (4.1) and (4.2) hold. If $X(\Psi)$ and $X(\tilde{\Psi})$ are dual frames and $P \in GL_n(\mathbb{Z})$ is an admissible oversampling matrix, then $X^P(\Psi)$ and $X^P(\tilde{\Psi})$ are dual frames with the same bounds as $X(\Psi)$ and $X(\tilde{\Psi})$, respectively. Moreover, for each $f \in L^2(\mathbb{R}^n)$ we have*

$$f = \lim_{j\to\infty} \mathcal{P}_j^P f = \sum_{j\in\mathbb{Z}} \mathcal{Q}_j^P f = f, \tag{4.6}$$

and

$$\lim_{j\to-\infty} \|\mathcal{P}_j^P f\| = 0. \tag{4.7}$$

Proof: For this class of scaling functions we use the corresponding properties of $\mathcal{P}_j$ and $\mathcal{Q}_j$ contained in Lemma 4.1. Let $f \in L^2(\mathbb{R}^n)$. Since $\mathcal{P}_j^P$ is the finite sum of translated versions of $\mathcal{P}_j$ we conclude (4.7) directly from Lemma 4.1 (c) and Proposition 4.2. By Lemma 4.1 (d) we also have that $\mathcal{P}_j f \to f$ in $L^2(\mathbb{R}^n)$ as $j \to \infty$, from which we will obtain the first equality of (4.6) by approximation. Indeed, for each $u \in \mathbb{R}^n$ we have the estimate

$$\begin{aligned}
\|T_{M^{-j}u}\mathcal{P}_j T_{-M^{-j}u} f - f\| &\le \|T_{M^{-j}u} f - f\| + \|T_{M^{-j}u}\mathcal{P}_j T_{-M^{-j}u} f - T_{M^{-j}u} f\| \\
&\le \|T_{M^{-j}u} f - f\| + \|\mathcal{P}_j T_{-M^{-j}u} f - f\| \\
&\le \|T_{M^{-j}u} f - f\| + \|\mathcal{P}_j f - f\| + \|\mathcal{P}_j T_{-M^{-j}u} f - \mathcal{P}_j f\| \\
&\le \|T_{M^{-j}u} f - f\| + \|\mathcal{P}_j f - f\| + C\|T_{-M^{-j}u} f - f\|.
\end{aligned}$$

Each of the three terms in this estimate tend to zero as $j \to \infty$ and, thus, the first equality of (4.6) follows by summing the above as $u = \theta_r$, $0 \le r \le p-1$.

By Theorem 3.2, we have that $X^P(\Psi)$ and $X^P(\Psi)$ are frames with the same bounds as their respective affine counterparts. Lastly, the second equality of (4.6) follows from a telescoping argument using Proposition 4.3 and the fact that the oversampled systems are Bessel, thereby implying that $X^P(\Psi)$ and $X^P(\tilde{\Psi})$ are indeed dual. $\square$

Remark: We should note that in many cases the assumption in Theorem 4.4 that the original affine systems are actually dual frames may be avoided. For example, in [8] Ron and Shen have derived sufficient conditions (assuming a weak smoothness condition on the refinable family) involving identities of the form (4.2) under which a pair of refinable affine Bessel systems will constitute dual frames.

4.2. Discrete Wavelet Transform. Throughout this section we assume that $P \in GL_n(\mathbb{Z})$ is an admissible oversampling matrix for the dilation M. Recall that the refinement equations (4.1) can be written in the space domain as

$$\psi_{\ell;j,k} = m^{\frac{1}{2}} \sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\varphi_{j+1,r+Mk},$$

for $0 \le \ell \le L$, $j \in \mathbb{Z}$, and $k \in \mathbb{Z}^n$, where $m_\ell(\xi) = \sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r} e^{-i\langle \xi, r\rangle}$. We omit the analogous formulas for the dual functions and filters. We can obtain a similar formula for the oversampled system by observing

$$\begin{aligned}
\psi^P_{\ell;j,k} = \frac{1}{\sqrt{p}}\psi_{\ell;j,P^{-1}k} &= \sqrt{m}\sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\frac{1}{\sqrt{p}}\varphi_{j+1,r+MP^{-1}k} \\
&= \sqrt{m}\sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\frac{1}{\sqrt{p}}\varphi_{j+1,P^{-1}(Pr+\tilde{M}k)} \\
&= \sqrt{m}\sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\varphi^P_{j+1,Pr+\tilde{M}k},
\end{aligned}$$

where $\tilde{M} := PMP^{-1}$. Notice that because P is admissible $\tilde{M}$ has integer entries. Letting $\alpha^P_{\ell;r}$ be the coefficient sequence given by

$$\alpha^P_{\ell;r} := \begin{cases} \alpha_{\ell;s} & r = Ps, s\in\mathbb{Z}^n \\ 0 & \text{otherwise,} \end{cases}$$

we arrive at

$$\psi^P_{\ell;j,k} = \sqrt{m}\sum_{r\in\mathbb{Z}^n} \alpha^P_{\ell;r}\varphi^P_{j+1,r+\tilde{M}k}.$$

Thus, given $f \in L^2(\mathbb{R}^n)$ the sequence of inner products $\{\langle f, \psi^P_{\ell;j,k}\rangle\}_{k\in\mathbb{Z}^n}$ is given by

$$\langle f, \psi^P_{\ell;j,k}\rangle = \sqrt{m}\sum_{r\in\mathbb{Z}^n}\sum_{r\in\mathbb{Z}^n} \overline{\alpha^P_{\ell;r}}\langle f, \varphi^P_{j+1,r+\tilde{M}k}\rangle,$$

for $0 \le \ell \le L$ and $j \in \mathbb{Z}$. To those familiar with subband coding theory, this is immediately recognizable as a convolution followed by a downsampling operation. Note that the downsampling is relative to $\tilde{M}$ rather than M as for the original affine system. We would like this decomposition to be reversible, meaning that the sequence $\big\{\langle f, \varphi_{j+1,k}\rangle\big\}_{k\in\mathbb{Z}^n}$ should be recoverable from the sequences $\big\{\langle f, \psi_{\ell;j,k}\rangle\big\}_{k\in\mathbb{Z}^n}$, $0 \le \ell \le L$, by first upsampling each sequence by $\tilde{M}$ and then summing the respective convolutions with the dual filter coefficient sequences $\tilde{\alpha}^P_{\ell;r}$. It is well known that this is equivalent to the coefficient sequences $\alpha^P_{\ell;r}$ and $\tilde{\alpha}^P_{\ell;r}$ satisfying the filter equations (4.2) with $\tilde{M}$ instead of M. Letting m^P_ℓ be defined by

$$m^P_\ell(\xi) = \sum_{r\in\mathbb{Z}^n} \alpha^P_{\ell;r} e^{-i\langle \xi, r\rangle},$$

for $0 \le \ell \le L$, we have $m^P_\ell(\xi) = m_\ell(P^T\xi)$ by the definition of $\alpha^p_{\ell;r}$. The necessary generalized Smith-Barnwell equations are thus

$$\sum_{\ell=0}^{L} \overline{m^P_\ell(\xi)}\tilde{m}^P_\ell(\xi + 2\pi(\tilde{M}^T)^{-1}\tilde{\vartheta}_s) = \delta_{0,s}, \tag{4.8}$$

for a.e. $\xi \in \mathbb{T}^n$ and $0 \le s \le m-1$, where $\{\tilde{\vartheta}_s\}_{s=0}^{m-1}$ is a complete set of coset representatives of $\mathbb{Z}^n/\tilde{M}^T\mathbb{Z}^n$ with $\tilde{\vartheta}_s = 0$. In terms of the original filters, (4.8) is equivalent to

$$\sum_{\ell=0}^{L} \overline{m_\ell(\xi)}\tilde{m}_\ell(\xi + 2\pi P^T(\tilde{M}^T)^{-1}\tilde{\vartheta}_s) = \delta_{0,s}, \tag{4.9}$$

for $0 \leq s \leq m-1$, because $m_\ell^P(\xi) = m_\ell(P^T\xi)$. With the following theorem we describe a condition on the oversampling matrix that reduces (4.9) to (4.2) giving a condition under which the dual oversampled affine systems have an associated DWT.

THEOREM 4.5. *Let $P \in GL_n(\mathbb{Z})$ be an admissible oversampling matrix and assume that P also satisfies $(P^T)^{-1}\mathbb{Z}^n \bigcap (\tilde{M}^T)^{-1}\mathbb{Z}^n = \mathbb{Z}^n$. Then (4.2) and (4.9) are equivalent.*

Proof: It is sufficient to prove that $\{M^TP^T(\tilde{M}^T)^{-1}\tilde{\vartheta}_s\}_{s=0}^{m-1}$ is a complete set of representatives for $\mathbb{Z}^n/M^T\mathbb{Z}^n$. In other words we need only show

$$(M^T)^{-1}\mathbb{Z}^n = \bigcup_{s=0}^{m-1} \left(P^T(\tilde{M}^T)^{-1}\tilde{\vartheta}_s + \mathbb{Z}^n\right).$$

Observing that $\{(\tilde{M}^T)^{-1}\tilde{\vartheta}_s\}_{s=0}^{m-1}$ is a complete set of representatives of $(\tilde{M}^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ our problem is equivalent to showing that if $\{\gamma_s\}_{s=0}^{m-1}$ is a complete set of distinct coset representatives for $(\tilde{M}^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ then $\{P^T\gamma_s\}_{s=0}^{m-1}$ is a compete set of coset representatives for $(M^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.

The first bit of business is to establish that $P^T\gamma_s \in (M^T)^{-1}\mathbb{Z}^n$. This requires for each $x \in \mathbb{Z}^n$ a corresponding $y \in \mathbb{Z}^n$ such that $P^T(\tilde{M}^T)^{-1}x = (M^T)^{-1}y$ or, equivalently, that $M^TP^T(\tilde{M}^T)^{-1}$ has integer entries. Computing this we see

$$M^TP^T(\tilde{M}^T)^{-1} = M^TP^T(P^T)^{-1}(M^T)^{-1}P^T = P^T,$$

which clearly has integer entries.

We now proceed by way of contradiction. Suppose that $\{P^T\gamma_s\}_{s=0}^{m-1}$ is not a complete set of coset representatives. Then $P^T\gamma_{s_0} \in \mathbb{Z}^n$ for some s_0, $1 \leq s_0 \leq m-1$. Since $\gamma_{s_0} \notin \mathbb{Z}$ this implies that $(P^T)^{-1}\mathbb{Z}^n \bigcap (\tilde{M}^T)^{-1}\mathbb{Z}^n \supsetneq \mathbb{Z}^n$, a contradiction. □

4.3. Reconciliation of the Hypotheses. In the last subsection we found a relationship between the dilation matrix, M, and the oversampling matrix, P, that is sufficient for the existence of a DWT for the oversampled system. This condition essentially ensures that the perfect reconstruction filter equations for the matrix $\tilde{M} := PMP^{-1}$ are equivalent to those associated with M. It turns out that this condition is automatically satisfied for all admissible oversampling matrices.

THEOREM 4.6. *Let $M, P \in GL_n(\mathbb{Z})$ and such that $\tilde{M} := PMP^{-1} \in GL_n(\mathbb{Z})$. Then*

$$P^{-1}\mathbb{Z}^n \bigcap M^{-1}\mathbb{Z}^n = \mathbb{Z}^n \tag{4.10}$$

and

$$(P^T)^{-1}\mathbb{Z}^n \bigcap (\tilde{M}^T)^{-1}\mathbb{Z}^n = \mathbb{Z}^n, \tag{4.11}$$

are equivalent.

Proof: We prove the result step by step.

1. By symmetry, it is sufficient to prove (4.10) implies (4.11). Indeed, letting $M' = \tilde{M}^T$ and $P' = P^T$ we have $(\tilde{M}')^T = M$ and $(P')^T = P$.

2. Consider (4.11). If $x \in (\tilde{M}^T)^{-1}\mathbb{Z}^n \bigcap (P^T)^{-1}\mathbb{Z}^n$, then $x = (\tilde{M}^T)^{-1}r = (P^T)^{-1}s$ for some $r, s \in \mathbb{Z}^n$. This allows us to write

$$(\tilde{M}^T)^{-1}\mathbb{Z}^n \bigcap (P^T)^{-1}\mathbb{Z}^n = \big\{(P^T)^{-1}s : s \in \mathbb{Z}^n \text{ and } (P^T)^{-1}M^T s \in \mathbb{Z}^n\big\}.$$

   Letting $S = \{s \in \mathbb{Z}^n : (P^T)^{-1}M^T s \in \mathbb{Z}^n\}$, we have $(\tilde{M}^T)^{-1}\mathbb{Z}^n \bigcap (P^T)^{-1}\mathbb{Z}^n = (P^T)^{-1}S$. Thus, (4.11) is equivalent to $S = P^T\mathbb{Z}^n$.

3. $P^T\mathbb{Z}^n \subseteq S$. Proof: Let $s \in P^T\mathbb{Z}^n$ and write $s = P^T x$, $x \in \mathbb{Z}^n$. Then $(P^T)^{-1}M^T P^T x = \tilde{M}^T x \in \mathbb{Z}^n$ because $\tilde{M}$ has integer entries. Hence, $s \in S$.

4. We now provide an unusual characterization of S. It is easy to see that $x \in \mathbb{Z}^n$ if and only if $\langle x, y\rangle \in \mathbb{Z}$ for all $y \in \mathbb{Z}^n$. Thus, $s \in S$ if and only if $\langle (P^T)^{-1}M^T s, y\rangle \in \mathbb{Z}$ for all $y \in \mathbb{Z}^n$. This, in turn, is equivalent to $s \in S$ if and only if $\langle s, My\rangle \in \mathbb{Z}$ for all $y \in P^{-1}\mathbb{Z}^n$.

5. Recall from Proposition 2.1 that (4.10) is equivalent to $\{M\theta_r\}_{r=0}^{p-1}$ being a complete set of coset representatives for $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. In other words, (4.10) allows us to write $u \in P^{-1}\mathbb{Z}^n$ as $u = y + Mv$, where $y \in \mathbb{Z}^n$ and $v \in P^{-1}\mathbb{Z}^n$. Notice that if $u \notin \mathbb{Z}^n$ then $v \notin \mathbb{Z}^n$. This will be used below.

6. $S \subseteq P^T\mathbb{Z}^n$. Proof: Let $s \in \mathbb{Z}^n$ and suppose that $s \notin P^T\mathbb{Z}^n$. Then there exists $u \in P^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n$ such that $\langle s, u\rangle \notin \mathbb{Z}$. As explained above, (4.10) allows us to write $u = y + Mv$, where $y \in \mathbb{Z}^n$ and $v \in P^{-1}\mathbb{Z}^n$. Since $u \notin \mathbb{Z}^n$, we must have $v \notin \mathbb{Z}^n$. Then $\langle s, u\rangle = \langle s, y\rangle + \langle s, Mv\rangle \notin \mathbb{Z}$ and since $\langle s, y\rangle \in \mathbb{Z}$ we conclude $\langle s, Mv\rangle \notin \mathbb{Z}$. Hence, $s \notin S$. □

Theorem 4.6 shows that the additional assumption of (4.11) in Theorem 4.5 is redundant and that the dual oversampled affine systems always have an associated DWT if P is admissible.

COROLLARY 4.7. *If $P \in GL_n(\mathbb{Z})$ is an admissible oversampling matrix, then (4.2) and (4.9) are equivalent.*

5. Discussion of Related Work. With the number of variations on this theme of bound-preserving oversampling, some comparisons are in order. In particular we will discuss in detail how our work relates to that of Chui and Shi, Ron and Shen, and Laugesen.

As mentioned in the introduction, the problem of identifying sufficient conditions for the bound-preserving oversampling of affine frames started with Chui and Shi in the one-dimensional setting with dyadic wavelets frames and oversampling by an odd integer [2]. Our proof for Theorem 3.2 is an adaptation of that given in [2] to the n-dimensional case for expansive dilations $M \in GL_n(\mathbb{Z})$ and admissible oversampling matrices P. Chui and Shi improved on their one-dimensional result in [3], extending the second oversampling theorem to expanding dilations $M \in GL_n(\mathbb{Z})$ (i.e. λ and eigenvalue of M then $|\lambda| > 1$) and $P = pI_n$, with $\gcd(p, \det M) = 1$. This version of the second oversampling theorem also allows for the replacement of the $\mathbb{Z}^n$-translations by $b\mathbb{Z}^n$-translations, where $b > 0$. We will see below, more generally, that this case actually follows from the $\mathbb{Z}^n$-translation case, which means that Theorem 3.2 implies each of the results of Chui and Shi.

We turn now to the Gramian analysis of Ron and Shen. The version of the second oversampling theorem offered by Ron and Shen in [7] achieves the result of Theorem

3.2 provided $M \in GL_n(\mathbb{Z})$ is expansive and $P \in GL_n(\mathbb{Z})$ satisfies

$$P^T\mathbb{Z}^n \bigcap (M^T)^j\mathbb{Z}^n = (M^T)^j P^T\mathbb{Z}^n, \tag{5.1}$$

for each $j \geq 0$. We will see that this rather complicated expression is actually equivalent to our notion of admissibility. Let us begin by showing that (5.1) implies that P is admissible in terms of Definition 2.2. The $j = 1$ statement of (5.1) says that

$$P^T\mathbb{Z}^n \bigcap M^T\mathbb{Z}^n = M^T P^T\mathbb{Z}^n,$$

which is equivalent to

$$\mathbb{Z}^n \bigcap (P^T)^{-1}M^T\mathbb{Z}^n = (P^T)^{-1}M^T P^T\mathbb{Z}^n =: \tilde{M}^T\mathbb{Z}^n,$$

from which we conclude that $\tilde{M} := PMP^{-1} \in GL_n(\mathbb{Z})$. Moreover, we have

$$P^T\mathbb{Z}^n \bigcap M^T\mathbb{Z}^n = M^T P^T\mathbb{Z}^n \Leftrightarrow (\tilde{M}^T)^{-1}\mathbb{Z}^n \bigcap (P^T)^{-1}\mathbb{Z}^n = \mathbb{Z}^n.$$

But by Theorem 4.6 we have

$$(\tilde{M}^T)^{-1}\mathbb{Z}^n \bigcap (P^T)^{-1}\mathbb{Z}^n = \mathbb{Z}^n \Leftrightarrow M^{-1}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n = \mathbb{Z}^n,$$

implying that P satisfies our admissibility condition.

For the reverse implication, assume P is admissible according to Definition 2.2. Using the notation of Proposition 2.1 we have $\{M^j\theta_r\}_{r=0}^{p-1}$ is a complete set of coset representatives for $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ for each $j \geq 0$. This is achieved by successively applying the proposition to the collection $\{M^{j-1}\theta_r\}_{r=0}^{p-1}$. Note that since $PMP^{-1} \in GL_n(\mathbb{Z})$, it follows that $PM^jP^{-1} = \tilde{M}^j \in GL_n(\mathbb{Z})$. Thus, Proposition 2.1 implies that

$$M^{-j}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n = \mathbb{Z}^n.$$

Theorem 4.6 now guarantees

$$M^{-j}\mathbb{Z}^n \bigcap P^{-1}\mathbb{Z}^n = \mathbb{Z}^n \Leftrightarrow (\tilde{M}^T)^{-j}\mathbb{Z}^n \bigcap (P^T)^{-1}\mathbb{Z}^n = \mathbb{Z}^n,$$

but we also have

$$(\tilde{M}^T)^{-j}\mathbb{Z}^n \bigcap (P^T)^{-1}\mathbb{Z}^n = \mathbb{Z}^n \Leftrightarrow (M^T)^j\mathbb{Z}^n \bigcap P^T\mathbb{Z}^n = (M^T)^j P^T\mathbb{Z}^n.$$

This string of equivalences shows that our notion of admissibility is equivalent to (5.1).

Finally, we compare our version of the Second Oversampling Theorem to that of Laugesen [6]. Laugesen handles two kinds of dilation matrices $M \in GL_n(\mathbb{Z})$, the *expanding* and *amplifying* dilations. The expanding dilations include, but are not limited to the expansive dilations considered in this work, while the class of amplifying matrices includes such dilations as $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, Laugesen's result applies to a larger class of dilations than Theorem 3.2.

Laugesen also considers translations over the lattice $b\mathbb{Z}^n$, where $b \in GL_n(\mathbb{R})$ commutes with M. This generalization is not essential as we will now describe, using notation as in Theorem 3.2. Let us denote the affine system generated by $\Psi \subset L^2(\mathbb{R}^n)$

relative to translations over $b\mathbb{Z}^n$ by $X_b(\Psi)$ and, similarly, the associated oversampled system $\{D^j T_{bP^{-1}k}\psi_\ell : 1 \leq \ell \leq L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ by $X_b^P(\Psi)$. For $b \in GL_n(\mathbb{R})$, let D_b be the unitary dilation operator mapping $f \in L^2(\mathbb{R}^n)$ to $D_b f := |\det b|^{\frac{1}{2}} f(b\cdot)$. Suppose that $X_b(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with lower bound A and upper bound B. We then have for each $f \in L^2(\mathbb{R}^n)$,

$$A\|f\|^2 \leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\langle D_b^{-1} f, D^j T_{bk}\psi_\ell\rangle\right|^2 \leq B\|f\|^2.$$

Using the fact that b and M commute we obtain for each $f \in L^2(\mathbb{R}^n)$,

$$A\|f\|^2 \leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\langle f, D^j T_k D_b\psi_\ell\rangle\right|^2 \leq B\|f\|^2.$$

This argument shows that $X_b(\Psi)$ is a frame with bounds A, B if and only if $X(D_b\Psi)$ is a frame with the same bounds. If P is admissible, then using Theorem 3.2 we conclude that $X^P(D_b\Psi)$ is a frame with bounds A, B. Thus, for each $f \in L^2(\mathbb{R}^n)$ we have

$$A\|f\|^2 \leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\langle D_b f, D^j T_{P^{-1}k} D_b\psi_\ell\rangle\right|^2 \leq B\|f\|^2,$$

from which it follows that

$$A\|f\|^2 \leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\langle f, D^j T_{bP^{-1}k}\psi_\ell\rangle\right|^2 \leq B\|f\|^2.$$

Hence, if M and P satisfy the hypotheses of Theorem 3.2 and $X_b(\Psi)$ is a frame with bounds A, B then the preceding argument shows that Theorem 3.2 is sufficient to conclude that $X_b^P(\Psi)$ is a frame with the same bounds.

We will now discuss how the notion of admissibility for an oversampling matrix P in [6] is equivalent to ours. Laugesen uses a notion of relative primality for $M, P \in GL_n(\mathbb{Z})$ in which M *is prime relative to* P if $M^T\mathbb{Z}^n \bigcap P^T\mathbb{Z}^n \subseteq M^T P^T \mathbb{Z}^n$. In [6], given a dilation M (which we will assume is expansive), an oversampling matrix $P \in GL_n(\mathbb{Z})$ is admissible if $PMP^{-1} \in GL_n(\mathbb{Z})$ and M is prime relative to P. Observe that

$$M^T\mathbb{Z}^n \bigcap P^T\mathbb{Z}^n \subseteq M^T P^T\mathbb{Z}^n \Leftrightarrow (P^T)^{-1}\mathbb{Z}^n \bigcap (\tilde{M}^T)^{-1}\mathbb{Z}^n \subseteq \mathbb{Z}^n, \tag{5.2}$$

where $\tilde{M} = PMP^{-1}$. On the other hand, since $P^T, \tilde{M}^T \in GL_n(\mathbb{Z})$ we have

$$\mathbb{Z}^n \subseteq (P^T)^{-1}\mathbb{Z}^n \bigcap (\tilde{M}^T)^{-1}\mathbb{Z}^n.$$

It follows that under the hypothesis that $PMP^{-1} \in GL_n(\mathbb{Z})$, M being prime relative to P requires equality rather than containment in (5.2). In light of Theorem 4.6 we see that the notion of admissibility in [6] is equivalent to ours.

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