Convolutional frames and the frame potential

Matthew Fickus [a,1], Brody D. Johnson [b,3], Keri Kornelson [c,2], Kasso A. Okoudjou [d,3]

[a] *Department of Mathematics and Statistics, Air Force Institute of Technology, Wright-Patterson AFB, OH 45433, USA*

[b] *Department of Mathematics and Computer Science, Saint Louis University, Saint Louis, MO 63103, USA*

[c] *Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA*

[d] *Department of Mathematics, Cornell University, Ithaca, NY 14853, USA*

Abstract

The recently introduced notion of frame potential has proven useful for the characterization of finite-dimensional tight frames. The present work represents an effort to similarly characterize finite-dimensional tight frames with additional imposed structure. In particular, it is shown that the frame potential still leads to a complete description of tight frames when restricted to the class of translation-invariant systems. It is natural to refer to such frames as convolutional because of the correspondence between translation-invariant systems and finite-dimensional filter banks. The fast algorithms associated with convolution represent one possible advantage over non-convolutional frames in applications.

Key words: filter bank, frame, potential, convolution
PACS: 42C40

Email addresses: `Matthew.Fickus@afit.edu` (Matthew Fickus), `brody@slu.edu` (Brody D. Johnson), `kornelso@math.grinnell.edu` (Keri Kornelson), `kasso@math.cornell.edu` (Kasso A. Okoudjou).

[1] Research partially supported by NSF DMS 0405376 and NSF VIGRE grant DMS 9983660 at Cornell University. The views expressed in this article are those of the author and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

[2] Research partially supported by the NSF VIGRE grant at Texas A&M University.

[3] Research partially supported by the National Science Foundation.

1 Preliminaries and Notation

Let $\ell(\mathbb{Z}_d)$ be the d-dimensional real or complex Hilbert space of functions on $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$, equipped with addition modulo $d > 0$. One may equivalently think of $\ell(\mathbb{Z}_d)$ as the set of d-periodic functions defined on the integers. Notice that $\ell(\mathbb{Z}_d)$ is endowed with a natural *translation*, T, which acts on $x \in \ell(\mathbb{Z}_d)$ via $(Tx)(k) = x(k-1)$, $k \in \mathbb{Z}_d$. The notational convention that $T_k := T^k$, $k \in \mathbb{Z}_d$, will be adopted hereafter. Given two positive integers m and n so that $mn = d$, $m\mathbb{Z}_d$ will denote the subgroup $\{0, m, 2m, \ldots, (n-1)m\} \subset \mathbb{Z}_d$.

The unitary Fourier transform of $x \in \ell(\mathbb{Z}_d)$ is denoted by $\hat{x}$ and defined by

$$\mathcal{F}_d x(k) := \hat{x}(k) = \frac{1}{\sqrt{d}} \sum_{\ell \in \mathbb{Z}_d} x(\ell)\, \mathrm{e}^{-2\pi \mathrm{i} \frac{\ell}{d} k}.$$

The *circular convolution* of $x, y \in \ell(\mathbb{Z}_d)$ is defined by

$$(x * y)(k) = \sum_{n \in \mathbb{Z}_d} x(n)\, y(k-n).$$

Hence, $\langle x, T_n y_m \rangle = (x * \tilde{y}_m)(n)$, where $\tilde{y}(k) := \overline{y(-k)}$ is the *involution* of $y \in \ell(\mathbb{Z}_d)$. It is routine to verify that $(x*y)\hat{\ }(k) = \sqrt{d}\hat{x}(k)\, \hat{y}(k)$. Given any positive integer N that divides d, henceforth denoted $N \,|\, d$, consider the corresponding *downsampling by* N operator,

$$\downarrow_N : \ell(\mathbb{Z}_d) \to \ell(\mathbb{Z}_{d/N}), \quad (\downarrow_N x)(k) = x(Nk),$$

and its adjoint, the *upsampling by* N operator,

$$\uparrow_N : \ell(\mathbb{Z}_{d/N}) \to \ell(\mathbb{Z}_d), \quad (\uparrow_N x)(k) = \begin{cases} x(k/N), & N \,|\, k, \\ 0, & N \nmid k. \end{cases}$$

The composition of upsampling by N with downsampling by N will be referred to as the *decimation by* N operator and denoted $\uparrow\downarrow_N$.

Recall that a finite collection $x_1, \ldots, x_J \in \ell(\mathbb{Z}_d)$ is a *frame* for $\ell(\mathbb{Z}_d)$ if and only if there exist constants $0 < B_1 \le B_2 < \infty$ such that for each $x \in \ell(\mathbb{Z}_d)$,

$$B_1 \|x\|^2 \le \sum_{j=1}^{J} |\langle x, x_j \rangle|^2 \le B_2 \|x\|^2. \tag{1}$$

In the event that B_1 and B_2 may be chosen to be equal the frame is said to be *tight*. Associated to any collection $X := \{x_j\}_{j=1}^J \subset \ell(\mathbb{Z}_d)$ is the corresponding *analysis operator*, $L : \ell(\mathbb{Z}_d) \to \ell(\mathbb{Z}_J)$, defined by $L_X x(j) := Lx(j) = \langle x, x_j \rangle$. The adjoint of the analysis operator is called the *synthesis operator* and acts on $y \in \ell(\mathbb{Z}_J)$ by $L_X^* y := L^* y = \sum_{j \in J} y(j) x_j$. By composing the synthesis and analysis operators one obtains the *frame operator*, $S : \ell(\mathbb{Z}_d) \to \ell(\mathbb{Z}_d)$, given by

$$S_X x := Sx = L^* Lx = \sum_{j=1}^{J} \langle x, x_j \rangle x_j.$$

The frame operator is well-defined whether or not X is a frame; however, in the event that X is a frame with bounds $A \leq B$ it follows that $AI \leq S_X \leq BI$; and conversely if the last inequality of operators holds then X is a frame. Finally, note that the *Gram operator* associated to X is defined as $G := LL^* : \ell(\mathbb{Z}_J) \to \ell(\mathbb{Z}_J)$.

2 Introduction

The notion of frame potential was introduced by Benedetto and Fickus [1] as a tool for characterizing sequences of unit-norm vectors that comprise tight frames for $\ell(\mathbb{Z}_d)$. In particular, they showed in this context that when the number of vectors exceeds the dimension of the space that each local minimizer of the frame potential gives rise to a tight frame. In essence, this result suggests that one may effectively search for tight frames of unit-norm vectors by minimizing the frame potential. Notice that the frame operator of a tight frame is simply a multiple of the identity operator, which leads to a simpler reconstruction procedure than what is generally available for non-tight frames.

Definition 1 ([1]) *Let $X := \{x_j\}_{j=0}^{J-1} \subset \ell(\mathbb{Z}_d)$. The* frame potential *of X is the quantity*

$$\mathrm{FP}(X) = \sum_{j,k=0}^{J-1} |\langle x_j, x_k \rangle|^2. \tag{2}$$

Following this characterization of tight frames of unit vectors, Casazza et al. [3] examined whether or not a similar result could hold for sequences of vectors with unequal norms. They found that if a sequence of vectors comprises a tight frame then, necessarily, the corresponding lengths of the vectors must satisfy the so-called *fundamental frame inequality*, cf. (3) below. Moreover, they also proved that under the restriction to sequences of vectors whose lengths satisfy the fundamental frame inequality, the local minimizers of the frame potential again provide a complete description of the tight frames. These results are collected below as Theorem 2; however, the reader is referred to [3] for further results as well as a detailed discussion of the physical interpretation of these findings.

Theorem 2 ([3]) *Let $\{a_j\}_{j=0}^{J-1} \subset \mathbb{R}$ be such that $a_0 \geq a_1 \geq \cdots \geq a_{J-1} > 0$. Let $d \leq J$ be a positive integer and denote by j_0 the smallest index $0 \leq j \leq d-1$ such that*

$$(d-j)a_j^2 \leq \sum_{m=j}^{J-1} a_m^2. \tag{3}$$

If $\{x_j\}_{j=0}^{J-1} \subset \ell(\mathbb{Z}_d)$ is a local minimizer of the frame potential over the set

$$\mathcal{A} = \{\{x_j\}_{j=0}^{J-1} \subset \ell(\mathbb{Z}_d) : \|x_j\|^2 = a_j^2, \quad 0 \leq j \leq J-1\},$$

then the collection $\{x_j\}_{j=0}^{J-1}$ may be divided into two mutually orthogonal subcollections: $\{x_j\}_{j=0}^{j_0-1}$, which consists of j_0 mutually orthogonal vectors, and $\{x_j\}_{j=j_0}^{J-1}$, which is a tight frame for its $(d-j_0)$-dimensional span. In particular, if $j_0 = 0$ then $\{x_j\}_{j=0}^{J-1}$ is a tight frame for $\ell(\mathbb{Z}_d)$.

Remark 3 *If $X := \{x_j\}_{j=0}^{J-1}$ is a local minimizer over $\mathcal{A}$ as in Theorem 2 then it follows that each x_j is an eigenvector of the associated frame operator S_X. Moreover, if $0 \leq j \leq j_0 - 1$ then the eigenvalue of x_j is $\|x_j\|^2$, whereas if $j_0 \leq j \leq J-1$ then the eigenvalue of x_j must be $\frac{1}{d-j_0}\sum_{j=j_0}^{J-1}\|x_j\|^2$. Similar reasoning leads to the following expressions for the frame potential of X:*

$$\mathrm{FP}(X) = \sum_{j=0}^{j_0-1} \mathrm{FP}(\{x_j\}) + \mathrm{FP}(\{x_j\}_{j=j_0}^{J-1}) = \sum_{j=0}^{j_0-1} a_j^4 + \frac{1}{d-j_0}\Big(\sum_{j=j_0}^{J-1} a_j^2\Big)^2.$$

The last expression implies that all local minimizers of the frame potential over $\mathcal{A}$ have the same frame potential, i.e., local minimizers are also global minimizers. Further explanation of these observations may be found in [3].

Finally, it is elementary to prove that if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ are the eigenvalues of S_X (listed according to multiplicity) then

$$\sum_{n=1}^{d} \lambda_n = \sum_{j=0}^{J-1} \|x_j\|^2.$$

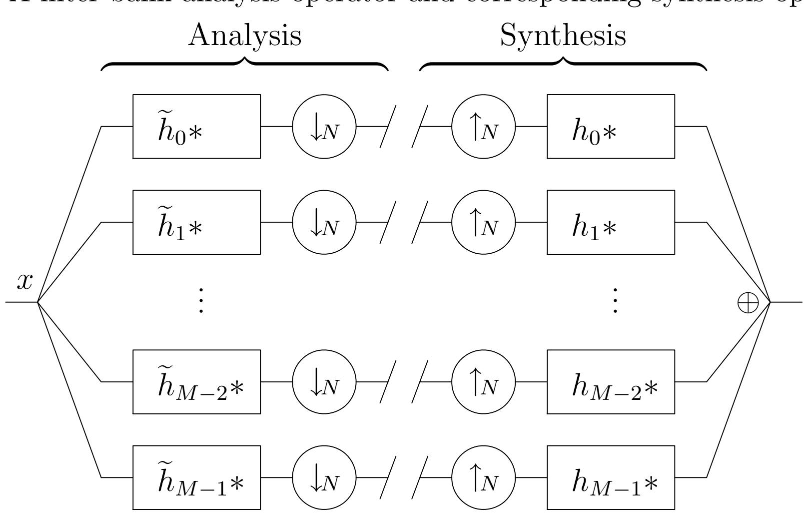
In addition to [1] and [3], there have been many other recent works devoted to the study of finite-dimensional frames [4–6,8]. One recurring theme in these works has been the careful attention paid to tight frames with additional structure. For example, in [6] a study of *ellipsoidal* tight frames was conducted, while in [8] various notions of symmetry were described for tight frames. It is therefore natural to ask whether the frame potential can still be used to characterize tight frames under the restriction to collections with a given structure. One specific structure that has found great use in applications is that of a filter bank. The main goal of this work is to provide a characterization of filter bank tight frames in terms of the frame potential analogous to Theorem 2.

Let $h_0, h_1, \ldots, h_{M-1} \in \ell(\mathbb{Z}_d)$ and consider the translation-invariant system $\{T_k h_m : k \in N\mathbb{Z}_d, 0 \leq m \leq M-1\}$ where $N \mid d$ is a positive integer. The frame operator of this collection, S, acts on $x \in \ell(\mathbb{Z}_d)$ by

$$\begin{aligned} Sx(\ell) &= \sum_{m=0}^{M-1} \sum_{k \in N\mathbb{Z}_d} \langle x, T_k h_m \rangle T_k h_m(\ell) \\ &= \sum_{m=0}^{M-1} \sum_{k \in N\mathbb{Z}_d} (x * \tilde{h}_m)(k)\, h_m(\ell - k) \\ &= \sum_{m=0}^{M-1} (\, \uparrow\downarrow_N (x * \tilde{h}_m) * h_m)(\ell),\ \ \ell \in \mathbb{Z}_d. \end{aligned}$$

In this sense, S may be regarded as arising from a convolutional system. It may also be thought of as a *filter bank frame operator*, induced by the *filters* $\{h_m\}_{m=0}^{M-1}$ with downsampling by N. A block-diagram representation of the filter bank frame operator is given as Figure 1. The latter expression for S above will be exploited further in the next section.

Fig. 1. A filter bank analysis operator and corresponding synthesis operator.



Definition 4 *Let $\{h_m\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_d)$ and suppose N and d are positive integers*

such that $N \mid d$. *The collection*

$$H_N(\{h_m\}_{m=0}^{M-1}) := \{T_k h_m : k \in N\mathbb{Z}_d, 0 \leq m \leq M-1\}$$

will be referred to as the convolutional system *generated by* $\{h_m\}_{m=0}^{M-1}$ *with downsampling* N.

Remark 5 *Let* $\mathcal{H} := \{h_m\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_d)$ *and suppose* $N \mid d$ *and denote the frame operator of* $H := H_N(\mathcal{H})$ *by* S_H. *Observe that*

$$S_H = \sum_{k \in N\mathbb{Z}_d} T_k S_{\mathcal{H}} T_{-k},$$

which implies that the matrix representing S_H *is a sum of diagonal shifts of the matrix representing* $S_{\mathcal{H}}$. *Many familiar examples of frames may be realized through this simple relation.*

(a) *Let* $d = 2^p$ *(p a positive integer) and* $N = 2$. *Define* $\mathcal{H} = \{h_0, h_1\} \subset \ell(\mathbb{Z}_d)$ *by* $h_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$ *and* $h_1 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0)$. *The matrix representation of* $S_{\mathcal{H}}$ *is zero everywhere except the first two diagonal entries, which are equal to* 1. *It is easy to see that this leads to* $S_H = I_d$. *In this case* $H_N(\mathcal{H})$ *corresponds to the Haar orthonormal basis for* $\ell(\mathbb{Z}_d)$.

(b) *Let* $d = 2^p$ *(p a positive integer) and* $N = 2$. *In this case, define* $\mathcal{H} = \{h_0, h_1, h_2\} \subset \ell(\mathbb{Z}_d)$ *by* $h_0 = (1, 0, 0, \ldots, 0)$, $h_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \ldots, 0)$, *and* $h_2 = (-\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0, \ldots, 0)$. *Again the matrix representation of* $S_{\mathcal{H}}$ *is zero everywhere except the first two diagonal entries, which in this case are equal to* $\frac{3}{2}$. *This leads to* $S_H = \frac{3}{2} I_d$, *which implies that* $H_N(\mathcal{H})$ *is a* $\frac{3}{2}$*-tight frame.*

The main result of this article is an analog to Theorem 2 characterizing convolutional tight frames in terms of the restriction of the frame potential to convolutional systems.

Theorem 6 *Let* $\{a_m\}_{m=0}^{M-1} \subset \mathbb{R}$ *be such that* $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$. *Let* d *and* N *be positive integers such that* $N \mid d$ *and* $N \leq M$. *Denote by* m_0 *the smallest index* $0 \leq m \leq N-1$ *such that*

$$(N - m)a_m^2 \leq \sum_{j=m}^{M-1} a_j^2. \tag{4}$$

If $H_N(\{h_m\}_{m=0}^{M-1}) \subset \ell(\mathbb{Z}_d)$ *is a local minimizer of the frame potential over the set*

$$\mathcal{A} = \{\{h_m\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_d) : \|h_m\|^2 = a_m^2, \quad 0 \leq m \leq M-1\},$$

then $H_N(\{h_m\}_{m=0}^{M-1})$ may be divided into two mutually orthogonal subcollections: $H_N(\{h_m\}_{m=0}^{m_0-1})$, which consists entirely of mutually orthogonal vectors, and $H_N(\{h_m\}_{m=m_0}^{M-1})$, which is a tight frame for its $\frac{d}{N}(N-m_0)$-dimensional span. In particular, if $m_0 = 0$ then $H_N(\{h_m\}_{m=0}^{M-1})$ is a tight frame for $\ell(\mathbb{Z}_d)$.

Remark 7 *A few remarks about the main theorem are in order.*

(a) The frame operator S_H satisfies $S_H T_n = T_n S_H$ for $n \in N\mathbb{Z}_d$:

$$S_H T_n x = T_n \sum_{m'=0}^{M-1} \sum_{k \in N\mathbb{Z}_d} \langle T_n x, T_k h_{m'} \rangle T_{k-n} h_{m'} = T_n S_H x.$$

(b) It is well known that if $\{f_n\}_n$ is a frame, then f_j is an eigenvector of the frame operator, S, with eigenvalue $\|f_j\|^2$ if and only if $f_j \perp f_k$, for all $k \neq j$. This follows from the equation

$$\|f_j\|^4 = \langle S f_j, f_j \rangle = \sum_n |\langle f_j, f_n \rangle|^2,$$

since the $n = j$ term in the sum at right already equals $\|f_j\|^4$ and, therefore, the remaining terms must vanish.

These two observations explain why, in the proof of Theorem 6, the family H_N is being split into groups consisting of $\frac{d}{N}$ vectors. Namely, if h_m is orthogonal to the remaining vectors, then h_m is an eigenvector of the frame operator, S_H, with eigenvalue $\|h_m\|^2$. By (a), above, each $T_n h_m$, $n \in N\mathbb{Z}_d$, is also an eigenvector of S_H with eigenvalue $\|T_n h_m\|^2 = \|h_m\|^2$ and, therefore, by (b) is also orthogonal to the remaining vectors in the collection.

The remainder of this article is devoted to building the machinery necessary to prove Theorem 6. Section 3 deals with the modulated filter representation of convolutional systems, which allows questions about the frame properties of convolutional systems to be examined in terms of associated non-convolutional systems via the action of the Fourier transform. Section 4 is devoted to the proof of Theorem 6, which relies heavily on the insight obtained through the modulated filter representation.

3 The modulated filter representation

As above, let N and d be positive integers such that $N \mid d$. Fix a sequence of real numbers, $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$ and consider the family of systems of the form $H := H_N(\{h_m\}_{m=0}^{M-1})$ where each filter $h_m \in \ell(\mathbb{Z}_d)$ satisfies $\|h_m\| = a_m$. In light of the observations preceding Definition 4, the synthesis operator

L^* associated to such a collection acts on a sequence $\oplus_m y_m \in \bigoplus_{m=0}^{M-1} \ell(\mathbb{Z}_{\frac{d}{N}})$ by

$$L^*(\oplus_m y_m) = \sum_{m=0}^{M-1} (\uparrow_N y_m) * h_m.$$

This realization of L^* does not impose a strict ordering on the vectors in H, but does associate $y_m(k)$ to $T_{kN} h_m$ in the linear combination given by L^*. Under the action of the Fourier transform upsampling becomes periodic extension, i.e.,

$$(\uparrow_N y_m)^\wedge(k) = \frac{1}{\sqrt{N}} \widehat{y}_m(k). \tag{5}$$

The reader should note that there are two different Fourier transforms used in (5) and, since the Fourier transform on the right hand side is defined only on $\ell(\mathbb{Z}_{\frac{d}{N}})$, there is an abuse of notation in (5) requiring one to consider its periodic extension to $\ell(\mathbb{Z}_d)$. One may verify that the Fourier transform of the synthesized signal is given by

$$\mathcal{F}_d L^*(\oplus_m y_m)(k) = \sqrt{\frac{d}{N}} \sum_{m=0}^{M-1} \widehat{h}_m(k) \widehat{y}_m(k), \ k \in \mathbb{Z}_d.$$

The component functions $\hat{y}_m$ are (d/N)-periodic, hence for any particular indices,

$$\mathcal{F}_d L^*(\oplus_m y_m)(k+\frac{nd}{N}) = \sqrt{\frac{d}{N}} \sum_{m=0}^{M-1} \widehat{h}_m(k+\frac{nd}{N}) \widehat{y}_m(k). \tag{6}$$

For any $k \in \mathbb{Z}_d$, stacking the N versions of (6) for $0 \leq n \leq N-1$ in the form of a matrix yields,

$$\begin{bmatrix} \mathcal{F}_d L^*(\oplus_m y_m)(k+\frac{0d}{N}) \\ \vdots \\ \mathcal{F}_d L^*(\oplus_m y_m)(k+\frac{(N-1)d}{N}) \end{bmatrix} = H^*_{\text{mod}}(k) \begin{bmatrix} \widehat{y}_0(k) \\ \vdots \\ \widehat{y}_{M-1}(k) \end{bmatrix}, \tag{7}$$

where $H^*_{\text{mod}}(k)$ is the $N \times M$ *adjoint modulated filter matrix*,

$$H^*_{\text{mod}}(k) = \sqrt{\frac{d}{N}} \begin{bmatrix} \widehat{h}_0(k+\frac{0d}{N}) & \ldots & \widehat{h}_{M-1}(k+\frac{0d}{N}) \\ \vdots & & \vdots \\ \widehat{h}_0(k+\frac{(N-1)d}{N}) & \ldots & \widehat{h}_{M-1}(k+\frac{(N-1)d}{N}) \end{bmatrix}. \tag{8}$$

That is, $H^*_{\rm mod}(k)$ is the $N \times M$ matrix whose (n, m)th entry is,

$$H^*_{\rm mod}(k)(n,m) = \sqrt{\frac{d}{N}}\,\widehat{h}_m\Big(k + \frac{nd}{N}\Big). \tag{9}$$

Stacking the d/N matrix-vector equations (7) that correspond to $0 \leq k \leq d/N - 1$, results in the $d \times (Md/N)$ block matrix-vector equation,

$$\left[\begin{array}{c}
\mathcal{F}_d L^*(\oplus_m y_m)(0+\frac{0d}{N})\\
\vdots\\
\mathcal{F}_d L^*(\oplus_m y_m)(0+\frac{(N-1)d}{N})\\ \hline
\mathcal{F}_d L^*(\oplus_m y_m)(1+\frac{0d}{N})\\
\vdots\\
\mathcal{F}_d L^*(\oplus_m y_m)(1+\frac{(N-1)d}{N})\\ \hline
\\
\vdots\\
\\ \hline
\mathcal{F}_d L^*(\oplus_m y_m)(\frac{d}{N}-1+\frac{0d}{N})\\
\vdots\\
\mathcal{F}_d L^*(\oplus_m y_m)(\frac{d}{N}-1+\frac{(N-1)d}{N})
\end{array}\right]
=
\left[\begin{array}{cccc}
H^*_{\rm mod}(0) & 0 & \dots & 0\\
0 & H^*_{\rm mod}(1) & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & H^*_{\rm mod}(\frac{d}{N}-1)
\end{array}\right]
\left[\begin{array}{c}
\widehat{y}_0(0)\\
\vdots\\
\widehat{y}_{M-1}(0)\\ \hline
\widehat{y}_0(1)\\
\vdots\\
\widehat{y}_{M-1}(1)\\ \hline
\\
\vdots\\
\\ \hline
\widehat{y}_0(\frac{d}{N}-1)\\
\vdots\\
\widehat{y}_{M-1}(\frac{d}{N}-1)
\end{array}\right].$$

Observe that the vectors in this equation on the left and right contain all the values of $\mathcal{F}_d L^*(\oplus_m y_m)$ and $\oplus_m \widehat{y}_m$, respectively, albeit in a permuted order. Thus, through the appropriate use of Fourier transforms and permutations, the synthesis operator of $H_N(\{h_m\}_{m=0}^{M-1})$ may be related to the *block adjoint modulated filter matrix* $H^*_{\rm mod}$ whose $(Nk{+}n, Mk{+}m)$th entry is given by,

$$H^*_{\rm mod}(Nk{+}n, Mk{+}m) = \sqrt{\frac{d}{N}}\,\widehat{h}_m\Big(k{+}\frac{nd}{N}\Big),$$

for all $k = 0, \dots, (d/N) - 1$ and $n = 0, \dots, N-1$, with the remaining entries all being 0.

The reordering of terms involved in the above factorization is formally described as a *perfect shuffle*, as noted in Strohmer's work on Gabor frames [7]. Given any positive integers $N \,|\, d$, the *mod N perfect shuffle* operator is

$$P_{N,d} : \ell(\mathbb{Z}_d) \to \ell(\mathbb{Z}_d), \quad (P_{N,d} f)(Nk{+}n) = f\Big(k + \frac{nd}{N}\Big),$$

where the indices are restricted to $k = 0, \dots, (d/N)–1$ and $n = 0, \dots, N–1$. For example, if $d = 15$ and $N = 3$, the effect of the mod 3 perfect shuffle $P_{3,15}$ upon the identity function $f \in \ell(\mathbb{Z}_{15})$ defined by $f(k) = k$, for all $k = 0, \ldots, 14$, is summarized in the following table:

$f(k)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$P_{3,15}f(k)$	0	5	10	1	6	11	2	7	12	3	8	13	4	9	14

Any such shuffle operator is clearly unitary, with $P^*_{N,d} = P^{-1}_{N,d} = P_{d/N,d}$. This leads to a formal factorization for the synthesis operator. For notational convenience, let $\mathcal{F}^M_{d/N}$ denote the direct sum of M copies of the discrete Fourier transform of size d/N.

Theorem 8 *Let L be the analysis operator associated to $H_N(\{h_m\}_{m=0}^{M-1})$ and let H_{mod} be the corresponding modulated filter operator. Then,*

$$L = (P_{M,Md/N}\mathcal{F}^M_{d/N})^* \, H_{\mathrm{mod}} \, (P_{N,d}\, \mathcal{F}_d).$$

PROOF. Let $\oplus_m y_m \in \bigoplus_{m=0}^{M-1} \ell(\mathbb{Z}_{d/N})$. By combining the definition of the shuffle $P_{N,d}$ with equation (6) and the definition of H^*_{mod}, one obtains

$$\begin{aligned} P_{N,d}\mathcal{F}_d L^*(\oplus_m y_m)(Nk+n) &= \mathcal{F}_d L^*(\oplus_m y_m)(k+\frac{nd}{N}) \\ &= \sqrt{\frac{d}{N}}\sum_{m=0}^{M-1} \widehat{h}_m(k+\frac{nd}{N})\widehat{y}_m(k) \\ &= \sum_{m=0}^{M-1} H^*_{\mathrm{mod}}(Nk+n, Mk+m)\widehat{y}_m(k), \end{aligned} \tag{10}$$

for all $k = 0, \ldots, d/N-1$, and all $n = 0, \ldots, N-1$.

Now observe that the $(Mk+m)$th entry of the standard column vector representation of

$$P_{M,Md/N}\, \mathcal{F}^M_{d/N}(\oplus_m y_m) \equiv P_{M,Md/N}(\oplus_m \widehat{y}_m)$$

may be obtained by extracting the kth entry of the mth block of $\oplus_m \widehat{y}_m$, namely the kth entry of $\widehat{y}_m$:

$$(P_{M,Md/N}\, \mathcal{F}^M_{d/N}(\oplus_m y_m))(Mk+m) = \oplus_m \widehat{y}_m(k+\frac{md}{N}).$$

In this light, the right hand side of (10) becomes the expression of one term of a matrix-vector product,

$$\sum_{m=0}^{M-1} H^*_{\text{mod}}(Nk+n, Mk+m)\ (P_{M,Md/N}\, \mathcal{F}^M_{d/N}(\oplus_m y_m))(Mk+m)$$
$$= (H^*_{\text{mod}} P_{M,Md/N}\, \mathcal{F}^M_{d/N}(\oplus_m y_m))(Nk+n).$$

As this holds for all $k = 0, \ldots, d/N-1$, and $n = 0, \ldots, N-1$, then,

$$P_{N,d}\mathcal{F}_d L^*(\oplus_m y_m) = H^*_{\text{mod}} P_{M,Md/N}\, \mathcal{F}^M_{d/N}(\oplus_m y_m). \qquad \square$$

In essence, the modulated filter representation decomposes $\ell(\mathbb{Z}_d)$ into the direct sum of $\frac{d}{N}$ copies of $\ell(\mathbb{Z}_N)$. In fact, the product $H^*_{\text{mod}} H_{\text{mod}}$ may be realized as the tensor product of frame operators acting on the respective copies of $\ell(\mathbb{Z}_N)$. Consider the $\frac{d}{N}$ collections

$$X_j = \{x_{m,j}\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_N), \quad 0 \le j \le \frac{d}{N} - 1, \tag{11}$$

where $x_{m,j}(n) := \sqrt{\frac{d}{N}}\hat{h}_m(j + \frac{nd}{N})$, $0 \le n \le N-1$. If we denote the frame operator of $X := \cup_j X_j$ by S_X, then it is apparent that

$$S_X = S_{X_0} \oplus S_{X_1} \oplus \cdots \oplus S_{X_{\frac{d}{N}-1}} = H^*_{\text{mod}} H_{\text{mod}}.$$

Theorem 8 implies that S_H, the frame operator of $H_N(\{h_m\}_{m=0}^{M-1})$, is unitarily equivalent to the block-diagonal frame operator S_X, associated with the collections X_j, $0 \le j \le \frac{d}{N} - 1$. Hence, frame-related computations involving, for instance, the eigenvalues of S_H, may be performed by computing the corresponding quantities for the collections X_j. This statement is made precise by the following corollary to Theorem 8.

Corollary 9 *Let $\{h_m\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_d)$ and let N and d be positive integers such that $N \mid d$. Then, defining the collections X_j, $0 \le j \le \frac{d}{N} - 1$, as in (11),*

- *(i) the frame bounds for $H_N(\{h_m\}_{m=0}^{M-1})$ are the minimum of the lower frame bounds and the maximum of the upper frame bounds for the collections X_j,*
- *(ii) $H_N(\{h_m\}_{m=0}^{M-1})$ is a tight frame for $\ell(\mathbb{Z}_d)$ if and only if for all j, X_j is a tight frame for $\ell(\mathbb{Z}_N)$ of common frame bound,*
- *(iii) the squares of the Hilbert-Schmidt norms of the analysis, synthesis, frame and Gram operators for the collection $H_N(\{h_m\}_{m=0}^{M-1})$ are equal to the sums of the squares of the Hilbert-Schmidt norms of the corresponding operators for the collections X_j.*

Remark 10 *In the $\ell^2(\mathbb{Z})$ setting, part (i) of Corollary 9 was known by Bölcskei, Hlawatsch and Feichtinger [2] in the context of the polyphase representation,* *while part (ii) was observed independently in both [2] and [4] and later used in [5]. It should be noted that Vetterli considered filter banks over finite fields in [9].*

Remark 11 *Corollary 9 suggests a natural approach for constructing convolutional tight frames that deserves brief mention here. Suppose a convolutional frame with M filters is desired for $\ell(\mathbb{Z}_d)$ with downsampling by $N \mid d$ under the constraint that $\|h_m\| = a_m$ with $a_0 \geq a_2 \cdots \geq a_{M-1} > 0$. Provided that*

$$Na_0^2 \leq \sum_{m=0}^{M-1} a_m^2,$$

Theorem 2 guarantees the existence of a tight frame for $\ell(\mathbb{Z}_N)$, $X = \{x_m\}_{m=0}^{M-1}$, where $\|x_m\|^2 = a_m^2$. One can construct the desired convolutional tight frame by filling in the columns of $H_{\text{mod}}^*(k)$ with the coordinates of the corresponding vector from X and computing the associated filters $\{h_m\}_{m=0}^{M-1}$.

4 Proof of Theorem 6

Again let N and d be positive integers where $N \mid d$. Fix a sequence of real numbers, $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$ and consider the family of systems of the form $H := H_N(\{h_m\}_{m=0}^{M-1})$ where each filter $h_m \in \ell(\mathbb{Z}_d)$ satisfies $\|h_m\| = a_m$. One important consequence of the modulated filter representation and, in particular, of Corollary 9 is the fact that the local minimizers of the frame potential over this family of convolutional systems are in direct correspondence with the local minimizers of the sum of the frame potentials over the family of systems of the form $\{X_j\}_{j=0}^{\frac{d}{N}-1}$ where each collection X_j is defined according to (11) and is regarded as a subset of an independent copy of $\ell(\mathbb{Z}_N)$. Through this correspondence, the constraints on the filter lengths, i.e., $\|h_m\| = a_m$, imply that

$$\sum_{j=0}^{\frac{d}{N}-1} \|x_{m,j}\|^2 = \sum_{j \in N\mathbb{Z}_d} \|T_j h_m\|^2 = \frac{d}{N} a_m^2, \quad 0 \leq m \leq M-1.$$

Moreover, the set of eigenvalues of the frame operator S_H is identical to the union of the sets of eigenvalues of the frame operators S_{X_j}, $0 \leq j \leq \frac{d}{N} - 1$. Thus, one may derive Theorem 6 from the following result.

Theorem 12 *Let $\{a_m\}_{m=0}^{M-1} \subset \mathbb{R}$ be such that $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$. Let d and N be positive integers such that $N \mid d$ and $N \leq M$. Denote by m_0 the* *smallest index* $0 \le m \le N-1$ *such that*

$$(N-m)a_m^2 \le \sum_{j=m}^{M-1} a_j^2. \tag{12}$$

If the collections $X_j := \{x_{m,j}\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_N)$ *form a local minimizer of the combined frame potential,* $\sum_{j=0}^{\frac{d}{N}-1} \mathrm{FP}(X_j)$*, under the constraint that*

$$\sum_{j=0}^{\frac{d}{N}-1} \|x_{m,j}\|^2 = \frac{d}{N} a_m^2, \quad 0 \le m \le M-1,$$

then each collection X_j *may be divided into two mutually orthogonal subcollections of* $\ell(\mathbb{Z}_N)$*:* $\{x_{m,j}\}_{m=0}^{m_0-1}$*, which consists of mutually orthogonal, nonzero vectors, and* $\{x_{m,j}\}_{m=m_0}^{M-1}$*, which is a tight frame for its* $(N-m_0)$*-dimensional span. Moreover, for each* j *the norms of the vectors of* X_j *must satisfy* $\|x_{m,j}\| = a_m$ *for* $0 \le m \le m_0 - 1$ *and* $\sum_{m=m_0}^{M-1} \|x_{m,j}\|^2 = \sum_{m=m_0}^{M-1} a_m^2$*. In the event that* $m_0 = 0$ *each collection* X_j*,* $0 \le j \le \frac{d}{N} - 1$*, is a tight frame for* $\ell(\mathbb{Z}_N)$ *with a common frame bound.*

The following technical lemmas will be used frequently in the proof of Theorem 12.

Lemma 13 *Let* $X_j = \{x_{m,j}\}_{m=0}^{M-1}$*,* $0 \le j \le \frac{d}{N} - 1$*. Suppose that* x_{m_0,j_1} *is an eigenvector of* $S_{X_{j_1}}$ *with eigenvalue* $\lambda_1 \ne 0$ *and that* $\|x_{m_0,j_2}\| = 0$*. Also suppose that there exists a unit eigenvector* u *of* $S_{X_{j_2}}$ *with eigenvalue* λ_2*. Define* $X_j^\varepsilon = \{x_{m,j}^\varepsilon\}_{m=0}^{M-1}$ *by*

$$x_{m,j}^\varepsilon = \begin{cases} ((1 - \frac{\varepsilon}{\|x_{m_0,j_1}\|^2})^{\frac{1}{2}} x_{m_0,j_1}, & (m,j) = (m_0,j_1), \\ \sqrt{\varepsilon} u, & (m,j) = (m_0,j_2), \\ x_{m,j}, & \text{otherwise.} \end{cases}$$

Then $P(\varepsilon) := \sum_{j=0}^{\frac{d}{N}-1} \mathrm{FP}(X_j^\varepsilon)$ *satisfies* $P'(\varepsilon) = 4\varepsilon + 2(\lambda_2 - \lambda_1)$*.*

PROOF. The only terms in the expression for $P(\varepsilon)$ that actually depend on ε are $\mathrm{FP}(X_{j_1}^\varepsilon)$ and $\mathrm{FP}(X_{j_2}^\varepsilon)$. Let $\alpha := \|x_{m_0,j_1}\|$ and observe that

$$\mathrm{FP}(X_{j_1}^\varepsilon) = (1 - \frac{\varepsilon}{\alpha^2})^2 \alpha^4 + 2(1 - \frac{\varepsilon}{\alpha^2}) \sum_{m=0, m\ne m_0}^{M-1} |\langle x_{m_0,j_1}, x_{m,j_1}\rangle|^2$$

$$+\sum_{m,n=0,m,n\neq m_0}^{M-1} |\langle x_{m,j_1}, x_{n,j_1}\rangle|^2,$$

while $\mathrm{FP}(X_{j_2}^{\varepsilon})$ is given by

$$\mathrm{FP}(X_{j_2}^{\varepsilon}) = \varepsilon^2 + 2\varepsilon \sum_{m=0,m\neq m_0}^{M-1} |\langle u, x_{m,j_2}\rangle|^2 + \sum_{m,n=0,m,n\neq m_0}^{M-1} |\langle x_{m,j_2}, x_{n,j_2}\rangle|^2.$$

By hypothesis $\|S_{X_{j_1}} x_{m_0,j_1}\|^2 = \lambda_1 \|x_{m_0,j_1}\|^2 = \lambda_1 \alpha^2$ and $\|S_{X_{j_2}} u\|^2 = \lambda_2 \|u\|^2$, from which one may deduce that $P'(\varepsilon) = 4\varepsilon + 2(\lambda_2 - \lambda_1)$ after differentiating the above expressions and appropriately interpreting the resulting terms. □

Lemma 14 *Let $X_j = \{x_{m,j}\}_{m=0}^{M-1}$, $0 \leq j \leq \frac{d}{N} - 1$. Suppose that x_{m_0,j_1} and x_{m_0,j_2} are non-zero eigenvectors of $S_{X_{j_1}}$ and $S_{X_{j_2}}$ with eigenvalues λ_1 and λ_2, respectively. Define $X_j^{\varepsilon} = \{x_{m,j}^{\varepsilon}\}_{m=0}^{M-1}$ by*

$$x_{m,j}^{\varepsilon} = \begin{cases} (1 - \frac{\varepsilon}{\|x_{m_0,j_1}\|^2})^{\frac{1}{2}} x_{m_0,j_1}, & (m,j) = (m_0, j_1), \\ \left(1 + \frac{\varepsilon}{\|x_{m_0,j_2}\|^2}\right)^{\frac{1}{2}} x_{m_0,j_2}, & (m,j) = (m_0, j_2), \\ x_{m,j}, & \text{otherwise.} \end{cases}$$

Then $P(\varepsilon) := \sum_{j=0}^{\frac{d}{N}-1} \mathrm{FP}(X_j^{\varepsilon})$ satisfies $P'(\varepsilon) = 4\varepsilon + 2(\lambda_2 - \lambda_1)$.

PROOF. The proof is analogous to that of Lemma 13 and the details are left to the reader. □

The proof of Theorem 12 will be accomplished through a sequence of steps, relying mainly on Theorem 2, Lemma 13, and Lemma 14.

Proof of Theorem 12 Assume that the collections X_j form a local minimizer of the combined frame potential, as described in the statement of the theorem. Then each collection X_j may be regarded as a local minimizer of $\mathrm{FP}(X_j)$ over the family of collections in $\ell(\mathbb{Z}_{N_j})$ with norms prescribed by those of X_j and where N_j is the minimum of N and the number of nonzero vectors in X_j. It will be shown below that $N_j = N$, but this is not clear a priori. In any case, X_j may be decomposed using Theorem 2 and each vector $x_{m,j}$ will be an eigenvector of the associated frame operator S_{X_j}. The presence of zero-norm vectors in X_j has no effect on the conclusion of Theorem 2. These facts will be used below.

1. *Each collection X_j is a frame for $\ell(\mathbb{Z}_N)$.* Assume by way of contradiction that X_{j_2} is not a frame for $\ell(\mathbb{Z}_N)$. It follows from Theorem 2 that if X_j consists of at least N nonzero vectors then it must be a frame for $\ell(\mathbb{Z}_N)$. Therefore, the contradiction hypothesis implies that X_{j_2} contains strictly fewer than than N nonzero vectors. Without loss of generality assume that $x_{m_1,j_2} = 0$. Since X_{j_2} is not a frame for $\ell(\mathbb{Z}_N)$ there exists $u \in \ell(\mathbb{Z}_N)$ such that u is orthogonal to X_{j_2}, i.e., u is a 0-eigenvector of $S_{X_{j_2}}$. Now observe that the constraints on the collections X_j require that at least one of $\{x_{m_1,j}\}_{0\leq j\leq \frac{d}{N}-1}$ is nonzero, say x_{m_1,j_1} where necessarily $j_1 \neq j_2$. As remarked above, each vector $x_{m,j}$ is an eigenvector of S_{X_j}, so Lemma 13 may be applied to x_{m_1,j_1},x_{m_1,j_2}, and u with $\lambda_2 = 0$ and $\lambda_1 > 0$. By considering sufficiently small ε this leads to a contradiction of the minimality of the combined frame potential.
2. *Nonzero siblings have identical eigenvalues.* The collection $\{x_{m,j}\}_{0\leq j\leq \frac{d}{N}-1}$ will be referred to as a collection of *siblings*, because these vectors are related to one another through the length constraints of the theorem. Suppose x_{m,j_1} and x_{m,j_2} are any two nonzero siblings. Applying Lemma 14 one finds that $P'(\varepsilon) = 4\varepsilon + 2(\lambda_2 - \lambda_1)$, where λ_1, λ_2 are the eigenvalues of x_{m_1,j_1} and x_{m_1,j_2}, respectively. If $\lambda_1 \neq \lambda_2$ then $|P'(\varepsilon)| > 0$ for sufficiently small ε, contradicting the minimality of the combined frame potential.
3. *The distribution of squared-norm is uniform across the collections X_j, i.e.,*

   $$\sum_{m=0}^{M-1} \|x_{m,j}\|^2 = C,$$

   where $C > 0$ is independent of j. Assume by contradiction that there exist $j_1 \neq j_2$ such that

   $$\sum_{m=0}^{M-1} \|x_{m,j_1}\|^2 > \sum_{m=0}^{M-1} \|x_{m,j_2}\|^2.$$

   List the eigenvalues of $S_{X_{j_1}}$ and $S_{X_{j_2}}$ according to multiplicity, $S_{X_{j_1}}$: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N > 0$ and $S_{X_{j_2}}$: $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N > 0$. The contradiction hypothesis implies that $\sum_{n=1}^{N} \lambda_n > \sum_{n=1}^{N} \gamma_n$, cf. Remark 3. Thus, there exists n such that $\lambda_n > \gamma_n$. Let n_0 be the smallest index for which $\lambda_{n_0} > \gamma_{n_0}$. In order to derive a contradiction it will be shown that there is a λ_{n_0}-eigenvector x_{m_1,j_1} such that $\|x_{m_1,j_2}\| = 0$. In this case one may apply Lemma 13 to show that the combined potential cannot be a local minimum.

   If λ_{n_0} is not an eigenvalue of $S_{X_{j_2}}$ then the existence of such an eigenvector is immediate. If λ_{n_0} is an eigenvalue of $S_{X_{j_2}}$ then one concludes that $\gamma_{n_0-1} = \lambda_{n_0}$, but since the eigenvalues are listed in decreasing order this implies that $\lambda_{n_0-1} = \lambda_{n_0}$ as well. Similar reasoning leads to the conclusion that if $\gamma_{n_0-k} = \lambda_{n_0}$ then so must λ_{n_0-k}, $1 \leq k \leq n_0 - 1$.

This means that the multiplicity of the eigenvalue λ_{n_0} is strictly greater for $S_{X_{j_1}}$ than it is for $S_{X_{j_2}}$ and, hence, there is at least one more λ_{n_0}-eigenvector among the collection X_{j_1} than among X_{j_2}. Moreover, the λ_{n_0}-eigenvectors of $S_{X_{j_2}}$ must be linearly independent because $\lambda_{n_0} > \gamma_N$. It follows from dimensional considerations that there exists some x_{m_1,j_1} which is a λ_{n_0}-eigenvector whose sibling $x_{m_1,j_2} = 0$.

4. *The index set $\mathcal{M} := \{0, 1, \dots, M-1\}$ may be partitioned into two subsets, $\mathcal{M}_1$ and $\mathcal{M}_2$, so that for each $0 \le j \le \frac{d}{N} - 1$ the collection $\{x_{m,j}\}_{m\in\mathcal{M}_1}$ consists of mutually orthogonal nonzero vectors and is orthogonal to $\{x_{m,j}\}_{m\in\mathcal{M}_2}$, which is a tight frame for its span.* It follows from Theorem 2 that such a partition, $\mathcal{M} = \mathcal{M}_{1,j} \cup \mathcal{M}_{2,j}$, exists for each j, but it remains to show that the same partition is valid for all $0 \le j \le \frac{d}{N} - 1$.

   Let x_{m_1,j_0} be the vector in $\{x_{m,j} : 0 \le j \le \frac{d}{N} - 1, m \in \mathcal{M}_{1,j}\}$ of maximum norm. It will be shown that each sibling of x_{m_1,j_0} is nonzero. Assume by way of contradiction that $x_{m_1,j_1} = 0$. If $S_{X_{j_1}}$ has no eigenvalue strictly less than $\|x_{m_1,j_0}\|^2$ then

$$\sum_{m=0}^{M-1} \|x_{m,j_0}\|^2 = \sum_{m=0}^{M-1} \|x_{m,j_1}\|^2 = \sum_{n=1}^{N} \lambda_n \ge N\|x_{m_1,j_0}\|^2,$$

   where $\{\lambda_n\}_{n=1}^N$ are the eigenvalues of $S_{X_{j_1}}$. This implies that $N\|x_{m_1,j_0}\|^2 \le \sum_{m=0}^{M-1} \|x_{m,j_0}\|^2$, which is a contradiction of the fact that $m_1 \in \mathcal{M}_{1,j_0}$. Hence, each sibling of x_{m_1,j_0} is nonzero.

   This argument may be repeated after the removal of x_{m_1,j_0} from each X_j until the remaining vectors in each X_j form a tight frame for their span. In other words, if $m \in \mathcal{M}_{1,j_0}$ then $m \in \mathcal{M}_{1,j}$ for each j and, therefore, the partition is independent of j as claimed.

5. *Let m_0 be as in the statement of the theorem. Then $\mathcal{M}_1 = \{0, 1, \dots, m_0 - 1\}$ and the norms of the vectors in X_j are as claimed.* If $m \in \mathcal{M}_1$ then the eigenvalue of $x_{m,j}$ must be $\|x_{m,j}\|^2$ for each j. Since each sibling has the same eigenvalue, the constraint on the norms of the siblings implies that $\|x_{m,j}\| = a_m$ for each j. Because $\sum_{m\in\mathcal{M}_1} \|x_{m,j}\|^2$ is then independent of j, so must $\sum_{m\in\mathcal{M}_2} \|x_{m,j}\|^2$ be. In particular, the tight frame constant of $\{x_{m,j}\}_{m\in\mathcal{M}_2}$ is also independent of j.

   In order to establish the fact that $\mathcal{M}_1 = \{0, 1, \dots, m_0 - 1\}$ first notice that Theorem 2 implies that the tight-frame constant of $\{x_{m,j}\}_{m\in\mathcal{M}_2}$ is strictly smaller than any eigenvalue associated with $x_{m,j}$, $m \in \mathcal{M}_1$. Suppose that $m_1 \in \mathcal{M}_2$, then the tight-frame constant must be greater than or equal to $\|x_{m_1,j}\|^2$ for each $0 \le j \le \frac{d}{N} - 1$. The norm constraint on the siblings $\{x_{m_1,j}\}_j$ implies that at least one of the siblings, say x_{m_1,j_0}, has norm greater than or equal to a_{m_1}. Hence the tight-frame constant is at least as big as a_{m_1}. Suppose that $m_2 \in \mathcal{M}_1$ with $m_2 > m_1$. The eigenvalue associated to each $x_{m_2,j}$ is a_{m_2} and, therefore, cannot be strictly larger than a_{m_1}, providing the desired contradiction. This shows that $\mathcal{M}_2$ must be of the form $\{m_1, m_1+1, \ldots, M-1\}$ for some $m_1 \geq m_0$. It remains to prove that $m_1 = m_0$.

Suppose that $m_1 > m_0$, then the associated tight-frame constant must be

$$\lambda := \frac{1}{D} \sum_{m=m_1}^{M-1} a_m^2,$$

where D is the dimension of the span of $\{x_{m,j}\}_{m \in \mathcal{M}_2}$. The contradiction hypothesis implies that $m_1 - 1 \geq m_0$, so

$$a_{m_1-1}^2 \leq \frac{1}{D+1} \sum_{m=m_1-1}^{M-1} a_m^2 = \frac{D\lambda + a_{m_1-1}^2}{D+1}.$$

It follows that $a_{m_1-1}^2 \leq \lambda$, which is again a contradiction of the tight-frame constant being strictly smaller than the eigenvalues associated with $\mathcal{M}_1$. Hence, $\mathcal{M}_2 = \{m_0, m_0+1, \ldots, M-1\}$, finishing the proof of this claim and the theorem. □

Proof of Theorem 6: Assume that $H := H_N(\{h_m\}_{m=0}^{M-1})$ is a local minimizer of the frame potential as described in the statement of the theorem. Define $\frac{d}{N}$ collections $X_j \subset \ell(\mathbb{Z}_N)$, $0 \leq j \leq \frac{d}{N} - 1$, by (11). By Corollary 9 (iii), it follows that the collections X_j satisfy the hypotheses of Theorem 12 with m_0 identical to that of the hypotheses of Theorem 6.

Let $S_X = \bigoplus_j S_{X_j}$ and observe by Theorem 8 that $S_H = (P_{N,d}\,\mathcal{F}_d)^* S_X (P_{N,d}\,\mathcal{F}_d)$. Hence, h_m is an eigenvector of S_H if and only if $P_{N,d}\,\mathcal{F}_d h_m$ is an eigenvector of S_X. By definition, $\bigoplus_j x_{m,j} = P_{N,d}\,\mathcal{F}_d h_m$ and since each nonzero sibling $x_{m,j}$ shares a common eigenvalue it is apparent that $\bigoplus_j x_{m,j}$ is an eigenvector of S_X as desired. If $m < m_0$, then the corresponding eigenvalue is a_m^2, while if $m \geq m_0$ then the eigenvalue is $\frac{1}{N-m_0} \sum_{m=m_0}^{M-1} a_m^2$. Finally, by Remark 7 (a), if h_m is a λ-eigenvector of S_H, then so is $T_n h_m$ for $n \in N\mathbb{Z}_d$. This completes the argument. □

5 Underdetermined Systems

The following result is the counterpart to Theorem 12 for underdetermined systems and actually follows very naturally from Theorem 12. The authors are grateful to the referee for pointing this out and supplying the current proof of Theorem 15.

Theorem 15 *Let $\{a_m\}_{m=0}^{M-1} \subset \mathbb{R}$ be such that $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$. Let d and N be positive integers such that $N \mid d$ and $N \geq M$. If the collections*

$X_j := \{x_{m,j}\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_N)$ form a local minimizer of $\sum_{j=0}^{\frac{d}{N}-1} \mathrm{FP}(X_j)$ under the constraint that $\sum_{j=0}^{\frac{d}{N}-1} \|x_{m,j}\|^2 = \frac{d}{N} a_m^2$, $0 \le m \le M-1$, then each collection X_j is an orthogonal sequence with $\|x_{m,j}\| = a_m$, $0 \le m \le M-1$.

PROOF. It is sufficient to prove the result for $N > M$, since $N = M$ is included in Theorem 12. By way of contradiction, fix $\{x_{m,j}\}_{m,j}$, $0 \le m \le M-1$, $0 \le j \le \frac{d}{N} - 1$, which is a local minimizer of the combined frame potential under the imposed constraint, but for which X_j is not an orthogonal sequence for some j, say j_0.

Observe that for each $0 \le m \le M-1$,

$$(N-m)a_m^2 > (M-m)a_m^2 \ge \sum_{j=m}^{M-1} a_j^2.$$

Let $\varepsilon > 0$ be chosen so that if $a_M = \cdots = a_{N-1} = \varepsilon$,

$$(N-m)a_m^2 > \sum_{j=m}^{N-1} a_j^2, \quad 0 \le m \le M-1.$$

Because $M < N$, one can choose $x_{m,j} \in \ell(\mathbb{Z}_N)$, $M \le m \le N-1$, $0 \le j \le \frac{d}{N}-1$, so that (a) each collection $\{x_{m,j}\}_{m=M}^{N-1}$ consists of mutually orthogonal vectors that lie in the orthogonal complement of the subspace spanned by $\{x_{m,j}\}_{m=0}^{M-1}$ and (b) each vector $x_{m,j}$ has norm $\frac{N}{d}\varepsilon^2$. Define $\tilde{X}_j$, $0 \le j \le \frac{d}{N} - 1$, by $\tilde{X}_j := \{x_{m,j}\}_{m=0}^{N-1}$. It must be noted that, by construction, our collections $\tilde{X}_j$ must also be local minimizers of the corresponding combined frame potential, because the collections X_j were local minimizers and the "new" vectors were all chosen to be both mutually orthogonal and orthogonal to the original collections X_j.

One can now apply Theorem 12 to the collections $\tilde{X}_j$ with $m_0 = M$, since

$$(N-M)a_M^2 = (N-M)\varepsilon^2 = \sum_{j=M}^{N-1} a_j^2.$$

In particular, Theorem 12 now implies that $X_j = \{x_{m,j}\}_{m=0}^{M-1}$, $0 \le j \le \frac{d}{N} - 1$, consists of mutually orthogonal, nonzero vectors. This is a contradiction, so the theorem is proven. □

The next corollary follows from Theorem 15 in the same way that Theorem 6 follows from Theorem 12.

Corollary 16 *Let* $\{a_m\}_{m=0}^{M-1} \subset \mathbb{R}$ *be such that* $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$. *Let* d *and* N *be positive integers such that* $N \mid d$ *and* $N \geq M$. *If* $H_N(\{h_m\}_{m=0}^{M-1}) \subset \ell(\mathbb{Z}_d)$ *is a local minimizer of the frame potential over the set*

$$\mathcal{A} = \{\{h_m\}_{m=0}^{M-1} \subset \ell(\mathbb{Z}_d) : \|h_m\|^2 = a_m^2, \quad 0 \leq m \leq M-1\},$$

then $H_N(\{h_m\}_{m=0}^{M-1})$ *is an orthogonal sequence.*

Acknowledgment

The third author would like to thank the School of Mathematics at the Georgia Institute of Technology for hosting her during the spring semester of 2003. The authors gratefully acknowledge the comments and suggestions of the anonymous referee.

References

[1] J.J. Benedetto, M. Fickus, Finite normalized tight frames, Adv. Comput. Math. 18 (2003) 357–385.

[2] H. Bölcskei, F. Hlawatsch, H.G. Feichtinger, Frame-theoretic analysis of oversampled filter banks, IEEE Trans. Signal Process. 46 (1998) 3256–3268.

[3] P.G. Casazza, M. Fickus, J. Kovačević, M. Leon, J. Tremain, A physical interpretation of tight frames, submitted.

[4] Z. Cvetković , A. Vetterli, Oversampled filter banks, IEEE Tran. Signal Process. 46 (1998) 1245–1255.

[5] P.L. Dragotti, V.K. Goyal, J. Kovačević, Filter bank frame expansions with erasures, IEEE Trans. Inform. Theory 48 (2002) 1439–1450.

[6] K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower, E. Weber, Ellipsoidal tight frames and projection decompositions of operators, Illinois J. Math. 48 (2004) 477–489.

[7] T. Strohmer, Numerical algorithms for discrete Gabor expansions, in: Gabor analysis and algorithms, Birkhauser, Boston, MA, 1998, 267–294.

[8] R. Vale, S. Waldron, Tight frames and their symmetries, Constr. Approx. 21 (2004) 83–112.

[9] M. Vetterli, Filter banks allowing perfect reconstruction, Signal Processing 10 (1986) 219–244.