THE NONHOLONOMY OF THE ROLLING SPHERE

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1. Introduction.

An old problem in the field of holonomy asks: *Given a pair of orientations for a sphere resting on a plane, is there a closed path along which one can roll the sphere (without slipping or twisting), starting with the first orientation, and return to the origin with the sphere in the second orientation?* (See Figure 1.) The answer is yes, and the goal of this article is to provide an elementary proof of this fact.

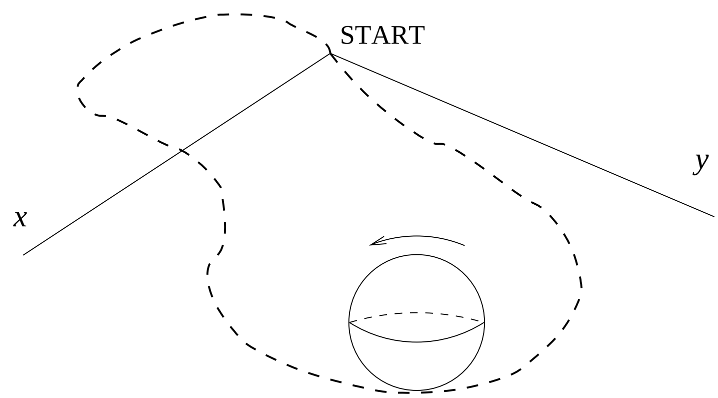

Figure 1. The rolling sphere.

In the late nineteenth century Heinrich Hertz introduced the term holonomy to describe the behavior of mechanical systems subject to velocity constraints. To be more specific, Hertz was interested in understanding whether the system could be moved between two arbitrary states without violating the velocity constraint. A mechanical system is said to be *nonholonomic* with respect to a given constrained motion if the system can move between any two states without violating the constraint. Otherwise, the system is said to be *holonomic* with respect to the constraint [2]. Holonomy remains an active area of research, and the interested reader is referred to [1] and [2] for detailed accounts of the history and present status of the subject.

In the case of the rolling sphere, the mechanical system is the sphere resting on the plane, and the velocity constraint comes from the requirement that the sphere may be moved only by rolling through a closed path with no slipping or twisting. It will be shown that we can move the sphere between any two states without violating the velocity constraint and, therefore, that this system is nonholonomic.

As a simpler example of this concept, consider the system that consists of a single particle in $\mathbb{R}^n$ whose motion is constrained so that its velocity vector must remain perpendicular to its position vector [2]. The coordinates of the particle describe its state and, thus, the collection of all possible states is $\mathbb{R}^n$. Letting $\vec{r}(t)$ denote the position vector of the particle, the velocity constraint states that $\vec{r}(t) \cdot \vec{r}'(t) = 0$, which implies that $d\|\vec{r}(t)\|^2/dt = 0$ or, equivalently, $\|\vec{r}(t)\|^2 = C$ for some nonnegative constant C. Therefore the system is holonomic, because the velocity constraint limits the motion of the particle to motion between points at the same distance from the origin.

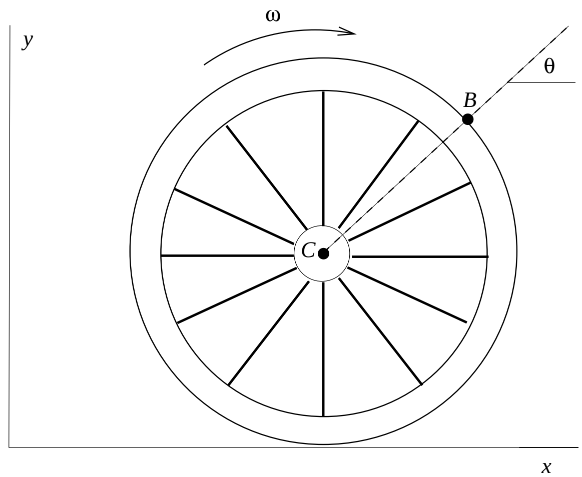

FIGURE 2. A rolling wheel.

The organization of the remaining sections is as follows. Section 2 introduces the necessary background material from dynamics and presents a derivation of the equations of motion for the rolling sphere. Section 3 develops a proof of the nonholonomy of the rolling sphere involving carefully chosen rectangular paths. Finally, section 4 offers an investigation of the rolling sphere under the constraint of circular motion in the plane.

2. Equations of Motion.

As a precursor to the more complicated rolling sphere, first consider a wheel of radius R rolling with angular velocity $d\theta/dt = -\omega$, as depicted in Figure 2. Assuming that the wheel does not slip, the displacement of the center of the wheel is given by

$$\vec{r}_C(t) = R(\theta_0 - \theta)\vec{\imath} + R\vec{\jmath},$$

where θ is the angle of rotation (in radians) and θ_0 is the initial angle of rotation of the wheel. Differentiating this relationship we see that the velocity of the center C is given in standard vector coordinates by

$$\vec{v}_C = \omega R\vec{\imath}.$$

Let B be an arbitrary point on the outside of the wheel, making an angle θ with respect to horizontal. The velocity at B satisfies

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B/C},$$

where the notation $\vec{v}_{B/C}$ indicates the *relative velocity* of B with respect to C (i.e., $\vec{v}_{B/C} := \vec{v}_B - \vec{v}_C$). Similarly, if $\vec{r}_{B/C} := \vec{r}_B - \vec{r}_C$, then $\vec{r}_{B/C} = R\cos\theta\,\vec{\imath} + R\sin\theta\,\vec{\jmath}$. The relative velocity $\vec{v}_{B/C}$ is tangent to the wheel and is expressed by

$$\vec{v}_{B/C} = \omega R\sin\theta\,\vec{\imath} - \omega R\cos\theta\,\vec{\jmath} = -\omega\vec{k} \times \vec{r}_{B/C},$$

where $\times$ represents the standard vector cross-product. The reader may find it helpful at this point to imagine the angular velocity as the vector quantity $d\vec{\theta}/dt = -\omega\vec{k}$ and observe that

$$\vec{v}_{B/C} = \frac{d\vec{\theta}}{dt} \times \vec{r}_{B/C}.$$

Now suppose that B is the point of contact between the wheel and the rolling surface. Then $\vec{r}_{B/C} = -R\vec{\jmath}$, so $\vec{v}_B = \vec{0}$. This observation characterizes rolling without slipping (i.e., *the contact point of the rolling object with the ground must have zero instantaneous velocity*).

The next step is to use this mathematical characterization of rolling without slipping to help derive the equations of motion for the rolling sphere. Let $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath}$ parameterize the path in the plane along which the sphere is to be rolled. The sphere is assumed to have unit radius, so the motion of its center C is described by $\vec{r}_C(t) = \vec{r}(t) + \vec{k}$.

The first task is to determine what rotation $\vec{\omega}(t)$ of the sphere corresponds to rolling the sphere according to the prescription of no slipping or twisting. It was seen earlier for the rolling wheel that no slippage required that the velocity at the bottom of the wheel be zero. Instantaneously, there is no difference between the rolling of the wheel and sphere, hence this characterization of rolling without slipping also applies to the rolling sphere. As with the wheel, we can compute the velocity of the bottom of the sphere by adding the velocity of the center to the relative velocity of the bottom with respect to the center. Denoting the bottom point by B, we arrive at

$$\begin{aligned}\vec{v}_B(t) &= \vec{r}_C{}'(t) + \vec{\omega}(t) \times \vec{r}_{B/C} \\ &= \vec{r}_C{}'(t) + \vec{\omega}(t) \times (-\vec{k}) \\ &= f'(t)\vec{\imath} + g'(t)\vec{\jmath} + \omega_x(t)\vec{\jmath} - \omega_y(t)\vec{\imath} \\ &= \vec{0}.\end{aligned}$$

This already fixes the $\vec{\imath}$ and $\vec{\jmath}$ components of $\vec{\omega}(t)$ to be

$$\omega_x(t) = -g'(t), \quad \omega_y = f'(t). \tag{1}$$

The second constraint of no twisting is intended to prevent one from simply spinning the sphere in place to achieve a desired orientation. Such a spinning results from rotation about the z-axis or, equivalently, from a nonzero $\vec{k}$ component of $\vec{\omega}(t)$. It follows that $\omega_z = 0$. The two previous observations now imply that

$$\vec{\omega}(t) = -g'(t)\vec{\imath} + f'(t)\vec{\jmath},$$

which completely characterizes the rolling behavior of the sphere along the trajectory of the path $\vec{r}(t)$. The reader may be interested in consulting a text on dynamics (for example, [4]) to obtain a better understanding of the foregoing analysis of the rolling sphere.

The next order of business is to construct equations of motion for an arbitrary point P on the sphere's surface, which will be simplified greatly by the following observation. The path $\vec{r}(t)$ is assumed to be closed, so the final position of the center C is unchanged by the rolling of the sphere. We can write the velocity at P as

$$\vec{v}_P(t) = \vec{v}_C(t) + \vec{v}_{P/C}(t).$$

We have the option of determining the final position of P using either $\vec{v}_P(t)$ and the initial position of P or $\vec{v}_{P/C}$ and the initial relative position of P with respect to C. The latter approach provides simpler equations of motions, namely,

$$\vec{v}_{P/C}(t) = \vec{\omega}(t) \times \vec{r}_{P/C}(t) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -g'(t) & f'(t) & 0 \\ x_{P/C}(t) & y_{P/C}(t) & z_{P/C}(t) \end{vmatrix}. \tag{2}$$

Abbreviating $x_{P/C}$, $y_{P/C}$, and $z_{P/C}$ to x, y, and z, equation (2) leads to the following first-order system of differential equations:

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} f'(t)z(t) \\ g'(t)z(t) \\ -f'(t)x(t) - g'(t)y(t) \end{pmatrix}. \tag{3}$$

We then hope to solve this system of equations given a path $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath}$ and an initial condition (x_0, y_0, z_0). As an example, consider a linear path $\vec{r}(t) = at\vec{\imath} + bt\vec{\jmath}$, where $a, b \in \mathbb{R}$. The resulting equations of motion become

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & a \\ -a & -b & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \tag{4}$$

which are easily solved due to the constant coefficient matrix. Outside the realm of piecewise linear paths, however, solution of the equations of motion will be more challenging.

3. Nonholonomy of the Rolling Sphere.

Ultimately, showing that the rolling sphere is a nonholonomic system requires a proof that we can move between two arbitrary orientations by rolling the sphere along a closed path. Determining the shape of such a path is, in general, a daunting task. However, the previous section provides some insight about the types of paths that should be considered first, namely, those paths which are piecewise linear. Moreover, we will divide the burden into two smaller loads:

- showing that any point on the sphere can be relocated to the North Pole;
- showing that we can achieve any desired rotation about the z-axis by rolling, without disturbing the North Pole.

Although the results will be proven by similar methods, the latter result is conceptually simpler than the former and, therefore, will be proven first. Recall that the no-twisting constraint prevents direct rotation of the sphere about the z-axis, but an equivalent transformation can be accomplished by rolling the sphere through an appropriate path.

Lemma 1. *Let $0 \le \theta \le 2\pi$ and define $\vec{r}(t)$ on $[0, 2\pi + \theta]$ by*

$$\vec{r}(t) = \begin{cases} 0\vec{\imath} + t\vec{\jmath} & \text{if } 0 \le t < \frac{\pi}{2}, \\ \left(t - \frac{\pi}{2}\right)\vec{\imath} + \frac{\pi}{2}\vec{\jmath} & \text{if } \frac{\pi}{2} \le t < \frac{\pi}{2} + \frac{\theta}{2}, \\ \frac{\theta}{2}\vec{\imath} + \left(\pi + \frac{\theta}{2} - t\right)\vec{\jmath} & \text{if } \frac{\pi}{2} + \frac{\theta}{2} \le t < \frac{3\pi}{2} + \frac{\theta}{2}, \\ \left(\theta + \frac{3\pi}{2} - t\right)\vec{\imath} - \frac{\pi}{2}\vec{\jmath} & \text{if } \frac{3\pi}{2} + \frac{\theta}{2} \le t < \frac{3\pi}{2} + \theta, \\ 0\vec{\imath} + \left(-2\pi - \theta + t\right)\vec{\jmath} & \text{if } \frac{3\pi}{2} + \theta \le t < 2\pi + \theta. \end{cases} \tag{5}$$

By rolling the sphere along the path $\vec{r}(t)$ the effect on the sphere is identical to a counterclockwise rotation (as viewed from above) θ radians about the z-axis.

Although quite tedious, the lemma can be proved by direct computation using the equations of motion (3). Rather than pursuing that approach, however, consider Figure 3, which depicts the path from the lemma (rotating the point P to the point Q) as well as four intermediate positions of the sphere. The five stages of motion depicted in the figure are: (1) clockwise rotation about the x-axis by $\pi/2$ radians, (2) counterclockwise rotation about the y-axis by $\theta/2$ radians, (3) counterclockwise rotation about the x-axis by π radians, (4) clockwise rotation about the y-axis by $\theta/2$ radians, and (5) clockwise rotation about the x-axis by $\pi/2$ radians. (The direction of motion in each case is described as viewed from the positive side of the axis of rotation.)

It remains to show that any point can be moved to the North Pole by means of rolling with no slipping or twisting.

Lemma 2. *For any given point P on the sphere there exists a piecewise differentiable path $\vec{r}(t)$ $(0 \le t \le T < \infty)$ such that rolling the sphere through the path $\vec{r}(t)$ relocates P to the North Pole.*

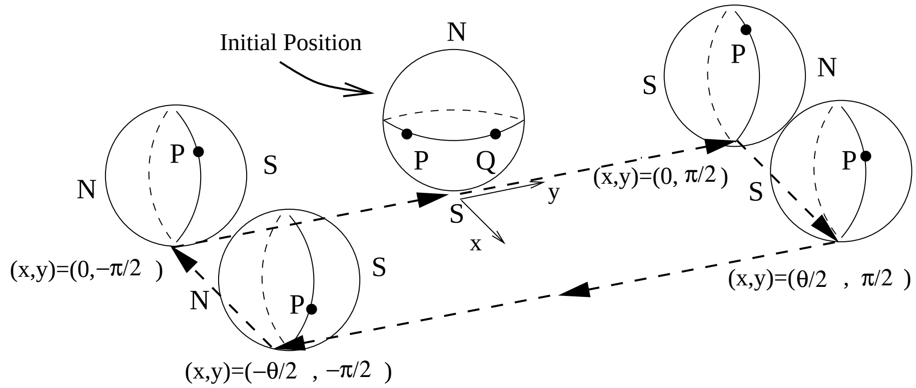

FIGURE 3. The path of Lemma 1.

Proof. Let $P = (x_0, y_0, z_0)$ and consider the great circle which passes through both the North Pole and the point P. Let ϕ denote the length of the shorter arc between P and the North Pole, which corresponds to the angle between the position vector of P and $\vec{k}$. Now, let $\vec{u}$ be a unit vector in the direction $-x_0\vec{\imath} - y_0\vec{\jmath}$ and denote by $\vec{v}$ a unit vector perpendicular to $\vec{u}$. Define $\vec{r}(t)$ as the rectangular path on $[0, 2\pi + \phi]$ given by

$$\vec{r}(t) = \begin{cases} t\vec{u} + 0\vec{v} & \text{if } 0 \le t < \frac{\phi}{2}, \\ \frac{\phi}{2}\vec{u} + (t - \frac{\phi}{2})\vec{v} & \text{if } \frac{\phi}{2} \le t < \pi + \frac{\phi}{2}, \\ (\phi + \pi - t)\,\vec{u} + \pi\vec{v} & \text{if } \pi + \frac{\phi}{2} \le t < \phi + \pi, \\ 0\vec{u} + (2\pi + \phi - t)\,\vec{v} & \text{if } \pi + \phi \le t \le 2\pi + \phi. \end{cases} \tag{6}$$

As with the proof of Lemma 1 we will again avoid direct solution of the equations of motion. Instead, we will describe the transformations resulting from each component of the rectangular path. Recall that the arc separating P from the North Pole is initially ϕ units in length. After rolling the sphere a distance $\phi/2$ in the direction $\vec{u}$, P is moved halfway between its original location and the North Pole. Next, the sphere is rolled a distance π in the perpendicular direction $\vec{v}$. The net result is that P lies on the same great circle, but is now separated from the South Pole by an arc of length $\phi/2$. At this point, rolling the sphere $\phi/2$ units in the direction $-\vec{u}$ relocates P to the South Pole. The final piece of the path returns the sphere to the origin by rolling a distance π in the direction $-\vec{v}$ while simultaneously moving P from the South Pole to the North Pole, thereby completing the desired transformation of the sphere. □

Together, Lemmas 1 and 2 demonstrate the main result of this article.

Theorem 3. *The sphere resting on the plane constrained to rolling through closed paths with no slipping or twisting is a nonholonomic system.*

In the next section the equations of motion will be put to better use as an independent and somewhat surprising proof of Lemma 2 is obtained exclusively from circular paths.

4. A Circular Path for the Rolling Sphere.

Suppose that the sphere is rolled along a circle of radius R that begins and ends at the origin, say with the parametrization

$$\vec{r}(t) = R(1 - \cos t)\vec{\imath} + R \sin t\vec{\jmath} \quad (0 \le t \le 2\pi). \tag{7}$$

The system of differential equations (3) becomes

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} R\sin t\, z(t) \\ R\cos t\, z(t) \\ -R\sin t\, x(t) - R\cos t\, y(t) \end{pmatrix},$$

which, at first glance, looks a bit unwieldy. However, it turns out that a fairly reasonable change of coordinates will lead to another linear system with a constant coefficient matrix.

4.1. **Solution of the Equations of Motion.** It is a simple observation that, regardless of the path, the sphere always rolls in the direction of the tangent vector to the path $\vec{r}(t)$. Thus, a natural alternate coordinate system is given by the instantaneous tangential and normal vectors to the path. In the case of the circular path this motivates the coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$, defined by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{2} - t\right) & -\sin\left(\frac{\pi}{2} - t\right) \\ \sin\left(\frac{\pi}{2} - t\right) & \cos\left(\frac{\pi}{2} - t\right) \end{pmatrix} \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix},$$

with $z = \tilde{z}$. The new *rotating* coordinate system adjusts the $\tilde{x}$- and $\tilde{y}$-axes so that they rotate with the perpendicular and tangential vectors on the circular path. Under this change of variables the equations of motion become

$$\begin{pmatrix} \tilde{x}'(t) \\ \tilde{y}'(t) \\ \tilde{z}'(t) \end{pmatrix} = \begin{pmatrix} -\tilde{y}(t) + R\tilde{z}(t) \\ \tilde{x}(t) \\ -R\tilde{x}(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 & R \\ 1 & 0 & 0 \\ -R & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \\ \tilde{z}(t) \end{pmatrix}. \tag{8}$$

The general solution of this new system is easily found using the eigenvalue method (see [3] for an explanation of this technique) to be

$$\vec{x}_r(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) + C_3\vec{x}_3(t), \tag{9}$$

where $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are given by

$$\vec{x}_1(t) = \frac{1}{\sqrt{R^2+1}} \begin{pmatrix} 0 \\ R \\ 1 \end{pmatrix},$$

$$\vec{x}_2(t) = \frac{1}{\sqrt{R^2+1}} \begin{pmatrix} 0 \\ 1 \\ -R \end{pmatrix} \cos\left(\sqrt{R^2+1}t\right) - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin\left(\sqrt{R^2+1}t\right),$$

and

$$\vec{x}_3(t) = \frac{1}{\sqrt{R^2+1}} \begin{pmatrix} 0 \\ 1 \\ -R \end{pmatrix} \sin\left(\sqrt{R^2+1}t\right) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos\left(\sqrt{R^2+1}t\right).$$

It should be noted that this solution is presented in the $(\tilde{x}, \tilde{y}, \tilde{z})$ coordinate system, but when $t = 0$ or $t = 2\pi$ the change of variables simply corresponds to $x = -\tilde{y}$, $y = \tilde{x}$, and $z = \tilde{z}$. One immediate consequence of the general solution is that if $R = \sqrt{n^2 - 1}$, where $n > 1$ is an integer, then rolling the sphere through the circular path with radius R will preserve the orientation of the sphere. This follows from the fact that each of the coordinate functions will be 2π periodic, implying that each point on the sphere will return to its original location at the end of the circular path.

The coefficients for the particular solution subject to given initial conditions are found by substituting $t = 0$ into (9) and performing a matrix inversion, as follows:

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{R}{\sqrt{R^2+1}} & \frac{1}{\sqrt{R^2+1}} \\ 0 & \frac{1}{\sqrt{R^2+1}} & -\frac{R}{\sqrt{R^2+1}} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}(0) \\ \tilde{y}(0) \\ \tilde{z}(0) \end{pmatrix}.$$

This information is useful because we can choose any initial point and determine its location at the end of the path by substituting $t = 2\pi$ into the corresponding general solution. In the next subsection it will be useful to have the particular solution that follows the motion of the North Pole along the circular path. The North Pole has initial coordinates $(\tilde{x}(0), \tilde{y}(0), \tilde{z}(0)) = (0, 0, 1)$ and leads to the solution

$$\begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \\ \tilde{z}(t) \end{pmatrix} = \frac{1}{R^2+1} \left[\begin{pmatrix} 0 \\ R \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ R \\ -R^2 \end{pmatrix} \cos(\omega t) + \sqrt{R^2+1} \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix} \sin(\omega t) \right], \tag{10}$$

where $\omega = \sqrt{R^2+1}$.

4.2. **Application to Holonomy.** The major tool in this section will be the particular solution, (10), found for the position of the North Pole after the sphere has been rolled through a circular path of radius R. However, before moving to apply the particular solution to questions regarding the holonomy of the system, we pause to consider the effect of a reorientation of the circular path in the xy-plane.

Let $\vec{u}$ and $\vec{v}$ be the images, respectively, of $\vec{\imath}$ and $\vec{\jmath}$ under a rotation by θ in the xy-plane. Replace the path, (7), by

$$\vec{r}(t) = R(1 - \cos t)\vec{u} + R \sin t \vec{v} \quad (0 \leq t \leq 2\pi). \tag{11}$$

Suppose that rolling the sphere through the path (7) moves a point P on the sphere to another point, say Q. If $\tilde{P}$ and $\tilde{Q}$ are the images of P and Q, respectively, under the same rotation used above, then rolling the sphere through the path (11) will move $\tilde{P}$ to $\tilde{Q}$. This simple observation will be useful in the proof of the next proposition.

Proposition 4. *For any given point P on the sphere there exists a piecewise differentiable path $\vec{r}(t)$ $(0 \leq t \leq T < \infty)$, composed of circular components, such that rolling the sphere through the path $\vec{r}(t)$ relocates P to the North Pole.*

Proof. We will begin by showing that any point $P = (x_0, y_0, z_0)$ in the closed northern hemisphere $(z_0 \geq 0)$ can be relocated to the North Pole via a single circular path. Observe that if P is moved to the North Pole, then the image of the North Pole must have altitude z_0. (The length of the arc between P and the North Pole is preserved by the transformation.) By (10), the z-coordinate of the image of the North Pole under the circular path (7) is

$$z(2\pi) = \left[1 + R^2 \cos(2\pi\sqrt{R^2+1})\right] / (R^2+1). \tag{12}$$

Let $f(R) = 1 + R^2 \cos(2\pi\sqrt{R^2+1})$ and observe that $f(0) = 1$, while

$$f(\frac{\sqrt{5}}{2}) = 1 + \frac{5}{4}\cos(2\pi\sqrt{\frac{9}{4}}) = 1 + \frac{5}{4}\cos(3\pi) = -0.25.$$

Since $f(R)$ varies continuously with R it follows from the intermediate value theorem that there is a single circular path which moves the North Pole to any desired nonnegative altitude. The upshot of this fact is that for every nonnegative altitude, there exists a point P with that altitude that can be moved to the North Pole by rolling through a single circular path. Moreover, if $\tilde{P}$ is another point on the sphere having the same altitude as P, there exists an angle θ such that rolling the sphere through the reoriented circular path, (11), will move $\tilde{P}$ to the North Pole. Thus, any point in the northern hemisphere may be relocated to the North Pole by rolling the sphere through a single circular path.

If P lies in the southern hemisphere then two circular paths will be used in sequence to move P to the North Pole. Because we have already shown that any point in the closed northern hemisphere can be relocated to the North Pole, it remains only to demonstrate that any point in the southern hemisphere can be moved to the closed northern hemisphere by a single circular path. Given a point in the southern hemisphere, consider the great circle containing both this point and the North Pole. The shorter arc between the point and the North Pole will have length less than or equal to π. The midpoint of this arc lies in the closed northern hemisphere, so there exists a circular path which relocates the midpoint of the arc to the North Pole. Since the arc between the original point and this midpoint must have length less than or equal to $\pi/2$, the original point will be relocated to a point somewhere in the closed northern hemisphere, finishing the proof. □

After reading the proof of Proposition 4 it is natural to wonder whether the North and South Poles may be exchanged when the sphere is rolled through a single circular path. It turns out that this is impossible, as is made clear by the following proposition.

Proposition 5. *The North and South Poles cannot be exchanged by rolling the sphere (with no slipping or twisting) through a single circular path.*

Proof. As in the proof of Proposition 4, the main tool is the general solution (3) obtained in section 4.1. Because of the symmetry of the situation, the parametrization (7) is sufficiently general for this analysis. For a fixed radius $R > 0$ the sphere returns to its starting point when $t = 2\pi$ and by (12) the image of the North Pole satisfies

$$-1 + \frac{2}{R^2+1} = \frac{1-R^2}{R^2+1} \leq z(2\pi) \leq \frac{1+R^2}{R^2+1} = 1.$$

Thus, $z(2\pi)$ can never equal -1, showing that the North and South Poles cannot be interchanged via rolling along a single circular path. □

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