Frame Potential & Finite Abelian Groups

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Notation & Definitions:

- $\mathbb{H}$ denotes a [finite-dimensional] real or complex Hilbert space

- $\{f_k\}_{k=1}^n \subseteq \mathbb{H}$ is a *frame* if $\exists\, 0 < A \leq B < \infty$ s.t.

$$A\|f\|^2 \leq \sum_{k=1}^{n} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathbb{H}.$$

  The frame is *tight* if it is possible to choose $A = B$.

- The *frame potential* of $\{f_k\}_{k=1}^n \subseteq \mathbb{H}$ is defined as

$$\mathrm{FP}(\{f_k\}_{k=1}^n) = \sum_{j,k=1}^{n} |\langle f_j, f_k\rangle|^2.$$

Background:

- Benedetto & Fickus (2000): Local minimizers of frame potential for overdetermined systems consisting of unit vectors characterize the tight frames [1].

- Casazza, Fickus, Kovačević , Leon, & Tremain (2004):
  - Considered collections constrained s.t. $\|x_n\| = a_n > 0$ with the convention that

$$a_0 \geq a_2 \geq \cdots \geq a_{N-1}.$$

  - Local minimizers of frame potential for overdetermined systems characterize the tight frames provided that the lengths satisfy the *fundamental frame inequality* [2].

Background:

Theorem 1 (Fundamental frame inequality [2]). *If* $\{x_n\}_{n=0}^{N-1} \in a_0\mathbb{S}^{d-1} \times \cdots \times a_{N-1}\mathbb{S}^{d-1}$ *is a tight frame and* $a_0 \geq a_1 \geq \cdots \geq a_{N-1} > 0$, *then*

$$da_0^2 \leq \sum_{n=0}^{N-1} a_n^2. \tag{1}$$

Notes:

- $\mathbb{H} = K^d$ where $K = \mathbb{R}$ or $\mathbb{C}$.
- $\mathbb{S}^{d-1} = \{x \in K^d : \|x\| = 1\}$.

Remark: The preceding characterization is thus complete since no tight frames exist when the fundamental frame inequality is not satisfied.

Background:

Casazza et al. also examine the properties of [overdetermined] minimizers of the frame potential when the lengths fail to satisfy the fundamental frame inequality.

- The largest vectors force smaller vectors into their orthogonal complement.
- At some point the remaining vectors satisfy the fundamental frame inequality in a lower-dimensional subspace and comprise a tight frame for this subspace.

In the underdetermined case local minimizers of the frame potential always consist of mutually orthogonal vectors.

Background:

Vale & Waldron (2004): examined the symmetries possessed by tight frames for finite-dimensional Hilbert spaces [5]. The *symmetry group* of a frame X is defined as

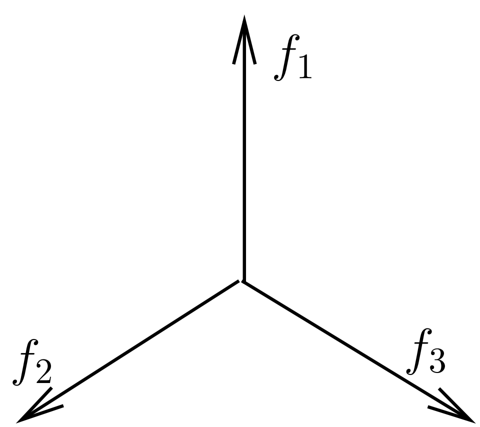
$$\mathrm{Sym}(X) = \{U \in \mathcal{U}(\mathbb{H}) : U(X) = X\}.$$

Note: $\mathcal{U}(\mathbb{H})$ is the group of unitary linear transformations on $\mathbb{H}$.

Example: Mercedes-Benz frame

$$\mathrm{Sym}(X) = D_3.$$

(dihedral group of order 6)



A Natural Question:

Under what conditions, if any, can tight frames with specified symmetries be characterized as local minimizers of the frame potential?

A partial answer to this question will be sought by considering collections of functions in the group algebra of a finite abelian group which possess the symmetries of a chosen subgroup.

More Notation:

- G will denote a finite abelian group

- $\ell(G)$ will denote the group algebra of G (real or complex valued functions on G)

- $(T_g f)(g') = f(g'g^{-1})$ will denote the *translation* operator on $\ell(G)$ induced by $g \in G$

- Given a subgroup H of G, collections of the form

$$X_H = \{T_h f_k : h \in H,\ f_k \in \ell(G),\ 0 \leq k \leq n-1\} \tag{2}$$

will be studied. By construction, $H \leq \mathrm{Sym}(X_H)$.

Convolution & Sampling:

- The *convolution* of $f_1, f_2 \in \ell(G)$ is given by

$$f_1 * f_2(g) = \sum_{x \in G} f_1(x) f_2(g^{-1}x), \quad g \in G.$$

- Sampling operator, $\mathcal{S}_H : \ell(G) \to \ell(H)$,

$$(\mathcal{S}_H f)(h) = f(h), \quad h \in H,$$

- Upsampling operator, $\mathcal{S}_H^* : \ell(H) \to \ell(G)$,

$$(\mathcal{S}_H^* f)(g) = \begin{cases} f(g), & g \in H \\ 0, & g \notin H, \end{cases} \quad g \in G.$$

Convolutional Systems for $\ell(G)$:

- Filters: $\{f_k\}_{k=0}^{n-1} \subseteq \ell(G)$
- Frame Operator of X_H:

$$
\begin{aligned}
Ff &= \sum_{k=0}^{n-1} \sum_{h \in H} \langle f, T_h f_k \rangle T_h f_k \\
&= \sum_{k=0}^{n-1} \Big[\mathcal{S}_H^* \mathcal{S}_H (f * \tilde{f}_k) \Big] * f_k
\end{aligned}
$$

The latter form reveals the convolutional nature of X_H. In this sense F may be thought of as a *filter bank frame operator*.

Note: $\tilde{f}_k$ denotes the *involution* of the filter f_k, given by

$$\tilde{f}_k(g) = \overline{f_k(g^{-1})}$$

Filterbanks on $\ell(G)$:

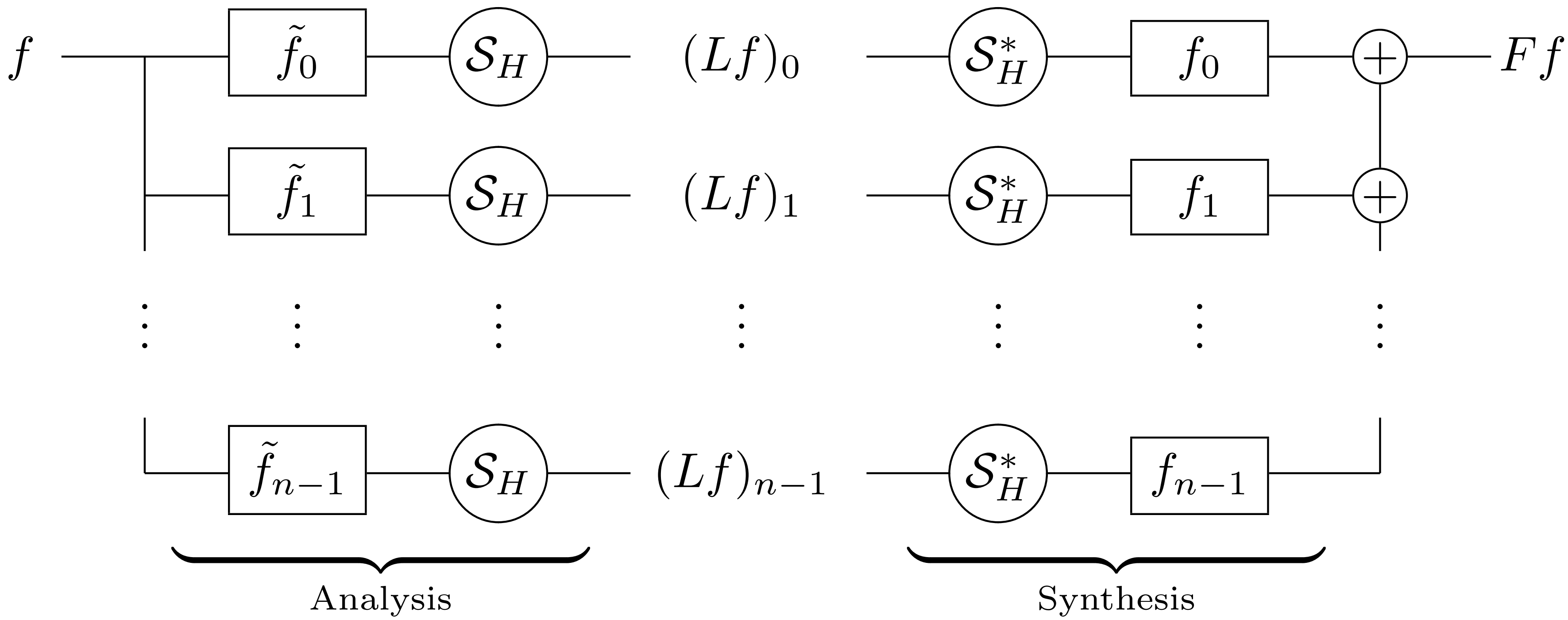


Figure 1: Block diagram of an n-channel filterbank on $\ell(G)$.

Prior Work:

Fickus, J–, Kornelson, Okoudjou (2004): Examined the case where $G = \mathbb{Z}/d\mathbb{Z}$ and H is a cyclic subgroup [3].

- Local minimizers of frame potential for overdetermined systems characterize the tight frames provided that the *lengths of the generators* satisfy the fundamental frame inequality.

- If the lengths of the generators do not satisfy the fundamental frame inequality then tight frames are not possible.

- The key tool behind the result is the *modulated filter representation* of the synthesis operator, which is essentially a block diagonalization in the Fourier domain making use of the relationship between sampling and the Fourier transform.

Modulated Filter Representation of L^*:

Under the Fourier transform L^* is unitarily equivalent to an operator of the form:

$$\begin{bmatrix} L^*_{X_1} & 0 & \cdots & 0 \\ 0 & L^*_{X_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L^*_{X_N} \end{bmatrix}$$

- Each collection X_j consists of n vectors, say $\{x_{j,k}\}_{k=0}^{n-1}$, whose lengths must satisfy

$$a_k^2 = \sum_{j=1}^{N} \|x_{j,k}\|^2.$$

- The frame bounds of the X_j determine the frame bounds of the convolutional system. The combined frame potential of the X_j is equal to the frame potential of the convolutional system.

Fourier Analysis on $\ell(G)$:

- The *discrete Fourier transform* (DFT) of $f \in \ell(G)$ is defined by

$$\mathcal{F}f(\chi) = \hat{f}(\chi) = \sum_{x \in G} f(x)\overline{\chi(x)}, \quad \chi \in \widehat{G}.$$

- $\widehat{G}$ is the *dual group* to G consisting of all characters of G under pointwise multiplication. ($\widehat{G}$ is isomorphic to G)

- A *character* of G is a group homomorphism $\chi : \ell(G) \to \mathbb{T}$. (Here, $\mathbb{T}$ represents the group of unimodular complex numbers.)

Downsampling/Upsampling in $\mathbb{Z}/d\mathbb{Z}$:

- Downsampling by 2: (periodization)

$$\widehat{\downarrow_2 f}(\ell) = \sum_{k=0}^{d/2-1} (\downarrow_2 f)(k) \exp\left(-2\pi i k\ell/(d/2)\right)$$
$$= \sum_{k=0}^{d-1} \left[\frac{1+(-1)^k}{2}\right] f(k) \exp\left(-2\pi i k\ell/d\right) = \frac{1}{2}\left[\hat{f}(\ell) + \hat{f}(\ell + d/2)\right].$$

- Upsampling by 2: (periodic extension)

$$\widehat{\uparrow_2 f}(\ell) = \sum_{k=0}^{d-1} (\uparrow_2 f)(k) \exp\left(-2\pi i k\ell/d\right)$$
$$= \sum_{k=0}^{d/2-1} f(k) \exp\left(-2\pi i 2k\ell/d\right) = \hat{f}(\ell).$$

Sampling over a Subgroup H in $\ell(G)$:

The following proposition relates a given character in $\ell(H)$ to its extensions in $\ell(G)$ and is based on a result of Serre [4].

Proposition 2. *Suppose $H \leq G$, let $x \in G \setminus H$, and denote by H^x the subgroup of G generated by H and x. Let $m_x = \min\{n \in \mathbb{N} : x^n \in H\}$. Then each $\chi \in \widehat{H}$ extends to m_x orthogonal characters in $\widehat{H^x}$, $\{\chi_j\}_{j=0}^{m_x-1}$, and*

$$\widehat{H^x} = \{\chi_j : \chi \in \widehat{H},\ 0 \leq j \leq m_x - 1\}.$$

Sampling over a Subgroup H in $\ell(G)$:

Fix $\chi \in \widehat{H}$. $\widehat{G}_\chi$ will denote the subset of $\widehat{G}$ consisting of characters ψ whose restrictions to H coincide with χ.

Corollary 3. *Let $H \leq G$ and $\chi \in \widehat{H}$. Then $|\widehat{G}_\chi| = [G : H]$.*

Corollary 4. *Let $H \leq G$ and $\chi \in \widehat{H}$. Then,*

$$\sum_{\psi \in \widehat{G}_\chi} \psi(g) = \begin{cases} [G : H]\chi(g), & g \in H, \\ 0, & \text{otherwise.} \end{cases}$$

These facts complete the picture for sampling and upsampling over a subgroup H.

Sampling/Upsampling in $\ell(G)$:

Proposition 5. *Let G be a finite abelian group with subgroup H. Then*

(i) For $f \in \ell(H)$,

$$\widehat{\mathcal{S}_H^* f}(\chi) = \hat{f}(\chi|_H), \quad \chi \in \widehat{G}.$$

(ii) For $f \in \ell(G)$,

$$\widehat{\mathcal{S}_H f}(\chi) = \frac{1}{[G:H]} \sum_{\psi \in \widehat{G}_\chi} \hat{f}(\psi), \quad \chi \in \widehat{H}.$$

With this information, the modulated filter representation can be extended to convolutional systems for $\ell(G)$.

The Main Result:

Theorem 6. *Let G be a finite abelian group and H a subgroup of G with $n \geq [G : H]$. If $X_H(\{f_m\}_{m=0}^{n-1}) \subset \ell(G)$ is a local minimizer of the frame potential over $a_0\mathbb{S}(G) \times \cdots \times a_{n-1}\mathbb{S}(G)$, where $a_0 \geq a_1 \geq \cdots \geq a_{n-1} > 0$ satisfy*

$$da_0^2 \leq \sum_{m=0}^{n-1} a_m^2,$$

then $X_H(\{f_m\}_{m=0}^{n-1})$ is a tight frame for $\ell(G)$.

Notation: $\mathbb{S}(G) = \{f \in \ell(G) : \|f\| = 1\}$.

Directions for Further Study:

- *Is it possible to extend the characterization of tight frames in terms of the frame potential to convolutional systems for $\ell(G)$, where G is an arbitrary finite group?*

- *What other symmetries or structures of systems in a finite dimensional Hilbert space lead to similar characterizations of tight frames in terms of the frame potential?*

References

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