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Co-affine systems in $\mathbb{R}^d$

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Abstract. The proof of non-existence for co-affine frames is extended from the one-dimensional setting [**GLWW**] to the case of expansive dilation matrices in $\mathbb{R}^d$. The problem of identifying subspaces on which co-affine systems may admit frame-type inequalities is then considered. In the context of multiresolution analysis it is shown that frame-type inequalities may hold on certain fundamental subspaces of an MRA. Finally, necessary conditions are given for a general co-affine system to admit frame-type inequalities on band-limited subspaces. In the case of the co-affine system generated by the Shannon wavelet these results dictate that Parseval's identity holds on the band-limited subspace having bandwidth 2, but cannot hold with any larger bandwidth.

1. Introduction

Let $A \in GL_d(\mathbb{R})$ be expansive, i.e., each eigenvalue of A is strictly larger than 1 in modulus. Let D be the dilation operator induced by A, given by $Df(x) = \sqrt{|\det A|}f(Ax)$. Let T_y, $y \in \mathbb{R}^d$, be the translation operator, defined by $T_yf(x) = f(x-y)$. Finally, Fourier transform, $\hat{f}$, of $f \in L^1 \cap L^2(\mathbb{R}^d)$ shall be taken to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i\xi\cdot x}dx.$$

Together, the dilation and translation operators form the backbone of wavelet theory. The classic notion of a *dyadic orthonormal wavelet* is a function $\psi \in L^2(\mathbb{R})$ for which the collection $\{2^{\frac{j}{2}}\psi(2^jx-k) : j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ [**D, HW**]. More generally, wavelet theory involves the study of collections composed of certain dilates and translates of a finite number of generating functions. Given a finite collection $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^d)$, the *affine system generated by* Ψ is defined to be the collection

$$X(\Psi) = \{D^jT_k\psi_\ell : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

As in the dyadic orthonormal case, one might be interested merely in those $\Psi \subset L^2(\mathbb{R}^d)$ for which $X(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, but in many instances this restriction is unnecessary and one may instead consider the more general class of affine systems which comprise frames for $L^2(\mathbb{R}^d)$.

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In a separable Hilbert space, $\mathbb{H}$, a collection $\{h_j\}_{j\in J} \subset \mathbb{H}$ is a *frame* if there exist constants $0 < B_1 \le B_2 < \infty$ such that for all $f \in \mathbb{H}$

$$B_1\|f\|_{\mathbb{H}}^2 \le \sum_{j\in J} |\langle f, h_j\rangle_{\mathbb{H}}|^2 \le B_2\|f\|_{\mathbb{H}}^2. \tag{1.1}$$

The constants B_1 and B_2 are referred to as the lower and upper frame bounds, respectively, and in the case that one may choose $B_1 = B_2$ the frame is said to be *tight*. In particular, when one may choose $B_1 = B_2 = 1$ the frame is said to be *Parseval*. Any collection possessing an upper frame bound is called a *Bessel sequence*.

The term *wavelet* will be used here to refer to a generator or set of generators Ψ (as above) for which $X(\Psi)$ is a frame for $L^2(\mathbb{R}^d)$. When $X(\Psi)$ is additionally an orthonormal collection the term *orthonormal wavelet* will also be used. The literature in wavelet theory has been dominated by the study of affine systems, but there are alternative systems that have also played important roles in the theory. The most notable alternative systems are the *quasi-affine* systems, which were introduced by Ron and Shen as a tool for characterizing affine frames using the theory of shift-invariant spaces [**RS**]. With Ψ as above, the *quasi-affine system generated by* Ψ is the collection

$$\begin{aligned} X^q(\Psi) = &\{D^jT_k\psi_\ell : 1 \le \ell \le L, j \ge 0, k \in \mathbb{Z}^d\} \cup \\ &\{|\det A|^{\frac{j}{2}} T_k D^j \psi_\ell : 1 \le \ell \le L, j < 0, k \in \mathbb{Z}^d\}. \end{aligned}$$

Notice that $X^q(\Psi)$ is $\mathbb{Z}^d$-shift invariant, a property stemming from the reversal of the dilation and translation operators for the scales $j < 0$. The principle result relating affine and quasi-affine systems says that when A is an integral, expansive matrix then $X(\Psi)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if $X^q(\Psi)$ is a frame for $L^2(\mathbb{R}^d)$ and, moreover, that in either case the frame bounds of the two systems are identical [**RS, CSS**]. The result also holds for Bessel sequences. Despite the prevalence of affine systems in the literature, there have been a number of other interesting works dealing with or related to quasi-affine systems [**Bo1, GLWW, HKMT, HLW, HLWW, La, MZ**]. In particular, the works [**HKMT**] and [**MZ**] deal with the à trous algorithm, which is a shift-invariant version of the discrete wavelet transform with strong ties to quasi-affine systems [**Jo**].

It is natural to wonder what would result from the reversal of the dilation and translation operators at *each* scale $j \in \mathbb{Z}$, as opposed to just the negative scales as in the quasi-affine systems.

Definition 1.1 ([**GLWW**]). Let $\Psi = \{\psi_\ell\}_{\ell=1}^L \subset L^2(\mathbb{R}^d)$ and let $c := \{c_{\ell;j}\}$, $1 \le \ell \le L$, $j \in \mathbb{Z}$, be any numerical sequence. The *weighted co-affine system generated by* Ψ *and* c, denoted $X^*(\Psi, c)$, is the collection

$$X^*(\Psi, c) = \{\psi^*_{\ell;j,k} := c_{\ell;j} T_k D^j \psi_\ell : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

Gressman *et al.* showed that when $d = 1$, $A = a > 1$, and $\Psi = \{\psi\}$, the weighted co-affine system $X^*(\Psi, c)$ can never comprise a frame for $L^2(\mathbb{R})$ [**GLWW**]. This suggests that finitely generated co-affine systems should always fail to constitute frames for $L^2(\mathbb{R}^d)$ (at least for expansive dilations), but as unions of shift-invariant systems one would still expect that co-affine systems should be able to provide stable, frame-like reconstruction formulas on proper subspaces of $L^2(\mathbb{R}^d)$.

The intent of this paper is threefold. First, the non-existence result of [**GLWW**] will be extended to the case of finitely generated co-affine systems associated with expansive dilation matrices in $\mathbb{R}^d$. Second, the framework of multiresolution analysis shall be used to develop frame-like reconstruction formulas for certain fundamental subspaces of an MRA. Finally, the technique of the non-existence result will be used to derive necessary conditions for a co-affine system to admit a frame-type inequality on a given band-limited subspace.

2. Non-existence of co-affine frames

The following lemma generalizes a useful calculation from [**GLWW**] to $\mathbb{R}^d$.

LEMMA 2.1. *If $X^*(\Psi, c)$ is a Bessel system for $L^2(\mathbb{R}^d)$ then for each $f \in L^2(\mathbb{R}^d)$,*

$$\int_{\mathcal{D}} w(x)dx = \int_{\mathbb{R}^d} \Big(\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2 |\det A|^{-j} |\hat{\psi}(A^{-j}\xi)|^2 \Big) |\hat{f}(\xi)|^2 d\xi, \tag{2.1}$$

where $w(x)$ is the $\mathbb{Z}^d$-periodic function defined by

$$w(x) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \big|\langle T_x f, \psi^*_{\ell;j,k}\rangle\big|^2,$$

and $\mathcal{D}$ represents the unit cube in $\mathbb{R}^d$.

PROOF. The fact that the system is Bessel implies that the integral in (2.1) is finite. Moreover, the integrand is non-negative, allowing one to interchange the order of the sums and the integral. Observe that

$$\begin{aligned}
\int_{\mathcal{D}} w(x)dx &= \int_{\mathcal{D}} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \big|\langle T_x f, \psi^*_{\ell;j,k}\rangle\big|^2 dx \\
&= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2 \int_{\mathcal{D}} \sum_{k\in\mathbb{Z}^d} \big|\langle f, T_{-x+k} D^j \psi_\ell\rangle\big|^2 dx \\
&= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2 \int_{\mathbb{R}^d} \big|\langle f, T_x D^j \psi_\ell\rangle\big|^2 dx \\
&= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2 \int_{\mathbb{R}^d} \big|\langle \hat{f}, |\det A|^{\frac{-j}{2}} \hat{\psi}_\ell\big((A^T)^{-j}\cdot\big) e^{-2\pi i\langle x,\cdot\rangle}\rangle\big|^2 dx \\
&= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2\, |\det A|^{-j} \int_{\mathbb{R}^d} \big|\big(\hat{f}\overline{\hat{\psi}_\ell\big((A^T)^{-j}\cdot\big)}\big)\check{}(x)\big|^2 dx \\
&= \int_{\mathbb{R}^d} \Big(\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2\, |\det A|^{-j}\, \big|\hat{\psi}_\ell\big((A^T)^{-j}\xi\big)\big|^2 \Big) |\hat{f}(\xi)|^2 dx.
\end{aligned}$$

□

The next lemma describes a well-known fact about matrices that plays an important role in the handling of expansive dilations. One can find a proof of the lemma in [**HLW**].

Lemma 2.2. *Suppose $A \in GL_d(\mathbb{R})$ and that there exist $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < |\lambda| < \beta < \infty$ for each eigenvalue λ of A. Then there exists $C = C(A, \alpha, \beta) \geq 1$ such that*

$$C^{-1}\alpha^j |x| \leq |A^j x| \leq C\beta^j |x|, \tag{2.2}$$

for all $x \in \mathbb{R}^d$ and each positive integer j.

Theorem 2.3. *There exist no weighted co-affine frames for $L^2(\mathbb{R}^d)$.*

Proof. Let $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R})$ and $c = \{c_{\ell;j}\}_{1 \leq \ell \leq L, j \in \mathbb{Z}}$ be a fixed numerical sequence, $c \neq 0$. Proceeding by contradiction, suppose that $X^*(\Psi, c)$ is a frame with bounds $0 < B_1 \leq B_2 < \infty$. Notice that for any fixed $j_0 \in \mathbb{Z}$, $k_0 \in \mathbb{Z}^d$, and $1 \leq \ell_0 \leq L$ one has

$$|c_{\ell_0;j_0}|^4 \|\psi^*_{\ell_0,j_0,k_0}\|^4 \leq \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left|\langle \psi^*_{\ell_0;j_0,k_0}, \psi^*_{\ell;j,k} \rangle\right|^2 \leq B_2 |c_{\ell_0;j_0}|^2 \|\psi_{\ell_0}\|^2,$$

which implies

$$|c_{\ell;j}|^2 \leq B_2 \|\psi_\ell\|^{-2}. \tag{2.3}$$

It follows from Lemma 2.1 that for a.e. $\xi \in \mathbb{R}^d$

$$B_1 \leq \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |c_{\ell;j}|^2 |\det A|^{-j} \left|\hat{\psi}_\ell\big((A^T)^{-j}\xi\big)\right|^2 \leq B_2. \tag{2.4}$$

It is by means of (2.4) that a contradiction will be derived.

Let $1 < \lambda \leq \Lambda < \infty$, respectively, be strict lower and upper bounds of the eigenvalues of A. Let $E_{a,b}$ be the set defined by

$$E = E_{a,b} = \{x \in \mathbb{R}^d : a < |x| < b\},$$

where $0 < a < b < \infty$. It is important to understand the overlap of $(A^T)^{j_1} E_{a,b}$ and $(A^T)^{j_2} E_{a,b}$ for $j_1, j_2 \in \mathbb{Z}$. Since $A \in GL_n(\mathbb{R})$ it is sufficient to consider $j_1 \geq 0$ and $j_2 = 0$. Observe that if $x \in E_{a,b}$ and $j > 0$ then

$$C^{-1}\lambda^j a < |(A^T)^j x| < C\Lambda^j b, \tag{2.5}$$

with C as in Lemma 2.2. Thus, in order for $(A^T)^j E$ to be disjoint from E one must have

$$\lambda^j \geq C\frac{b}{a}.$$

Fixing $a = 1$ and $b = \lambda$ there exists $J \geq 1$ (finite) such the last inequality is satisfied for all $j \geq J$. Let $E := E_{1,\lambda}$. The upshot of these observations is that the sets $\{(A^T)^j E\}_{j \in \mathbb{Z}}$ have finite overlap, i.e., $A^j E \cap E = \emptyset$ for sufficiently large integers j. This property is crucial for the following argument. In fact, if $x \in (A^T)^{j_0} E$, $j_0 \in \mathbb{Z}$, then x belongs to at most $2J - 1$ sets in the collection $\{(A^T)^j E\}_{j \in \mathbb{Z}}$. Let $n \in \mathbb{Z}$ and observe by (2.4) that

$$\begin{aligned}
\int_{(A^T)^{-n}E} B_1 d\xi &\leq \int_{(A^T)^{-n}E} \Big(\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |c_{\ell;j}|^2 \ |\det A|^{-j} \ \left|\hat{\psi}_\ell\big((A^T)^{-j}\xi\big)\right|^2\Big) d\xi \\
&= \int_E \Big(\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |c_{\ell;j}|^2 \ |\det A|^{-(j+n)} \ \left|\hat{\psi}_\ell\big((A^T)^{-(j+n)}\xi\big)\right|^2\Big) d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \int_E \Big(\sum_{\ell=1}^{L} \sum_{m\in\mathbb{Z}} |c_{\ell;-(m+n)}|^2 \, |\det A|^m \, \big|\hat{\psi}_\ell\big((A^T)^m\xi\big)\big|^2 \Big) d\xi \\
&\le \sum_{\ell=1}^{L} B_2 \|\psi_\ell\|^{-2} \sum_{m\in\mathbb{Z}} \int_{(A^T)^m E} |\hat{\psi}_\ell(\xi)|^2 d\xi \\
&\le (2J-1) \sum_{\ell=1}^{L} B_2 \|\psi_\ell\|^{-2} \int_{\mathbb{R}^d} |\hat{\psi}_\ell(\xi)|^2 d\xi \\
&\le (2J-1)\, B_2\, L.
\end{aligned}
$$

Letting μ denote the volume of the unit sphere in $\mathbb{R}^d$, the left-hand side of the last chain of inequalities becomes

$$\int_{A^{-n}E} B_1 d\xi = |\det A|^{-n} \mu(\lambda^d - 1) B_1,$$

from which it follows that

$$\mu(\lambda^d - 1) B_1 \le |\det A|^n \, (2J-1)\, B_2\, L \tag{2.6}$$

for each $n \in \mathbb{Z}$. By choosing $-n$ large enough one obtains the desired contradiction. □

3. Multiresolution analysis and co-affine systems

In the last section it was seen that co-affine systems can never constitute frames for $L^2(\mathbb{R}^d)$. This leads naturally to questions about which subspaces admit frame-like reconstructions in terms of co-affine systems. One setting in which this will be possible is that of the familiar multiresolution analysis, a tool that has fueled much research in wavelet theory. Accordingly, it will be assumed throughout this section not only that A is expansive, but also that A is integral, i.e., $A\mathbb{Z}^d \subset \mathbb{Z}^d$.

Recall that a *multiresolution analysis* (MRA) is a collection of closed subspaces $\{V_j\}_{j\in\mathbb{Z}} \subseteq L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subseteq V_{j+1}$, $j \in \mathbb{Z}$;
(ii) $f \in V_j$ if and only if $Df \in V_{j+1}$, $j \in \mathbb{Z}$;
(iii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;
(v) There exists $\varphi \in V_0$ (a *scaling function*) such that $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis for V_0.

The subspace V_0 is often referred to as the *core subspace* of the MRA. Notice that V_0 is invariant under integral translations.

The importance of multiresolution analysis in wavelet theory is immense. The early works of Mallat [**Ma**] and Daubechies [**D**] offered constructions of dyadic orthonormal wavelets relying on the structure of MRAs and since that time there have been generalizations and extensions too numerous to mention here in any detail. One example of a multi-dimensional construction of MRA wavelets (tight affine frames) may be found in [**Bo1**] and a nice treatment of the one-dimensional theory of dyadic orthonormal wavelets may be found in [**HW**].

Example 3.1. Consider the Shannon wavelet,

$$\psi(x) = -2\frac{\sin(2\pi x) + \cos(\pi x)}{\pi(2x+1)},$$

and the associated scaling function

$$\varphi(x) = \frac{\sin(\pi x)}{\pi x}.$$

Notice that $\hat{\varphi}(\xi) = \chi_{[-2^{-1},2^{-1}]}(\xi)$ and thus the principal shift-invariant space V_0 generated by φ consists precisely of those $L^2(\mathbb{R})$ functions whose Fourier transforms are supported in $[-2^{-1}, 2^{-1}]$. The collection $\{T_k\varphi\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0. Defining $V_j = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [-2^{j-1}, 2^{j-1}]\}$ it is easy to see that $\{V_j\}_{j\in\mathbb{Z}}$ is a dyadic MRA. The subspaces V_j are merely dyadic band-limited subspaces with bandwidth 2^j.

Given a scaling function φ for an MRA, it is an elementary fact that there exists $m_0 \in L^\infty(\mathbb{T}^d)$ (the *low-pass filter* associated to φ) such that $\hat{\varphi}(A^T\xi) = m_0(\xi)\,\hat{\varphi}(\xi)$. The $\mathbb{Z}^d$-periodic low-pass filter must also satisfy

$$\sum_{s=0}^{|\det A|-1} |m_0(\xi + (A^T)^{-1}\vartheta_s)|^2 = 1, \quad \text{a.e.}\xi \in \mathbb{T}^d,$$

where $\{\vartheta_s\}_{s=0}^{|\det A|-1}$ is a complete set of distinct coset representatives of $\mathbb{Z}^d/A^T\mathbb{Z}^d$ with $\vartheta_0 = 0$. Given $m_1, \ldots, m_L \in L^\infty(\mathbb{T}^d)$ (*high-pass filters*) one can construct $\Psi = \{\psi_\ell\}_{1\le\ell\le L}$ via the refinement equations

$$\hat{\psi}_\ell(A^T\xi) = m_\ell(\xi)\,\hat{\varphi}(\xi), \quad 1 \le \ell \le L.$$

Given a low-pass filter, m_0, there are two conditions on the high-pass filters that will be useful below, namely,

$$\sum_{\ell=0}^{L} |m_\ell(\xi)|^2 = 1, \quad \text{a.e. } \xi \in \mathbb{T}^d, \tag{3.1}$$

and

$$\sum_{\ell=0}^{L} |m_\ell(\xi + (A^T)^{-1}\vartheta_s)|^2 = \delta_{0,s}, \quad \text{a.e.}\xi \in \mathbb{T}^d. \tag{3.2}$$

Observe that (3.2) implies (3.1). It will be seen below how these assumptions on the high-pass filters will lead to two different reconstruction formulas for fundamental subspaces of a given MRA in terms of appropriate co-affine systems.

Associated with an MRA $\{V_j\}_{j\in\mathbb{Z}}$, the multiresolution approximation operators P_j are defined by

$$P_j f = \sum_{k\in\mathbb{Z}^d} \langle f, D^jT_k\varphi\rangle D^jT_k\varphi \tag{3.3}$$

for $f \in L^2(\mathbb{R}^d)$ and $j \in \mathbb{Z}$. The operator P_j is actually the orthogonal projection onto V_j, since φ is a scaling function for the MRA.

One may also construct multiresolution operators using the co-affine structure. Let P_j^* and Q_j^*, respectively, be the co-affine approximation and detail operators at the scale $j \in \mathbb{Z}$. For $f \in L^2(\mathbb{R}^d)$ these operators are given by

$$P_j^* f := \sum_{k\in\mathbb{Z}^d} \langle f, \varphi_{j,k}^*\rangle \varphi_{j,k}^* \text{ and } Q_j^* f := \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d} \langle f, \psi_{\ell;j,k}^*\rangle \psi_{\ell;j,k}^*, \tag{3.4}$$

where $\psi^*_{\ell;j,k} := c_{\ell;j} T_k D^j \psi_\ell$ and $\varphi^*_{j,k} = T_k D^j \varphi$. For $j \le 0$ these operators resemble the analogous multiresolution operators for the refinable quasi-affine systems as defined in [**Jo**] and, through this association, it is easy to see that P^*_j and Q^*_j are bounded when φ is a scaling function for an MRA, see, e.g., Proposition 6 of [**Jo**]. The following lemma from [**Jo**] summarizes the key properties of the co-affine multiresolution operators.

LEMMA 3.2. [**Jo**] *Let φ and Ψ as above and suppose that the coefficient sequence associated to $X^*(\Psi, c)$ is such that $c_{\ell;j} = |\det A|^{\frac{j}{2}}$, $1 \le \ell \le L$, $j < 0$.*

(a) *For each $f \in L^2(\mathbb{R}^d)$, $\|P^*_j f\| \to 0$ as $j \to -\infty$.*
(b) *If the filters $m_0, \dots, m_L$ satisfy* (3.1)*, then $P^*_j + Q^*_j = P^*_{j+1}$, $j < 0$.*
(c) *If the filters $m_0, \dots, m_L$ satisfy* (3.2) *and $c_{\ell;0} = 1$, $1 \le \ell \le L$, then $P^*_0 + Q^*_0 = P_1$, where, as above, P_1 is the orthogonal projection onto V_1.*

REMARK 3.3. The conditions on $\{c_{\ell;j}\}_{\ell,j}$ in Lemma 3.2 owe some explanation. Notice that Lemma 3.2 (b) depends only on $\{c_{\ell;j}\}$, $j < 0$, while Lemma 3.2 (c) additionally employs the coefficients $\{c_{\ell;0}\}$. These normalizations are borrowed from the quasi-affine structure and, moreover, the properties presented in Lemma 3.2 are essentially a restatement of the analogous properties of quasi-affine systems (see [**Jo**]).

The next proposition shows how one may decompose V_0 using a *truncated* co-affine system under the fairly weak filter condition (3.1).

PROPOSITION 3.4. *Let φ and Ψ as above and suppose that $c = \{c_{\ell;j}\}_{j\in\mathbb{Z}}$ is defined by*

$$c_{\ell;j} = \begin{cases} 0, & j \ge 0 \\ |\det A|^{\frac{j}{2}}, & j < 0. \end{cases}$$

If the filters $m_1, \dots, m_L$ satisfy (3.1)*, then the collection $P_0\big(X^*(\Psi, c)\big)$ is a Parseval frame for V_0, i.e., for each $f \in V_0$,*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi^*_{\ell;j,k}\rangle|^2 = \|f\|^2. \tag{3.5}$$

PROOF. Applying Lemma 3.2 (b) one finds that for $f \in V_0$,

$$f = P_0 f = \lim_{J\to\infty} \sum_{j=-J}^{-1} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} \langle f, \psi^*_{\ell;j,k}\rangle \psi^*_{\ell;j,k},$$

where the limit converges in $L^2(\mathbb{R}^d)$. By taking inner products with f on each side of this equation it follows that

$$\|f\|^2 = \lim_{J\to\infty} \sum_{\ell=1}^{L} \sum_{j=-J}^{-1} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi^*_{\ell;j,k}\rangle|^2 = \sum_{j<0} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi^*_{\ell;j,k}\rangle|^2.$$

Because $c_{\ell;j} = 0$ whenever $j \ge 0$, (3.5) now follows. Finally, that (3.5) is equivalent to $P_0\big(X^*(\Psi, c)\big)$ being a Parseval frame for V_0 follows from the fact that $P_0 f = f$ for $f \in V_0$. $\square$

Remark 3.5. One must include the projection P_0 in the statement of Proposition 3.4 because, even after truncation, the collection $X^*(\Psi, c)$ is not necessarily contained in V_0.

By additionally imposing the filter conditions (3.2) one may marginally increase the utility of co-affine systems by truncating the system at $j = 1$ rather than $j = 0$.

Proposition 3.6. *Let φ and Ψ as above and suppose that $c = \{c_{\ell;j}\}_{j\in\mathbb{Z}}$ is defined by*

$$c_{\ell;j} = \begin{cases} 0, & j > 0 \\ |\det A|^{\frac{j}{2}}, & j \leq 0. \end{cases}$$

If the filters $m_1, \dots, m_L$ satisfy (3.2)*, then the collection $P_1\big(X^*(\Psi, c)\big)$ is a Parseval frame for V_1.*

Proof. The proof follows that of Proposition 3.4, except in this case one may also appeal to Lemma 3.2 (c). □

Remark 3.7. For the sake of simplicity this discussion has been limited to the context of MRAs, but one could prove results similar to Propositions 3.4 and 3.6 for properly defined *dual* systems in which separate filters were used for analysis and synthesis, e.g., biorthogonal wavelets.

Example 3.8. Again consider the Shannon wavelet of Example 3.1. By defining $m_0(\xi)$ by

$$m_0(\xi) = \begin{cases} 1, & 0 \leq |\xi| \leq \frac{1}{4} \\ 0, & \frac{1}{4} < |\xi| < \frac{1}{2}. \end{cases}$$

and $m_1(\xi)$ by

$$m_1(\xi) = \begin{cases} 0, & 0 \leq |\xi| \leq \frac{1}{4} \\ e^{2\pi i\xi}, & \frac{1}{4} < |\xi| < \frac{1}{2}. \end{cases}$$

one can verify the refinement equations $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$ and $\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi)$. Moreover, the filter equations (3.2) are satisfied and, thus, Proposition 3.6 applies. In this case the collection $\{2^j\psi(2^j(x-k)\}_{j\leq 0, k\in\mathbb{Z}^d}$ is itself a Parseval frame for $V_1 = \big\{f \in L^2(\mathbb{R}) : \text{supp}(f) \subseteq [-1, 1]\big\}$. The projection P_1 is unnecessary because of the fact that V_1 is band-limited and, hence, shift-invariant under all $\mathbb{R}^d$ translations.

Propositions 3.4 and 3.6 indicate that truncated co-affine systems may provide stable reconstruction formulas on useful subspaces of $L^2(\mathbb{R}^d)$, admitting frame-like inequalities on core subspaces of a given MRA. One deficit of these results is that the elements of $X^*(\Psi, c)$ for scales $j \geq 1$ are ignored. In the next section, this problem will be approached from another angle, where necessary conditions will be considered for a co-affine system to admit a frame-type inequality on band-limited subspaces of $L^2(\mathbb{R}^d)$. The goal is to quantify the contribution of the positive scales.

4. Band-limited subspaces spanned by co-affine systems

Example 3.8 provides motivation for understanding necessary conditions under which frame inequalities (or equalities) of the sort (3.5) may hold for a given co-affine system on a band-limited subspace of $L^2(\mathbb{R}^d)$. For $\Omega \in \mathbb{R}_+$, let $\mathcal{B}_\Omega$ be the band-limited subspace

$$\mathcal{B}_\Omega = \big\{f \in L^2(\mathbb{R}^d) : \text{supp}(\hat{f}) \subseteq [-\Omega, \Omega]^d\big\}.$$

Let $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^d)$ and $c = \{c_{\ell;j}\}_{1\le\ell\le L, j\in\mathbb{Z}}$ be such that $X^*(\Psi, c)$ is a Bessel system with bound B_2 relative to an expanding dilation matrix $A \in GL_d(\mathbb{R})$. Suppose further that for all $f \in \mathcal{B}_\Omega$ one has

$$B_1\|f\|^2 \le \sum_{\ell=1}^{L} \sum_{j,k\in\mathbb{Z}} \left|\langle f, \psi^*_{\ell;j,k}\rangle\right|^2.$$

It follows from Lemma 2.1 that

$$\sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2 \,|\det A|^j \left|\hat{\psi}\big((A^T)^j\xi\big)\right|^2 \le B_2, \quad \text{a.e.}\, \xi \in \mathbb{R}^d \tag{4.1}$$

and

$$\sum_{j\in\mathbb{Z}} |c_{\ell;j}|^2 \,|\det A|^j \left|\hat{\psi}\big((A^T)^j\xi\big)\right|^2 \ge B_1, \quad \text{a.e.}\, \xi \in [-\Omega, \Omega]^d. \tag{4.2}$$

The main idea behind Theorem 2.3 was to contradict (4.2) by integrating the inequality over a carefully chosen set E. In particular, the argument hinged on the fact that the dilates of E have finite overlap, i.e., $A^jE \cap E = \emptyset$ for sufficiently large integers j. In order to apply this argument to the current problem one must additionally assume that E be contained in $[-\Omega, \Omega]$. The result of these observations is the following proposition.

Proposition 4.1. *Suppose that $X^*(\Psi, c)$ is a Bessel system with bound $B_2 > 0$ relative to an expanding dilation matrix $A \in GL_d(\mathbb{R})$. Suppose further that there exists $0 < B_1 < B_2$ such that for each $f \in \mathcal{B}_\Omega$,*

$$B_1\|f\|^2 \le \sum_{\ell=1}^{L} \sum_{j,k\in\mathbb{Z}} \left|\langle f, \psi^*_{\ell;j,k}\rangle\right|^2.$$

If $E \subseteq [-\Omega, \Omega]$ and $A^jE \cap E = \emptyset$ for each integer $j \ge J$, then, necessarily,

$$|E|B_1 \le (2J - 1)B_2\, L, \tag{4.3}$$

where $|E|$ is the Lebesgue measure of $E \subset \mathbb{R}^d$.

As an application of Proposition 4.1 the situation of the Shannon wavelet will be reexamined under the umbrella of all dyadic orthonormal wavelets for $L^2(\mathbb{R})$.

Corollary 4.2. *Let $\psi \in L^2(\mathbb{R})$ be a dyadic orthonormal wavelet and fix $c = \{c_j\}_{j\in\mathbb{Z}}$ so that $c_j = 1$ if $j > 0$ and $c_j = 2^{\frac{j}{2}}$ if $j \le 0$. Then $X^*(\psi, c)$ does not satisfy Parseval's identity on any subspace $\mathcal{B}_\Omega$, $\Omega > 1$.*

Proof. In the terminology of Proposition 4.1, $L = 1$ and $B_1 = B_2 = 1$. Two separate cases will be considered: $\Omega \ge 2$ and $1 < \Omega < 2$. Case 1: $\Omega \ge 2$. Letting $E = [-2, -1] \cup [1, 2]$, one has that $E \subset [-\Omega, \Omega]$ as well as the fact that $2^jE \cap E = \emptyset$ whenever $j \ge 1$ (j an integer). Hence, $J = 1$ and $|E| = 2$. The necessary condition (4.3) then becomes $2 \le 1$, a contradiction.

Case 2: $1 < \Omega < 2$. Let $\Omega = 1 + t$, where $0 < t < 1$. Suppose by way of contradiction that $X^*(\psi, c)$ does yield Parseval's identity on $\mathcal{B}_\Omega$. Let $E = [-(1 + t), -\frac{1+t}{2}] \cup [\frac{1+t}{2}, 1 + t]$. The sets 2^jE, $j \in \mathbb{Z}$, are pairwise disjoint (J=1) and, thus, Proposition 4.1 implies

$$|E| = 2\frac{t + 1}{2} = t + 1 \le 1,$$

which is a contradiction since $0 < t < 1$. $\square$

In the case of the co-affine system generated by the Shannon wavelet, Corollary 4.2 implies that Parseval's identity cannot hold on any band-limited subspace $\mathcal{B}_\Omega$, $\Omega > 1$. Thus, the result of Proposition 3.6 is in some sense sharp.

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