STABLE FILTERING SCHEMES WITH RATIONAL DILATIONS

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Abstract. The relationship between multiresolution analysis and filtering schemes is a well-known facet of wavelet theory. However, in the case of rational dilation factors, the wavelet literature is somewhat lacking in its treatment of this relationship. This work seeks to establish a means for the construction of stable filtering schemes with rational dilations through the theory of shift-invariant spaces. In particular, principal shift-invariant spaces will be shown to offer frame wavelet decompositions for rational dilations even when the associated scaling function is not refinable. Moreover, it will be shown that such decompositions give rise to stable filtering schemes with finitely supported filters, reminiscent of those studied by Kovačević and Vetterli.

1. Introduction

In the case of an integer dilation $a > 1$, multiresolution analysis (MRA) has played a central role in the development of wavelet frames and associated filtering schemes. Recall that a multiresolution analysis consists of an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ and a scaling function $\varphi \in V_0$ satisfying: (1) $f \in V_j$ if and only if $D_a^{-j} f \in V_0$ for each $j \in \mathbb{Z}$; (2) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$; (3) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$; and (4) $\{T^k\varphi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0. In some instances the last property is loosened, allowing for the shifts of the scaling function to comprise a Riesz basis for V_0 rather than an orthonormal basis, but this generalization is not essential.

Many constructions of MRAs begin with a *refinable* scaling function φ, which means

$$D_a^{-1}\varphi = \sum_{k\in\mathbb{Z}} c_k T^k \varphi,$$

for some sequence of coefficients $\{c_k\}_{k\in\mathbb{Z}}$. Given such a refinable scaling function, one defines V_j to be the closed linear span of $\{D_a^j T^k \varphi : k \in \mathbb{Z}\}$. The refinability of φ implies $D_a^{-1}\varphi \in V_0$ and, moreover, the shift invariance of V_0 then guarantees that $T^{ak} D_a^{-1}\varphi = D_a^{-1} T^k \varphi \in V_0$. In other words, refinability of φ implies $V_j \subseteq V_{j+1}$, $j \in \mathbb{Z}$, provided that $a > 1$ is an integer. This innocent consequence of refinability does not hold, in general, for non-integer dilation factors and presents perhaps the most challenging obstacle in generalizing MRA constructions to include rational dilation factors. This can be seen in the work of Auscher [1] as well as the exposition of MRAs for the dilation $a = \frac{3}{2}$ offered by Daubechies in [6]. Furthermore, in [1], Auscher proved:

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If φ is a scaling function for an MRA with dilation $a = \frac{p}{q}$ ($p, q > 1$ relatively prime integers), then φ has neither compact support nor exponential decay at ∞.

One reason compactly supported scaling functions are desirable is that they give rise to filtering schemes with finitely supported filters. Thus, Auscher's result suggests that one must look beyond the usual MRA structure to achieve compatibility of rational dilation factors with compactly supported functions and finitely supported filters. This idea was hinted at in Daubechies' treatment of the $a = \frac{3}{2}$ case, where it was pointed out that the rational filtering schemes studied by Kovačević and Vetterli in [9] could not arise from standard MRA constructions. More recently, refinability with rational dilations has been studied by Dai, Feng, and Wang with particular attention paid to the regularity of the refinable functions and distributions [5].

The goal of this work is to describe frame decompositions of principal shift-invariant (PSI) spaces using rational dilations that give rise to stable filtering schemes with rational sampling factors. Moreover, these decompositions will not require the associated scaling functions to be refinable and will achieve compatibility between rational dilations and compactly supported functions. Towards this end, the remaining sections are organized as follows. Preliminary notation, definitions, and results will be introduced in Section 2. Section 3 provides a brief account of the necessary theory of shift-invariant spaces, while Section 4 investigates certain frame decompositions of PSI spaces for arbitrary rational dilations. In Section 5, the decompositions of Section 4 are shown to give rise to stable filtering schemes. Finally, Section 6 is dedicated to a pair of relatively simple examples that illustrate the theory of Sections 4 and 5.

2. Preliminaries

Throughout this work, $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $a > 1$, and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ will denote the unitary dilation and translation operators, respectively, given by

$$D_a f(x) = \sqrt{a} f(ax) \quad \text{and} \quad Tf(x) = f(x-1).$$

For $f \in L^1 \cap L^2(\mathbb{R})$ the Fourier transform of f is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx.$$

The discrete Fourier transform on $\ell^2(\mathbb{Z})$ will also be useful and for $\mathbf{f} = \{\mathbf{f}_k\}_{k\in\mathbb{Z}}$ is given by

$$\hat{\mathbf{f}}(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{f}_k e^{-2\pi i k \xi}, \quad \xi \in \mathbb{T}.$$

A boldface letter will be used to distinguish an element of $\ell^2(\mathbb{Z})$ from a function in $L^2(\mathbb{R})$, while the choice of Fourier transform will be made clear from context. Downsampling and upsampling operators on $\ell^2(\mathbb{Z})$ are fundamental to the description of subband filtering schemes. Let $\downarrow_p$ represent *downsampling by* $p \in \mathbb{N}$, given by $(\downarrow_p \mathbf{f})_k = \mathbf{f}_{pk}$. Observe that

$$\widehat{\downarrow_p \mathbf{f}}(\xi) = \frac{1}{p} \sum_{j=0}^{p-1} \hat{\mathbf{f}}(\xi/p + j/p). \tag{1}$$

Similarly, let $\uparrow_p \mathbf{f}$ denote *upsampling by* $p \in \mathbb{N}$, defined by

$$(\uparrow_p \mathbf{f})_k = \begin{cases} \mathbf{f}_{k/p}, & k \in p\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

from which it follows that

$$\widehat{\uparrow_p \mathbf{f}}(\xi) = \hat{\mathbf{f}}(p\xi). \tag{2}$$

Translation in $\ell^2(\mathbb{Z})$ is described by the operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, given by $(T\mathbf{f})_k = \mathbf{f}_{k-1}$, $k \in \mathbb{Z}$. As with the Fourier transform the choice of translation will be clear from context.

A collection $E = \{e_j\}_{j\in\mathbb{J}}$ in a separable Hilbert space $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $0 < A \le B < \infty$ such that for all $f \in \mathbb{H}$,

$$A\|f\|^2 \le \sum_{j\in\mathbb{J}} |\langle f, e_j\rangle|^2 \le B\|f\|^2. \tag{3}$$

The constants A, B are called the lower and upper frame bounds, respectively. If it is possible to choose $A = B$ the frame is said to be *tight*. A collection is a *Bessel system* when only the right-hand inequality holds. When E is a frame for $\mathbb{H}$ it follows that the *frame operator*, $S_E : \mathbb{H} \to \mathbb{H}$ described by $f \mapsto \sum_{j\in\mathbb{J}} \langle f, e_j\rangle e_j$, is bounded and satisfies $A \le \langle S_E f, f\rangle \le B$ for all $f \in \mathbb{H}$ with $\|f\| = 1$. The following theorem, equivalent to the *frame algorithm* presented in [6] and [10], describes an iterative inversion property of frames.

Theorem 1 (Frame Algorithm). *Let S be a self-adjoint operator acting on a Hilbert space $\mathbb{H}$ and satisfying $AI \le S \le BI$, where $0 < A \le B < \infty$. Fix $0 < \gamma \le 2/(A + B)$. Given $x \in \mathbb{H}$, let $x_0 = 0$ and define x_n, $n \in \mathbb{N}$ by*

$$x_n = x_{n-1} + \gamma S(x - x_{n-1}).$$

Then, $x_n \to x$ in $\mathbb{H}$ and $\|x - x_n\| \le \delta^n \|x\|$, $n \in \mathbb{N}$, where $\delta = \max\{|\gamma A - 1|, |\gamma B - 1|\}$.

In order that the frame algorithm provide an efficient means for inverting the frame operator, it is apparent that the frame should be nearly tight. Moreover, in order to achieve the optimal convergence rate a precise knowledge of the frame bounds is required. The conjugate-gradient method, described for frames by Gröchenig in [7], is an improved algorithm for inverting a frame operator that does not require estimates of the frame bounds.

3. Shift-invariant spaces

The theory of shift-invariant spaces has been extensively described in the literature, e.g., [2, 3, 4, 11], yet it will be convenient to establish a minimal amount of machinery in order to naturally develop the results of subsequent sections.

Definition 1. Given $\Phi = \{\phi_1, \dots, \phi_n\} \in L^2(\mathbb{R})$ and $p \in \mathbb{N}$ the $p\mathbb{Z}$ *shift-invariant space* generated by Φ is

$$V(\Phi; p) = \overline{\operatorname{span}} \left\{ T^{pk}\phi_\ell : 1 \le \ell \le n, k \in \mathbb{Z} \right\}.$$

The collection $\{T^{pk}\phi_\ell : 1 \le \ell \le n, k \in \mathbb{Z}\}$ will be denoted by $X(\Phi; p)$ and the functions $\phi_1, \dots, \phi_n$ will be referred to as the *generators* of $V(\Phi; p)$.

When Φ consists of just one generating function, the space $V(\Phi; p)$ is termed a *principal shift-invariant* (PSI) space. An essential tool in the analysis of shift-invariant spaces is the bracket product.

Definition 2. Fix $p \in \mathbb{N}$. Define the *p-bracket product* of $f, g \in L^2(\mathbb{R})$ by

$$[\hat{f}, \hat{g}]_p(\xi) = \sum_{k \in \mathbb{Z}} \hat{f}(\xi + k/p)\overline{\hat{g}(\xi + k/p)}. \tag{4}$$

When $p = 1$ the subscript p will often be omitted.

The p-bracket product of $f, g \in L^2(\mathbb{R})$ belongs to $L^1(\mathbb{T}_p)$, where $\mathbb{T}_p$ is identified with $[0, \frac{1}{p})$. Thus, the Fourier coefficients of $[\hat{f}, \hat{g}]_p$ are well-defined and

$$\left\langle [\hat{f}, \hat{g}]_p, \sqrt{p} e^{2\pi i p k \xi} \right\rangle_{L^2(\mathbb{T}_p)} = \sqrt{p} \langle f, T^{-pk} g \rangle_{L^2(\mathbb{R})}.$$

If g additionally satisfies $[\hat{g}, \hat{g}]_p \in L^\infty(\mathbb{T}_p)$, then $[\hat{f}, \hat{g}]_p \in L^2(\mathbb{T}_p)$ and

$$\| [\hat{f}, \hat{g}]_p \|_2 \leq \| [\hat{g}, \hat{g}]_p \|_\infty \, \| f \|_2.$$

A short calculation reveals the following relationship between the p-bracket product of two functions and the ordinary bracket product of their p-dilates,

$$[\widehat{D_p f}, \widehat{D_p g}](\xi) = \frac{1}{p} [\hat{f}, \hat{g}]_p(\xi/p), \quad \xi \in \mathbb{T}. \tag{5}$$

The identity (5) will be useful in reducing questions about $p\mathbb{Z}$-invariant spaces to equivalent questions about corresponding $\mathbb{Z}$-invariant spaces.

Let $\Phi = \{\phi_1, \ldots, \phi_n\} \subset L^2(\mathbb{R})$. The *p-spectrum of* ϕ_ℓ, $1 \leq \ell \leq n$, is defined by

$$\sigma_{\phi_\ell; p} = \left\{ \xi \in [0, 1/p] : [\hat{\phi}_\ell, \hat{\phi}_\ell]_p(\xi) \neq 0 \right\},$$

while the *p-spectrum of* Φ is defined as

$$\sigma_{\Phi; p} = \bigcup_{n=1}^{N} \sigma_{\phi_n; p}.$$

Finally, the p-Gramian matrix of Φ is defined by

$$G_{\Phi; p}(\xi) = \frac{1}{p} \begin{pmatrix} [\hat{\phi}_1, \hat{\phi}_1]_p(\xi) & \cdots & [\hat{\phi}_N, \hat{\phi}_1]_p(\xi) \\ \vdots & \ddots & \vdots \\ [\hat{\phi}_1, \hat{\phi}_N]_p(\xi) & \cdots & [\hat{\phi}_N, \hat{\phi}_N]_p(\xi) \end{pmatrix}.$$

Notice that the entries of $G_{\Phi;p}(\xi)$ belong to $L^1(\mathbb{T}_p)$. Let $\lambda(\xi)$ be the smallest eigenvalue of $G_{\Phi;p}(\xi)$, $\Lambda(\xi)$ be the largest eigenvalue of $G_{\Phi;p}(\xi)$, and $\lambda^+(\xi)$ be the smallest *nonzero* eigenvalue of $G_{\Phi;p}(\xi)$. The following theorem provides a characterization of frames for finitely generated p-invariant spaces, which, for $p = 1$, is due to Ron and Shen [11]. The generalization to $p \in \mathbb{N}$ is straightforward, but will be presented for completeness.

Theorem 2 ([11])**.** *Fix $p \in \mathbb{N}$ and let $\Phi = \{\phi_1, \ldots, \phi_n\} \in L^2(\mathbb{R})$. Then $X(\Phi; p)$ is a frame for $V(\Phi; p)$ if and only if $1/\lambda^+$ and Λ are essentially bounded on $\sigma_{\Phi;p}$. If either condition holds, then*

$$A = \operatorname*{ess\,inf}_{\xi \in \sigma_{\Phi;p}} \lambda^+(\xi) \quad \text{and} \quad B = \operatorname*{ess\,sup}_{\xi \in \sigma_{\Phi;p}} \Lambda(\xi),$$

respectively, are the lower and upper frame bounds of $X(\Phi; p)$.

Proof. The $p = 1$ case is a subset of Theorem 2.3.6 in [11]. If $p \neq 1$, define $\Psi = \{\psi_1, \ldots, \psi_n\}$ by $\psi_\ell = D_p\phi_\ell$. Notice that $f \in V(\Phi; p)$ if and only if $D_p f \in V(\Psi)$. Moreover, $X(\Psi)$ is a frame for $V(\Psi)$ if and only if $X(\Phi; p)$ is a frame for $V(\Phi; p)$ and in either case the frame bounds are identical. The frame bounds of $V(\Psi)$ are determined by the eigenvalues of $G_\Psi(\xi)$ over σ_Ψ, which, by (5), are seen to be identical to the eigenvalues of $G_{\Phi;p}(\xi)$ over $\sigma_{\Phi;p}$. □

The following lemma describes conditions under which a principal $q\mathbb{Z}$ shift-invariant space may be recovered as a finitely generated $p\mathbb{Z}$ shift-invariant space.

Lemma 3. *Let $\varphi \in L^2(\mathbb{R})$, fix $n, p, q \in \mathbb{N}$ such that $p = nq$, and let $\Phi = \{\phi_1, \ldots \phi_k\} \subseteq V(\varphi; q)$. Suppose that $\sigma_{\varphi;q} = \mathbb{T}_q$, then $V(\Phi; p) = V(\varphi; q)$ if and only if the rank of*

$$\mathcal{M}(\xi) = \begin{pmatrix} m_1(\xi) & \cdots & m_k(\xi) \\ \vdots & \ddots & \vdots \\ m_1(\xi + \frac{n-1}{p}) & \cdots & m_k(\xi + \frac{n-1}{p}) \end{pmatrix}$$

is n for almost every $\xi \in \mathbb{T}_p$. Here, m_j is the $1/q$-periodic function such that $\hat{\phi}_j(\xi) = m_j(\xi)\hat{\varphi}(\xi)$, $1 \leq j \leq k$.

Proof. Because $p = nq$ and each $\phi_k \in V(\varphi; q)$ it is immediate that $V(\Phi; p) \subseteq V(\varphi; q)$. To establish the reverse containment let $f \in V(\varphi; q)$. Then $\hat{f}(\xi) = m_f(\xi)\hat{\varphi}(\xi)$ for some $1/q$-periodic function m_f. If $f \in V(\Phi; p)$ then, similarly, there must exist $1/p$-periodic functions η_j, $1 \leq j \leq k$, such that

$$\hat{f}(\xi) = \sum_{j=1}^{k} \eta_j(\xi)\hat{\phi}_j(\xi) = \sum_{j=1}^{n} \eta_j(\xi) m_j(\xi)\hat{\varphi}(\xi).$$

The period of m_f as well as each m_j is $1/q$, while the period of each η_j is $1/p$. Hence, by considering shifts of the form $\xi + r/p$, $0 \leq r \leq n-1$, one obtains n independent relations,

$$m_f(\xi + r/p)\hat{\varphi}(\xi + r/p) = \sum_{j=1}^{k} \eta_j(\xi) m_j(\xi + r/p)\hat{\varphi}(\xi + r/p), \quad 0 \leq r \leq n-1,$$

each of which must hold for $a.e.$ $\xi \in \mathbb{R}$. Since $\sigma_{\varphi;q} = \mathbb{T}_q$ for each $\xi \in \mathbb{R}$ there exists $\tilde{\xi} \in \xi + \frac{1}{q}\mathbb{Z}$ such that $\hat{\varphi}(\tilde{\xi}) \neq 0$. Therefore, the above system of equations is equivalent to

$$m_f(\xi + r/p) = \sum_{j=1}^{k} \eta_j(\xi) m_j(\xi + r/p), \quad 0 \leq r \leq n-1, \quad \xi \in \mathbb{T}_p.$$

This, in turn, is equivalent to the matrix equation

$$\begin{pmatrix} m_f(\xi) \\ \vdots \\ m_f(\xi + \frac{n-1}{p}) \end{pmatrix} = \begin{pmatrix} m_1(\xi) & \cdots & m_k(\xi) \\ \vdots & \ddots & \vdots \\ m_1(\xi + \frac{n-1}{p}) & \cdots & m_k(\xi + \frac{n-1}{p}) \end{pmatrix} \begin{pmatrix} \eta_1(\xi) \\ \vdots \\ \eta_k(\xi) \end{pmatrix}. \tag{6}$$

The sufficiency of the claimed rank condition follows immediately, since (6) will admit a solution for any $f \in V(\varphi; q)$ provided that $\mathcal{M}(\xi)$ has rank n almost everywhere on $\mathbb{T}_p$.

To establish the necessity of the condition, consider the n distinct elements of $V(\varphi;q)$, f_j, defined by

$$\hat{f}_j(\xi) = \chi_{[\frac{j-1}{p},\frac{j}{p})}(\xi)\,\hat{\varphi}(\xi), \quad 1 \le j \le n.$$

The n vector-valued functions corresponding to the left-hand side of (6) for these n functions are linearly independent on $\mathbb{T}_p$ and, therefore, if $V(\Phi;p) = V(\varphi;q)$ it follows from (6) that $\mathcal{M}(\xi)$ must have rank n almost everywhere on $\mathbb{T}_p$. □

4. Wavelet decomposition of PSI spaces

Fix $a = p/q > 1$, where $p, q \in \mathbb{N}$ are relatively prime and let $\varphi \in L^2(\mathbb{R})$ such that

$$0 < A \le [\hat{\varphi},\hat{\varphi}](\xi) \le B, \quad a.e.\ \xi \in \mathbb{T}, \tag{7}$$

i.e., $X(\varphi)$ is a Riesz basis for $V(\varphi)$. Let S denote the frame operator of $X(\varphi)$, given by

$$Sf = \sum_{k\in\mathbb{Z}} \langle f, T^k\varphi\rangle T^k\varphi$$

which under the Fourier transform is equivalent to $\widehat{Sf} = [\hat{f},\hat{\varphi}]\,\hat{\varphi}$.

Following the refinable case, one would attempt to decompose $V(\varphi)$ in terms of the collection $\{D_a^{-1}T^k\varphi : k \in \mathbb{Z}\}$. Unfortunately, there is no guarantee that any of the functions in this collection even belong to $V(\varphi)$, but this obstacle can be overcome by mapping each function into $V(\varphi)$ via the frame operator S. One could also use the orthogonal projection onto $V(\varphi)$, but this operation could destroy desirable properties of the generating functions, e.g., compact support.

Notice that $D_a^{-1}T^{qk} = T^{pk}D_a^{-1}$, $k \in \mathbb{Z}$, and

$$\{D_a^{-1}T^k\varphi : k \in \mathbb{Z}\} = \bigcup_{\ell=0}^{q-1}\{D_a^{-1}T^{\ell+qk}\varphi : k \in \mathbb{Z}\} = \bigcup_{\ell=0}^{q-1} X(D_a^{-1}T^\ell\varphi;p).$$

Because S commutes with T it follows that the image of $X(D_a^{-1}T^\ell\varphi;p)$ in $V(\varphi)$ under S is $X(SD_a^{-1}T^\ell\varphi;p)$. In order to simplify notation, let ϕ_ℓ be given by

$$\phi_\ell = D_a^{-1}T^\ell\varphi, \quad 0 \le \ell \le q-1. \tag{8}$$

Further, let $\Phi_0 = \{\phi_0,\dots,\phi_{q-1}\}$ and consider $V(S\Phi_0;p) \subseteq V(\varphi)$. The space $V(\varphi)$ can be naturally interpreted as the $p\mathbb{Z}$ shift-invariant space generated by $\{T^r\varphi : 0 \le r \le p-1\}$. This fact provides heuristic evidence that $V(S\Phi_0;p)$ is $p-q$ generators short of what will be required to span $V(\varphi)$, the idea being that these $p-q$ generators will play the role of wavelets. Let $\Phi_1 = \{D_a^{-1}\psi_1,\dots,D_a^{-1}\psi_{p-q}\}$ be a given collection of potential wavelet generators. The goal is to characterize when these generators fill out Φ_0 to provide a set of spanning generators for $V(\varphi)$. Towards this end, adopt the notational convention that

$$\phi_\ell = D_a^{-1}\psi_{\ell-q+1}, \quad q \le \ell \le p-1, \tag{9}$$

and define $\Phi = \{\phi_0,\dots,\phi_{p-1}\}$, i.e., $\Phi = \Phi_0 \cup \Phi_1$. The following theorem describes conditions under which $X(S\Phi;p)$ is a frame for $V(\varphi)$.

Theorem 4. *Let $\varphi \in L^2(\mathbb{R})$ such that (7) holds and let $\Phi = \{\phi_0, \dots, \phi_{p-1}\}$ be given by (8) for $0 \le \ell \le q-1$ and (9) for $q \le \ell \le p-1$. Moreover, for $0 \le \ell \le p-1$, let m_ℓ be the 1-periodic function $[\hat{\phi}_\ell, \hat{\varphi}]$ so that $\widehat{S\phi_\ell} = m_\ell \hat{\varphi}$. Define $\mathcal{M}(\xi)$ by*

$$\mathcal{M}(\xi) = \frac{1}{\sqrt{p}} \begin{pmatrix} m_0(\xi) & \cdots & m_{p-1}(\xi) \\ \vdots & \ddots & \vdots \\ m_0(\xi + \frac{p-1}{p}) & \cdots & m_{p-1}(\xi + \frac{p-1}{p}) \end{pmatrix} \tag{10}$$

and let $\lambda_\mathcal{M}$ and $\Lambda_\mathcal{M}$ be the smallest and largest eigenvalue functions of $\mathcal{M}^*\mathcal{M}$ over $\mathbb{T}_p$. Then $X(S\Phi; p)$ is a frame for $V(\varphi)$ if and only if $1/\lambda_\mathcal{M}$ and $\Lambda_\mathcal{M}$ are essentially bounded on $\mathbb{T}_p$. If either condition holds, let

$$\lambda_A = \operatorname*{ess\,inf}_{\xi \in \mathbb{T}_p} \lambda_\mathcal{M}(\xi) \quad \textit{and} \quad \lambda_B = \operatorname*{ess\,sup}_{\xi \in \mathbb{T}} \Lambda_\mathcal{M}(\xi),$$

then $X(S\Phi; p)$ is a frame for $V(\varphi)$ with lower and upper bounds $\lambda_A A$ and $\lambda_B B$, respectively.

Proof. Lemma 3 implies that $V(S\Phi; p) = V(\varphi)$ if and only if $\mathcal{M}(\xi)$ is nonsingular for *a.e.* $\xi \in \mathbb{T}_p$, while Theorem 2 describes when $X(S\Phi; p)$ is a frame for $V(S\Phi; p)$. Of particular interest is the Gramian matrix $G_{S\Phi;p}$ over $\mathbb{T}_p$. Notice that

$$\begin{aligned} [\widehat{S\phi_\ell}, \widehat{S\phi_k}]_p(\xi) &= [m_\ell \hat{\varphi}, m_k \hat{\varphi}]_p(\xi) \\ &= \sum_{k \in \mathbb{Z}} m_\ell(\xi + k/p)\overline{m_k(\xi + k/p)} |\hat{\varphi}(\xi + k/p)|^2 \\ &= \sum_{r=0}^{p-1} m_\ell(\xi + r/p)\overline{m_k(\xi + r/p)} [\hat{\varphi}, \hat{\varphi}](\xi + r/p). \end{aligned}$$

Now, let $G(\xi)$, $\xi \in \mathbb{T}_p$, be the $p \times p$ diagonal matrix such that $G_{jj}(\xi) = [\hat{\varphi}, \hat{\varphi}](\xi + \frac{j}{p})$, $0 \le j \le p-1$, and observe that

$$G_{S\Phi;p}(\xi) = \mathcal{M}(\xi)^* G(\xi) \mathcal{M}(\xi). \tag{11}$$

Given $v \in \mathbb{C}^p$, the above relationship implies

$$\langle G_{S\Phi;p}(\xi)v, v\rangle = \langle G(\xi)\mathcal{M}(\xi)v, \mathcal{M}(\xi)v\rangle,$$

so it follows that

$$\lambda_\mathcal{M}(\xi) A \|v\|^2 \le A \|\mathcal{M}(\xi)v\|^2 \le \langle G_{S\Phi;p}(\xi)v, v\rangle \le B\|\mathcal{M}(\xi)v\|^2 \le \Lambda_M(\xi) B \|v\|^2. \tag{12}$$

Thus, if $1/\lambda_\mathcal{M}$ and $\Lambda_\mathcal{M}$ are essentially bounded on $\mathbb{T}_p$, then $\mathcal{M}(\xi)$ is necessarily nonsingular on $\mathbb{T}_p$, implying that $V(S\Phi; p) = V(\varphi)$, and the claimed frame bounds follow from (12).

On the other hand, if $X(S\Phi; p)$ is a frame for $V(\varphi)$ then, by Lemma 3, $\mathcal{M}(\xi)$ must be nonsingular on $\mathbb{T}_p$ so (11) can be reformulated as

$$G(\xi) = [\mathcal{M}(\xi)^*]^{-1} G_{S\Phi;p}(\xi) [\mathcal{M}(\xi)]^{-1}.$$

Similarly to (12), it follows that

$$\tilde{A} \|\mathcal{M}(\xi)^{-1}v\|^2 \le \langle G(\xi)v, v\rangle \le \tilde{B} \|\mathcal{M}(\xi)^{-1}v\|^2, \tag{13}$$

where $\tilde{A}$ and $\tilde{B}$ are the frame bounds of $X(S\Phi;p)$. Suppose that $\Lambda_{\mathcal{M}}$ is not essentially bounded, then there exists a set of positive measure $\sigma \subseteq \mathbb{T}_p$ and $v : \sigma \to \mathbb{C}^p$ such that for $\xi \in \sigma$,

$$\|\mathcal{M}(\xi)^{-1}v(\xi)\|^2 < \frac{A}{\tilde{B}}\|v(\xi)\|^2,$$

which, by (13) implies that

$$\langle G(\xi)v(\xi), v(\xi)\rangle \leq \tilde{B}\|\mathcal{M}(\xi)^{-1}v(\xi)\|^2 < A\|v(\xi)\|^2.$$

Notice, however, that the hypothesis on $[\hat{\varphi}, \hat{\varphi}]$, given by (7), implies for *a.e.* $\xi \in \sigma$ that

$$A\|v(\xi)\|^2 \leq \langle G(\xi)v(\xi), v(\xi)\rangle \leq B\|v(\xi)\|^2, \tag{14}$$

which leads to a contradiction. Therefore, $\Lambda_{\mathcal{M}}$ must be essentially bounded. A similar argument can be used to prove the essential boundedness of $\lambda_{\mathcal{M}}^{-1}$ and, given the boundedness of the eigenvalue functions, the claimed formulas for the frame bounds of $X(S\Phi;p)$ follow. □

5. Stable filtering schemes for PSI spaces

Let Φ, Φ_0, and Φ_1 be defined as in Section 4 and assume that the hypotheses of Theorem 4 hold. This means that $X(S\Phi;p)$ is a frame for $V(\varphi)$. Given $f \in V(\varphi)$ there is a unique sequence $\mathbf{f} = \{\mathbf{f}_k\}_{k\in\mathbb{Z}}$ such that

$$f = \sum_{k\in\mathbb{Z}} \mathbf{f}_k T^k \varphi.$$

Consider the frame operator of $X(\Phi_0;p)$ applied to f,

$$g_0 := S_{X(\Phi_0;p)}f = \sum_{\ell=0}^{q-1}\sum_{k\in\mathbb{Z}} \langle f, T^{pk}\phi_\ell\rangle T^{pk}\phi_\ell = \sum_{\ell=0}^{q-1}\sum_{k\in\mathbb{Z}} \left[\sum_{m\in\mathbb{Z}} \mathbf{f}_m \langle \varphi, T^{pk-m}\phi_\ell\rangle\right] T^{pk}\phi_\ell.$$

Recall that, for $0 \leq \ell \leq q-1$, $\phi_\ell = D^{-1}T^\ell\varphi$, so

$$g_0 = \sum_{k\in\mathbb{Z}}\sum_{\ell=0}^{q-1} \left[\sum_{m\in\mathbb{Z}} \mathbf{f}_m \langle \varphi, T^{pk-m}\phi_\ell\rangle\right] D^{-1}T^{kq+\ell}\varphi.$$

Let $\{\mathbf{g}_{0,k}\}_{k\in\mathbb{Z}}$ be defined so that $g_0 = \sum_{k\in\mathbb{Z}} \mathbf{g}_{0,k} D^{-1}T^k\varphi$, then the subsequence $\{\mathbf{g}_{0,qk+\ell}\}_{k\in\mathbb{Z}}$ is obtained through convolution,

$$\mathbf{g}_{0,qk+\ell} = \sum_{m\in\mathbb{Z}} \mathbf{f}_m \langle \varphi, T^{pk-m}\phi_\ell\rangle = \left(\{\mathbf{f}_m\} * \{\langle \varphi, T^m\phi_\ell\rangle\}\right)(pk). \tag{15}$$

This can be rephrased using operator notation for $\ell^2(\mathbb{Z})$ as

$$\{\mathbf{g}_{0,k}\}_{k\in\mathbb{Z}} = \sum_{\ell=0}^{q-1} T^\ell \uparrow_q\downarrow_p \left(\{\mathbf{f}_m\} * \{\langle \varphi, T^m\phi_\ell\rangle\}\right)$$

Similarly, applying the frame operator of $X(D^{-1}\psi_\ell;p)$ to f leads to functions g_ℓ, $1 \leq \ell \leq p-q$,

$$g_\ell := S_{\psi_\ell;p}f = \sum_{k\in\mathbb{Z}} \left[\sum_{m\in\mathbb{Z}} \mathbf{f}_m \langle \varphi, T^{pk-m}D^{-1}\psi_\ell\rangle\right] D^{-1}T^{qk}\psi_\ell$$

and letting $g_\ell = \sum_{k\in\mathbb{Z}} \mathbf{g}_{\ell,k} T^{qk} D^{-1}\psi$ it follows that

$$\mathbf{g}_{\ell,k} = \left(\{\mathbf{f}_m\} * \{\langle\varphi, T^m D^{-1}\psi\rangle\}\right)(pk), \quad 1 \le \ell \le p-q. \tag{16}$$

This can be restated as

$$\{\mathbf{g}_{\ell,k}\}_{k\in\mathbb{Z}} = \downarrow_p (\{\mathbf{f}_m\} * \{\langle\varphi, T^m \phi_{\ell+q}\rangle\}), \quad 1 \le \ell \le p-q,$$

under the notational equivalence, $D^{-1}\psi_\ell = \phi_{\ell+q-1}$, $1 \le \ell \le p-q$. Thus, the action of the frame operators for $X(\Phi_0; p)$ and $X(D^{-1}\psi_\ell; p)$, $1 \le \ell \le p-q$, on $f \in V(\varphi)$ is described by a certain subband filtering scheme. There are a number of filters behind this filtering scheme. The low-pass portion of the filtering scheme, described by (15), uses the q filters $\overline{m_0}, \ldots, \overline{m_{q-1}}$ which are given by

$$\overline{m_\ell(\xi)} = \sum_{k\in\mathbb{Z}} \overline{\langle\varphi, T^k\phi_\ell, \varphi\rangle e^{-2\pi i k\xi}} = [\hat{\phi}_\ell, \hat{\varphi}](\xi).$$

Similarly, the high-pass portion of the filtering scheme, described by (16), uses the $p-q$ filters $\overline{m}_q, \ldots, \overline{m}_{p-1}$ where m_ℓ is given by $m_\ell = [\hat{\phi}_\ell, \hat{\varphi}]$, $q \le \ell \le p-1$. Thus the filters m_ℓ used here correspond exactly with those in Theorem 4. Figure 1 depicts the block-diagram corresponding to this filtering scheme, in which the rectangles represent convolution with the filter whose Fourier transform is given.

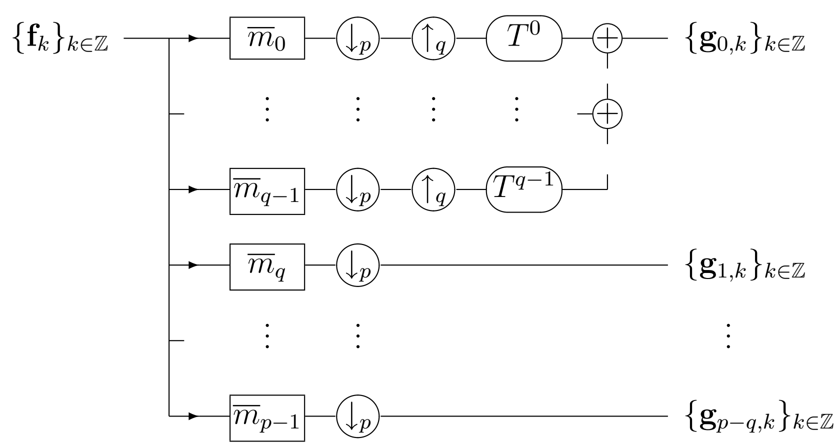

FIGURE 1. Analysis filtering scheme for $V(\varphi)$.

Define the *filterbank analysis operator* by

$$F : \ell^2(\mathbb{Z}) \to \bigoplus_{\ell=0}^{p-q} \ell^2(\mathbb{Z})$$
$$\{\mathbf{f}_k\}_{k\in\mathbb{Z}} \mapsto \oplus_{\ell=0}^{p-q} \{\mathbf{g}_{\ell,k}\}_{k\in\mathbb{Z}},$$

which follows the notation used above as well as in Figure 1. The filtering scheme of Figure 1 will be referred to as *stable* if there exist constants A, B such that for any $\{\mathbf{f}_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$,

$$0 < A \left\|\{\mathbf{f}_k\}_k\right\|^2 \le \left\|F\{\mathbf{f}_k\}_k\right\|^2 \le B \left\|\{\mathbf{f}_k\}_k\right\|^2 \le \infty. \tag{17}$$

It should be noted that the above inequality can be reformulated as a frame identity for a specific system of translates in $\ell^2(\mathbb{Z})$; however, it is more natural in this context to describe the stability of the filtering scheme through (17).

Theorem 5. *Let Φ, $\mathcal{M}$, $\lambda_{\mathcal{M}}$, and $\Lambda_{\mathcal{M}}$ be as in Theorem 4. Then the induced filtering scheme of Figure 1 is stable if and only if $1/\lambda_{\mathcal{M}}$ and $\Lambda_{\mathcal{M}}$ are essentially bounded on $\mathbb{T}_p$. If either condition holds, let*

$$\lambda_A = \operatorname*{ess\,inf}_{\xi\in\mathbb{T}_p} \lambda_{\mathcal{M}}(\xi) \quad \textit{and} \quad \lambda_B = \operatorname*{ess\,sup}_{\xi\in\mathbb{T}_p} \Lambda_{\mathcal{M}}(\xi),$$

then the filterbank analysis operator satisfies

$$\lambda_A \left\|\{\mathbf{f}_k\}_k\right\|^2 \le \left\|F\{\mathbf{f}_k\}_k\right\|^2 \le \lambda_B \left\|\{\mathbf{f}_k\}_k\right\|^2, \quad \forall\ \{\mathbf{f}_k\}_k \in \ell^2(\mathbb{Z}).$$

Proof. Let $\mathbf{f} = \{\mathbf{f}_k\}_k$ and $\mathbf{g}_\ell = \{\mathbf{g}_{\ell,k}\}_k$, where the sequences correspond to those in Figure 1. For any $\mathbf{f} \in \ell^2(\mathbb{Z})$, one has

$$\left\|F\{\mathbf{f}_k\}_k\right\|^2 = \sum_{\ell=0}^{p-q} \|\mathbf{g}_\ell\|^2 = \sum_{\ell=0}^{p-q} \|\hat{\mathbf{g}}_\ell\|^2.$$

For $1 \le \ell \le p-q$, the norm of $\mathbf{g}_\ell$ is given by

$$\|\hat{\mathbf{g}}_\ell\|^2 = \frac{1}{p^2}\int_{\mathbb{T}_p} \left|\sum_{r=0}^{p-1} m_{\ell+q-1}(\xi/p + r/p)\hat{\mathbf{f}}(\xi/p + r/p)\right|^2 \, d\xi,$$

by (1). Similarly, the norm of $\mathbf{g}_0$ is given by

$$\|\hat{\mathbf{g}}_0\|^2 = \sum_{\ell=0}^{q-1} \frac{1}{p^2}\int_{\mathbb{T}_p} \left|\sum_{r=0}^{p-1} m_{\ell}(\xi/p + r/p)\hat{\mathbf{f}}(\xi/p + r/p)\right|^2 \, d\xi.$$

Let $v_{\mathbf{f}}(\xi) = \begin{pmatrix}\hat{\mathbf{f}}(\xi/p + 0/p) & \cdots & \hat{\mathbf{f}}(\xi/p + (p-1)/p)\end{pmatrix}^T$, then

$$\begin{aligned}\left\|F\{\mathbf{f}_k\}_k\right\|^2 &= \sum_{\ell=0}^{p-1} \frac{1}{p^2}\int_{\mathbb{T}_p} \left|\sum_{r=0}^{p-1} m_{\ell}(\xi/p + r/p)\hat{\mathbf{f}}(\xi/p + r/p)\right| \, d\xi \\ &= \frac{1}{p}\int_{\mathbb{T}_p} \left\|\mathcal{M}(\xi/p)v_{\mathbf{f}}(\xi)\right\|^2 \, d\xi \\ &= \frac{1}{p}\int_{\mathbb{T}_p} \langle \mathcal{M}^*(\xi/p)\mathcal{M}(\xi/p)v_{\mathbf{f}}(\xi), v_{\mathbf{f}}(\xi)\rangle \, d\xi.\end{aligned}$$

It follows that

$$\frac{\lambda_{\mathcal{M}}}{p}\int_{\mathbb{T}_p} \|v_{\mathbf{f}}(\xi)\|^2 \, d\xi \le \left\|F\{\mathbf{f}_k\}_k\right\|^2 \le \frac{\Lambda_{\mathcal{M}}}{p}\int_{\mathbb{T}_p} \|v_{\mathbf{f}}(\xi)\|^2 \, d\xi,$$

which, under the change of variables $\xi \mapsto p\xi$, is equivalent to

$$\lambda_{\mathcal{M}}\int_{\mathbb{T}} |\hat{\mathbf{f}}(\xi)|^2 \, d\xi \le \left\|F\{\mathbf{f}_k\}_k\right\|^2 \le \Lambda_{\mathcal{M}}\int_{\mathbb{T}} |\hat{\mathbf{f}}(\xi)|^2 \, d\xi,$$

finishing the proof. $\square$

Reconstruction after analysis with a stable filterbank is equivalent to reconstruction from frame coefficients. Consider the synthesis filtering scheme of Figure 2, which describes the action of the *filterbank synthesis operator*,

$$F^* : \bigoplus_{\ell=0}^{p-q} \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$$
$$\oplus_{\ell=0}^{p-q}\{\mathbf{g}_{\ell,k}\}_{k\in\mathbb{Z}} \mapsto \{\tilde{\mathbf{f}}_k\}_{k\in\mathbb{Z}}.$$

The image of $\oplus_{\ell=0}^{p-q}\{\mathbf{g}_{\ell,k}\}_{k\in\mathbb{Z}}$ under F^* is given by

$$\{\tilde{\mathbf{f}}_k\}_k = \sum_{\ell=0}^{q-1} \left(\uparrow_p\downarrow_q T^{-\ell}\{\mathbf{g}_{0,k}\}_k\right) * \{\langle \phi_\ell, T^{-k}\varphi\rangle\}_k + \sum_{\ell=q}^{p-1} (\uparrow_p \{\mathbf{g}_{\ell-q+1,k}\}_k) * \{\langle \phi_\ell, T^{-k}\varphi\rangle\}_k$$

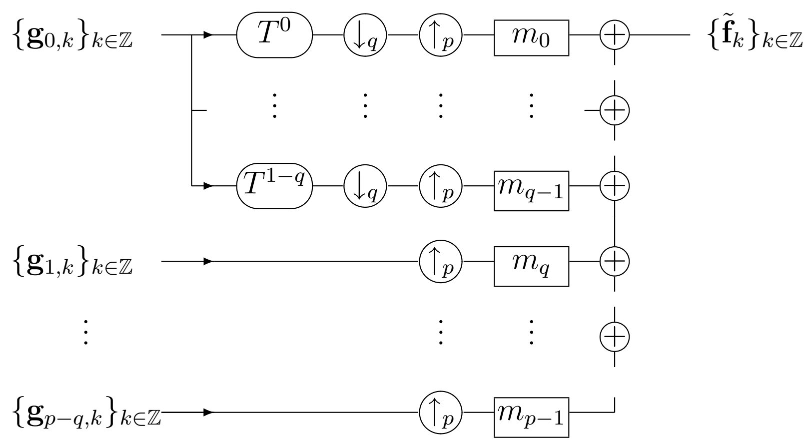

FIGURE 2. Synthesis filtering scheme for $V(\varphi)$.

Notice that the filters in the synthesis scheme are conjugate to those of the analysis stage. A routine calculation shows that, when bounded, the operators F and F^* are indeed adjoint. Assume that F and F^* are associated with a stable filtering scheme and consider the composition $S = F^*F : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$. In this case S satisfies

$$A\|\mathbf{f}\|^2 \le \langle S\mathbf{f}, \mathbf{f}\rangle \le B\|\mathbf{f}\|^2, \quad \forall\, \mathbf{f} \in \ell^2(\mathbb{Z}),$$

and where A and B are the bounds for F as in (17). Therefore, Theorem 1 provides an iterative inversion of F via the frame algorithm.

6. Examples

The notational conventions of Section 4 shall be observed throughout this section.

6.1. **Haar example.** Let $a = \frac{3}{2}$ and let $\varphi = \chi_{[0,1)}$, the scaling function of the familiar Haar wavelet. Observe that

$$D_a^{-1}\varphi = \sqrt{\frac{2}{3}}\chi_{[0,\frac{3}{2})} \quad \text{and} \quad D_a^{-1}T\varphi = \sqrt{\frac{2}{3}}\chi_{[\frac{3}{2},3)}.$$

Let $\phi_0 = D_a^{-1}\varphi$ and $\phi_1 = D_a^{-1}T\varphi$, which leads to the filters

$$m_0(\xi) = [\hat{\phi}_0, \hat{\varphi}](\xi) = \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{6}}e^{-2\pi i\xi},$$

$$m_1(\xi) = [\hat{\phi}_1, \hat{\varphi}](\xi) = \sqrt{\frac{1}{6}}e^{-2\pi i\xi} + \sqrt{\frac{2}{3}}e^{-2\pi i2\xi}.$$

Choose $\hat{\phi}_2(\xi) = \widehat{D^{-1}\psi}(\xi) = m_2(\xi)\hat{\varphi}(\xi)$, where

$$m_2(\xi) = -\sqrt{\frac{1}{6}} + \sqrt{\frac{2}{3}}e^{-2\pi i\xi} - \sqrt{\frac{1}{6}}e^{-2\pi i2\xi}.$$

It is easy to see that $m_2(\xi) = [\hat{\phi}_2, \hat{\varphi}]$. Moreover, ϕ_2 satisfies $\hat{\phi}_2(0) = 0$ and $V(\phi_2; 3)$ is perpendicular to $V(\phi_0, \phi_1; 3)$. Following Theorem 4, let $\Phi = \{\phi_0, \phi_1, \phi_2\}$ and consider the collection $X(S\Phi; 3)$ as well as the space $V(S\Phi; 3)$ it generates in $V(\varphi)$, where S is the frame operator of $X(\varphi)$. (Notice that S is the orthogonal projection onto $V(\varphi)$ in this case.) Because of the relative simplicity of this example it can be verified directly that $X(S\Phi; 3)$ is a frame for $V(\varphi)$. Indeed, let $f = \sum_{k\in\mathbb{Z}} \mathbf{f}_k T^k\varphi$ be an arbitrary element of $V(\varphi)$, then a simple calculation shows that

$$\sum_{\ell=0}^{2}\sum_{k\in\mathbb{Z}} |\langle f, T^{3k}S\phi_\ell\rangle|^2 = \sum_{k\in 3\mathbb{Z}} \frac{5}{6}\mathbf{f}_k^2 + \mathbf{f}_{k+1}^2 + \frac{5}{6}\mathbf{f}_{k+2}^2 + \frac{1}{3}\mathbf{f}_k\mathbf{f}_{k+2}.$$

It follows that

$$\frac{2}{3}\|f\|^2 \le \sum_{\ell=0}^{2}\sum_{k\in\mathbb{Z}} |\langle f, T^{3k}S\phi_\ell\rangle|^2 \le \|f\|^2,$$

i.e., $X(S\Phi; 3)$ is a frame for $V(\varphi)$ with lower bound $\frac{2}{3}$ and upper bound 1. It follows from Theorems 4 and 5 that m_0, m_1, and m_2 induce a stable filtering scheme for the dilation $a = \frac{3}{2}$. This filtering scheme will satisfy (17) with $A = \frac{2}{3}$ and $B = 1$.

6.2. **Numerical example.** Again let $a = \frac{3}{2}$ and for this example, let φ be given by

$$\varphi(x) = \begin{cases} \frac{1}{2}\left(x + \frac{3}{2}\right) + \frac{11}{10\pi}\cos(\pi x), & -\frac{3}{2} \le x < -\frac{1}{2}, \\ \frac{1}{2} + \frac{11}{5\pi}\cos(\pi x), & -\frac{1}{2} \le x < \frac{1}{2}, \\ \frac{1}{2}\left(\frac{3}{2} - x\right) + \frac{11}{10\pi}\cos(\pi x), & \frac{1}{2} \le x \le \frac{3}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

which is depicted in Figure 3 (a). The bracket product $[\hat{\varphi}, \hat{\varphi}]$ is shown in Figure 3 (b) and indicates that $X(\varphi)$ is a near-tight frame for $V(\varphi)$.

As in the first example, set $\phi_0 = D_a^{-1}\varphi$ and $\phi_1 = D_a^{-1}T\varphi$. Define $\phi_2 = D^{-1}\psi$ by

$$\phi_2(x) = \begin{cases} -\sqrt{\frac{1}{3}}, & -\frac{3}{4} \le x < \frac{1}{4}, \\ 2\sqrt{\frac{1}{3}}, & \frac{1}{4} \le x < \frac{5}{4}, \\ -\sqrt{\frac{1}{3}}, & \frac{5}{4} \le x \le 9/4, \\ 0, & \text{otherwise}. \end{cases}$$

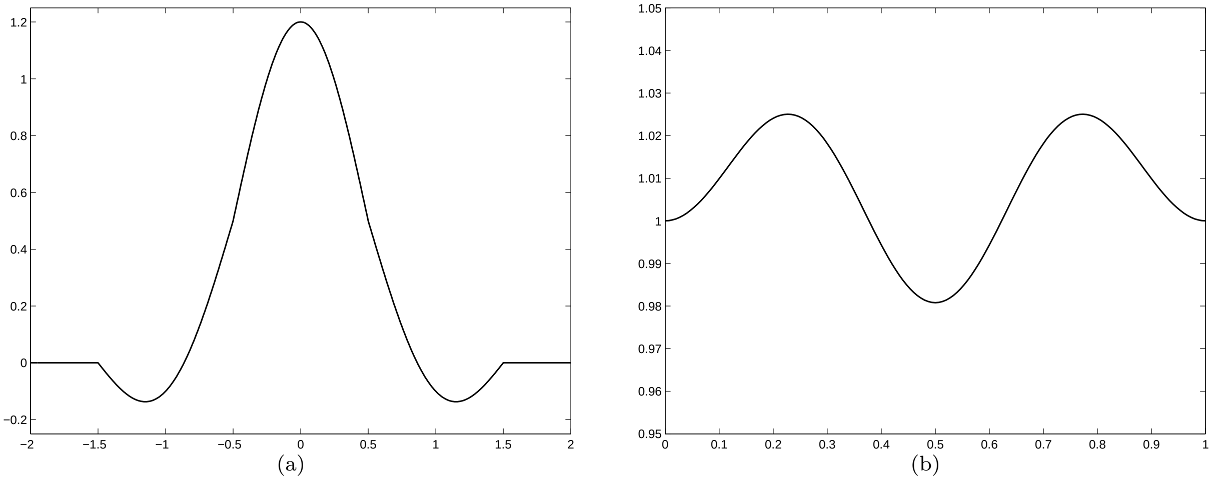

FIGURE 3. (a) The scaling function, φ; (b) The bracket product, $[\hat{\varphi}, \hat{\varphi}]$.

TABLE 1. The approximate Fourier coefficients of m_k, $0 \leq k \leq 2$.

n	m_0	m_1	m_2
-5		0.0004694751361	
-4		-0.01422253308	
-3	0.005361711649	-0.06569968109	0.02865833229
-2	-0.08156253964	0.6918251749	-0.5378201552
-1	0.2199870039	0.6918251749	0.8543343266
0	0.9371725199	-0.06569968109	-0.2000156407
1	0.2199870039	-0.01422253308	-0.1549827389
2	-0.08156253964	0.0004694751361	0.009825875967
3	0.005361711649		

Let $\mathcal{M}(\xi)$ be defined as in (10), where $m_k(\xi) = [\hat{\phi}_k, \hat{\varphi}]$, $0 \leq k \leq 2$. In light of Theorem 4, one need only examine the eigenvalues of $\mathcal{M}(\xi)^*\mathcal{M}(\xi)$ over $\mathbb{T}_3$ to determine whether or not $X(S\Phi; 3)$ is a frame for $V(\varphi)$. The approximate Fourier coefficients of the filters m_k are given in Table 1, while Figure 4 depicts the function ϕ_2 as well as the eigenvalue functions $\lambda_{\mathcal{M}}$ and $\Lambda_{\mathcal{M}}$ over $\mathbb{T}_3$. This information leads to approximate values $\lambda_A \approx 0.59$ and $\lambda_B \approx 1.34$, suggesting that $X(S\Phi; 3)$ is a frame for $V(\varphi)$. Note that the frame bounds for $X(S\Phi; 3)$ will differ slightly from λ_A and λ_B, due to the fact that $X(\varphi)$ is not an orthonormal basis for $V(\varphi)$. This example illustrates how Theorem 4 can be used to generate examples of stable filtering schemes with rational dilations which do not depend on the refinability of the scaling function.

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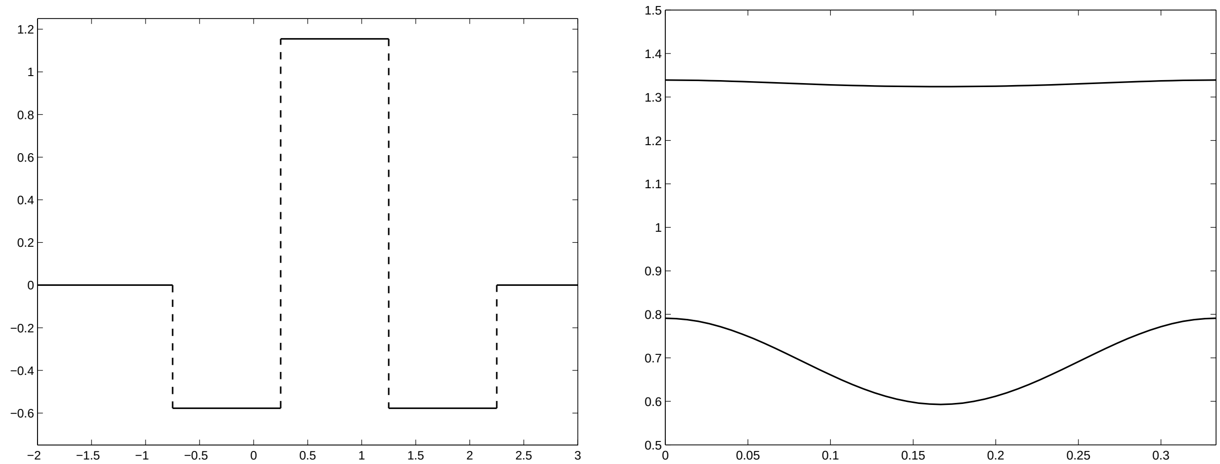

Figure 4. (a) The function, ϕ_2; (b) The eigenvalue functions of $\mathcal{M}(\xi)^*\mathcal{M}(\xi)$, $\lambda_{\mathcal{M}}$, $\Lambda_{\mathcal{M}}$.

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