SPARSE RECOVERY USING THE DISCRETE COSINE TRANSFORM

BENJAMIN BARROS AND BRODY DYLAN JOHNSON

Abstract. This note considers the problem of sparse recovery in $\mathbb{R}^n$ from linear measurements associated with a discrete cosine transform. The main theorem shows that an s-sparse vector in $\mathbb{R}^n$ can be recovered from the first $2s$ coefficients of its discrete cosine transform. This theorem is a real-valued analog of a result in [1] concerned with sparse recovery in $\mathbb{C}^n$ based on linear measurements via the discrete Fourier transform.

1. Introduction

This article is concerned with the problem of recovering a sparse vector $x \in \mathbb{R}^N$ from its linear measurements, $y = Ax \in \mathbb{R}^m$, associated with a *measurement matrix* $A \in \mathbb{R}^{m\times N}$. It will be convenient to establish some notation (mostly following that of [1]) surrounding vectors in $\mathbb{R}^N$ and their sparsity.

Let $[N]$ represent the index set $\{0, 1, 2, \dots, N-1\}$ and define the support of a vector $x = (x(0), x(1), \dots, x(N-1)) \in \mathbb{R}^N$ by

$$\operatorname{supp}(x) := \{j \in [N] : x(j) \neq 0\}.$$

The sparsity of a vector in $\mathbb{R}^N$ is measured in terms of the cardinality of its support set and for $x \in \mathbb{R}^N$ will be denoted by

$$\|x\|_0 := \operatorname{card}(\operatorname{supp}(x)),$$

despite the fact that this definition does not produce a proper norm. A vector $x \in \mathbb{R}^N$ is said to be *s-sparse* if $\|x\|_0 \leq s$.

The starting point for the present work is Theorem 2.15 of [1], which describes a procedure for the recovery of an s-sparse vector in $\mathbb{C}^N$ from the first two $2s$ coefficients of its discrete Fourier transform. The discrete Fourier transform of $x \in \mathbb{C}^N$ will be denoted by $\hat{x}$ and is defined by

$$\hat{x}(j) = \sum_{k=0}^{N-1} x(k)e^{-2\pi ijk/N}, \quad 0 \leq j \leq N-1.$$

The DFT-based sparse recovery procedure described in [1] essentially involves two steps. First, the support set of the s-sparse vector is determined from a matrix equation involving the discrete Fourier coefficients. Second, the nonzero elements of the s-sparse vector are determined from the $2s$ linear measurements, $\hat{x}(0), \hat{x}(1), \dots, \hat{x}(2s-1)$. The statement presented here focuses on the first step, which relies on the special structure of the discrete Fourier transform.

Theorem 1 (See Theorem 2.15 of [1])**.** *Let $x \in \mathbb{C}^N$ be s-sparse with $2s \leq N$. Suppose that $q \in \mathbb{C}^N$ is such that its discrete Fourier transform satisfies $\hat{q}(0) = 1$, $\hat{q}(j) = 0$ for $j > s$,*

1991 *Mathematics Subject Classification.* 94A12, 41A60, 42C15.

Key words and phrases. sparse recovery, discrete cosine transform, compressive sensing.

and

$$\begin{bmatrix} \hat{x}(s-1) & \hat{x}(s-2) & \cdots & \hat{x}(0) \\ \hat{x}(s) & \hat{x}(s-1) & \cdots & \hat{x}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{x}(2s-2) & \hat{x}(2s-3) & \cdots & \hat{x}(s-1) \end{bmatrix} \begin{bmatrix} \hat{q}(1) \\ \hat{q}(2) \\ \vdots \\ \hat{q}(s) \end{bmatrix} = - \begin{bmatrix} \hat{x}(s) \\ \hat{x}(s+1) \\ \vdots \\ \hat{x}(2s-1) \end{bmatrix},$$

where $\hat{x}$ *denotes the discrete Fourier transform of* x*. Then,*

$$supp(x) \subseteq \{j \in [N] : q(j) = 0\}.$$

The main result of this article is a real-valued version of Theorem 1 based on the discrete cosine transform, which will be developed in the remaining sections. Section 2 is devoted to a brief review of the discrete cosine transform, including a convolution formula which plays a critical role in the proof of the main result. The main result will be presented and proven in Section 3.

2. Discrete Cosine Transform

The goal of this section is to introduce a specific version of the discrete cosine transform (DCT) that will be used in the next section to prove a real-valued counterpart to Theorem 1. A convolution formula for the DCT will be reviewed, following the treatment of Martucci in his systematic study of convolution formulas for discrete sine and cosine transforms [2].

The discrete cosine transform of $x \in \mathbb{R}^N$ will be denoted by $\hat{x} \in \mathbb{R}^N$ and is defined by

$$\hat{x}(j) = x(0) + 2\sum_{k=1}^{N-1} x(k) \cos\left(\frac{2\pi}{2N-1} jk\right)$$

for $0 \le j \le N-1$. The distinction between the DCT and DFT will be made clear in context. The DCT can be implemented via matrix multiplication by $C_1 \in \mathbb{R}^{N\times N}$ with entries

$$(C_1)_{jk} = \begin{cases} \cos\left(\frac{2\pi}{2N-1} jk\right) & k = 0 \\ 2\cos\left(\frac{2\pi}{2N-1} jk\right) & 1 \le k \le N-1, \end{cases}$$

where $0 \le j, k \le N-1$. This variety of the DCT is known in the literature as the odd version of the DCT-I and its inverse transform corresponds to multiplication by $(2N-1)^{-1}C_1$.

Loosely speaking, the DCT of $x \in \mathbb{R}^N$ can be realized as the DFT of a symmetric extension of x to $\mathbb{R}^{2N-1}$. The convolution formula for the DFT in $\mathbb{R}^{2N-1}$ then induces a convolution formula for the DCT in $\mathbb{R}^N$. Towards this end, denote the symmetric extension of $x \in \mathbb{R}^N$ to $\mathbb{R}^{2N-1}$ by X, which is defined by

$$X(n) = \begin{cases} x(n) & 0 \le n \le N-1 \\ x(2N-1-n) & N \le n \le 2N-2. \end{cases}$$

It is often convenient to interpret the indices modulo $2N-1$. This extension imposes a whole-index symmetry at $n = 0$ in that $X(-n) = X(n)$ for $n \in \mathbb{Z}$. Similarly, the extension forces a half-index symmetry at $n = N - \frac{1}{2}$ in that $X(n) = X(2N-1-n)$. Notice that the average of the two indices in this latter identity is $N - \frac{1}{2}$. As mentioned above, the DFT of

X is related to the DCT of x:

$$\begin{aligned}\hat{X}(m) &= \sum_{n=0}^{2N-2} X(n)e^{-\frac{2\pi i}{2N-1}mn} \\ &= x(0) + \sum_{n=1}^{N-1} X(n)e^{-\frac{2\pi i}{2N-1}mn} + \sum_{n=N}^{2N-2} X(2N-1-n)e^{-\frac{2\pi i}{2N-1}mn} \\ &= x(0) + \sum_{n=1}^{N-1} x(n) 2\cos\left(\frac{2\pi}{2N-1}mn\right) \\ &= \hat{x}(m).\end{aligned}$$

The symmetric convolution of $X, Y \in \mathbb{R}^{2N-1}$ is denoted by $X * Y \in \mathbb{R}^{2N-1}$ and is defined by

$$(X * Y)(n) = \sum_{k=0}^{2N-2} X(k)Y(n-k),$$

where the indices should be interpreted modulo $2N-1$. It is straightforward to verify that if $X, Y \in \mathbb{R}^{2N-1}$ are symmetric with respect to $n = 0$ and $n = N - \frac{1}{2}$, then $X * Y$ will be as well. Hence, it becomes possible to define a convolution operation on $x, y \in \mathbb{R}^n$ by restricting the symmetric convolution of X and Y to the indices $0, 1, \ldots, N-1$; namely, for $x, y \in \mathbb{R}^N$, let $x * y \in \mathbb{R}^N$ be defined by

$$(x * y)(n) = (X * Y)(n), \quad 0 \le n \le N-1,$$

where X and Y are the symmetric extensions of x and y, respectively. The final piece of the puzzle comes from the usual convolution formula of the DFT. If $0 \le m \le N-1$, then

$$\begin{aligned}(\widehat{x * y})(m) &= (\widehat{X * Y})(m) \\ &= \sum_{n=0}^{2N-1} (X * Y)(n)e^{-\frac{2\pi i mn}{2N-1}} \\ &= \sum_{n=0}^{2N-2} \sum_{k=0}^{2N-2} X(k)Y(n-k)e^{-\frac{2\pi i mn}{2N-1}} e^{-\frac{2\pi i m}{2N-1}(-k+k)} \\ &= \sum_{k=0}^{2N-2} X(k)e^{-\frac{2\pi i mk}{2N-1}} \sum_{n=0}^{2N-2} Y(n-k)e^{-\frac{2\pi i m(n-k)}{2N-1}} \\ &= \hat{X}(m)\,\hat{Y}(m) \\ &= \hat{x}(m)\,\hat{y}(m).\end{aligned}$$

Note that, in this calculation, $\hat{}$ corresponds to the DFT in $\mathbb{R}^{2N-1}$ and the DCT in $\mathbb{R}^N$.

3. Sparse Recovery via the Discrete Cosine Transform

The convolution formula of the last section will now be put to use in the development of a sparse recovery algorithm based on the DCT. The success of this algorithm will rely on a uniqueness result for the DCT that generalizes an analogous property of the DFT.

Let $x \in \mathbb{R}^N$ be an s-sparse vector and suppose that the first s discrete Fourier coefficients of x vanish, i.e., $\hat{x}(m) = 0$ for $0 \le m \le s-1$. Suppose further that the support of x is contained in $S = \{n_1, n_2, \ldots, n_s\} \subset [N]$, so that

$$\begin{bmatrix} \hat{x}(0) \\ \hat{x}(1) \\ \vdots \\ \hat{x}(s-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-\frac{2\pi i n_1}{N-1}} & e^{-\frac{2\pi i n_2}{N-1}} & \cdots & e^{-\frac{2\pi i n_s}{N-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{2\pi i (s-1) n_1}{N-1}} & e^{-\frac{2\pi i (s-1) n_2}{N-1}} & \cdots & e^{-\frac{2\pi i (s-1) n_s}{N-1}} \end{bmatrix} \begin{bmatrix} x_{n_1} \\ x_{n_2} \\ \vdots \\ x_{n_s} \end{bmatrix}.$$

The $s \times s$ matrix in this equation is Vandermonde and has a nonzero determinant, so it follows that $x_{n_k} = 0$, $1 \le k \le s$. In other words, if the first s discrete Fourier coefficients of an s-sparse vector vanish, then the s-sparse vector is identically zero.

The following lemma will lead to a similar result for the DCT of s-sparse vectors and corresponds to an exercise on double alternants from Muir's treatise on determinants [3].

Lemma 2. *Let $x_1, x_2, \ldots, x_s$ be real numbers. Then,*

$$\begin{vmatrix} 1 & \cdots & 1 \\ \cos x_1 & \cdots & \cos x_s \\ \vdots & \ddots & \vdots \\ \cos((s-1)x_1) & \cdots & \cos((s-1)x_s) \end{vmatrix} = 2^{\frac{(s-1)(s-2)}{2}} \begin{vmatrix} 1 & \cdots & 1 \\ \cos x_1 & \cdots & \cos x_s \\ \vdots & \ddots & \vdots \\ \cos^{s-1} x_1 & \cdots & \cos^{s-1} x_s \end{vmatrix}.$$

Proof. The basic idea is to convert products of cosines into sums using the angle addition formula,

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right).$$

Notice, for example, that

$$\cos^2 x = \cos x \, \cos x = \frac{1}{2}(\cos 2x + 1)$$

and, after applying the angle addition to each term in this last sum, one finds that

$$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x).$$

In general, $\cos^n x$ can be written as a linear combination of $\cos kx$, $0 \le k \le n$, as follows,

$$\cos^n x = 2^{-(n-1)}\left(\alpha_{n,0} + \alpha_{n,1}\cos x + \cdots + \alpha_{n,n}\cos nx\right),$$

where $\alpha_{n,n} = 1$, $\alpha_{n,n-1} = 0$, and the remaining coefficients satisfy the recursive formulas

$$\begin{aligned} \alpha_{n,0} &= \alpha_{n-1,1} \\ \alpha_{n,1} &= 2\alpha_{n-1,0} + \alpha_{n-1,2} \\ \alpha_{n,k} &= \alpha_{n-1,k-1} + \alpha_{n-1,k+1}, \quad 2 \le k \le n-2. \end{aligned}$$

Thus, the rows of the matrix of powers in cosine can be written as linear combinations of the rows of the matrix of frequencies of cosine. Only the highest frequency in each row contributes to the determinant due to the linear dependence of the lower frequency terms on the previous rows. Finally, each frequency n greater than or equal to 2 introduces a factor of 2^{n-1} into the determinant, leading to an overall factor of $2^{\frac{(s-2)(s-1)}{2}}$, as claimed. □

Corollary 3. *If $x \in \mathbb{R}^N$ is s-sparse and the discrete cosine coefficients $\hat{x}(0)$, $\hat{x}(1)$, …, $\hat{x}(s-1)$ all vanish, then $x \equiv 0$.*

Proof. Let $x \in \mathbb{R}^N$ be an s-sparse vector for which the first s discrete cosine coefficients of x vanish, i.e., $\hat{x}(m) = 0$ for $0 \le m \le s-1$. Assume that the support of x is contained in

$S = \{n_1, n_2, \dots, n_s\} \subset [N]$, where $n_1 < n_2 < \cdots < n_s$. In order to apply Lemma 2, define x_k, $1 \le k \le s$, by

$$x_k = \frac{2\pi n_k}{(2N-1)},$$

which leads to the matrix equation

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & \cdots & 2 \\ 2\cos x_1 & 2\cos x_2 & \cdots & 2\cos x_3 \\ \vdots & \vdots & \ddots & \vdots \\ 2\cos((s-1)x_1) & 2\cos((s-1)x_2) & \cdots & 2\cos((s-1)x_s) \end{bmatrix} \begin{bmatrix} x_{n_1} \\ x_{n_2} \\ \vdots \\ x_{n_s} \end{bmatrix}.$$

Note that the first column in the matrix must be multiplied by a factor of $\frac{1}{2}$ if $n_1 = 0$, but this is not essential to the argument. Because $\cos x_k \neq \cos x_{k'}$ when $k \neq k'$, Lemma 2 guarantees that the above matrix is invertible, forcing $x_{n_k} = 0$, $1 \le k \le s$. ☐

This corollary is essential to the success of the main theorem, which states that the support of an s-sparse vector can be determined from the first $2s$ coefficients of its discrete cosine transform.

Theorem 4. *Let $x \in \mathbb{R}^N$ be s-sparse with $2s \le N$. Suppose that $q \in \mathbb{R}^N$ is such that its discrete cosine transform satisfies $\hat{q}(0) = 1$, $\hat{q}(j) = 0$ for $j > s$, and*

$$\begin{bmatrix} \hat{x}(1)+\hat{x}(1) & \hat{x}(2)+\hat{x}(2) & \cdots & \hat{x}(s)+\hat{x}(s) \\ \hat{x}(0)+\hat{x}(2) & \hat{x}(1)+\hat{x}(3) & \cdots & \hat{x}(s-1)+\hat{x}(s+1) \\ \hat{x}(1)+\hat{x}(3) & \hat{x}(0)+\hat{x}(4) & \cdots & \hat{x}(s-2)+\hat{x}(s+2) \\ \vdots & \vdots & & \vdots \\ \hat{x}(s-3)+\hat{x}(s-1) & \hat{x}(s-4)+\hat{x}(s) & \cdots & \hat{x}(2)+\hat{x}(2s-2) \\ \hat{x}(s-2)+\hat{x}(s) & \hat{x}(s-3)+\hat{x}(s+1) & \cdots & \hat{x}(1)+\hat{x}(2s-1) \end{bmatrix} \begin{bmatrix} \hat{q}(1) \\ \hat{q}(2) \\ \vdots \\ \hat{q}(s) \end{bmatrix} = - \begin{bmatrix} \hat{x}(0) \\ \hat{x}(1) \\ \vdots \\ \hat{x}(s-1) \end{bmatrix}, \tag{1}$$

where $\hat{x}$ denotes the discrete cosine transform of x. Then,

$$supp(x) \subseteq \{j \in [N] : q(j) = 0\}.$$

Proof. The proof is an adaptation of the proof of Theorem 1, as presented in [1]. Let S represent the support of x and define $p \in \mathbb{R}^N$ by

$$p(j) = \prod_{k \in S} \left[1 - \sec\left(\frac{2\pi}{2N-1}k\right)\cos\left(\frac{2\pi}{2N-1}j\right)\right].$$

The reader can verify that $\cos\left(\frac{2\pi}{2N-1}j\right)$ can never vanish for $k \in S$, so p is well-defined. Moreover, by construction, p is a polynomial in $\cos\left(\frac{2\pi}{2N-1}t\right)$ of degree at most s and satisfies $p(j) = 0$ for $j \in S$.

Let $\hat{x}$ and $\hat{p}$ represent the discrete cosine transforms of x and p, respectively. Notice that $\hat{p}(m) = 0$ for $m > s$. Recall that x and p are, up to a scalar multiple, equal to the discrete cosine transforms of $\hat{x}$ and $\hat{p}$, so it is also possible to apply the convolution formula of Section 2 to $\hat{x}$ and $\hat{p}$. Because $p \cdot x \equiv 0$, it follows that $\hat{x} * \hat{p} \equiv 0$, which leads to

$$0 = \sum_{k=0}^{2N-2} \hat{P}(k)\hat{X}(m-k), \quad 0 \le m \le 2N-2, \tag{2}$$

where $\hat{P}$ and $\hat{X}$ denote the symmetric extensions of $\hat{p}$ and $\hat{x}$ to $\mathbb{R}^{2N-1}$. Using the symmetry properties of $\hat{P}$ and $\hat{X}$, along with the fact that $\hat{p}(m) = 0$ for $m > s$, equation (2) can be rewritten for $0 \leq m \leq s$ as

$$\begin{aligned} 0 &= \hat{P}(0)\hat{X}(m) + \sum_{k=1}^{s} \hat{P}(k)\hat{X}(m-k) + \sum_{k=2N-s-1}^{2N-2} \hat{P}(k)\hat{X}(m-k) \\ &= \hat{p}(0)\hat{x}(m) + \sum_{k=1}^{s} \hat{p}(k)\hat{X}(m-k) + \sum_{k=2N-s-1}^{2N-2} \hat{p}(2N-1-k)\hat{X}(m-k) \\ &= \hat{p}(0)\hat{x}(m) + \sum_{k=1}^{s} \left[\hat{X}(m-k) + \hat{X}(m+k)\right]\hat{p}(k) \\ &= \hat{p}(0)\hat{x}(m) + \sum_{k=1}^{m} \left[\hat{x}(m-k) + \hat{x}(m+k)\right]\hat{p}(k) + \sum_{k=m+1}^{s} \left[\hat{x}(m-k) + \hat{x}(m+k)\right]\hat{p}(k). \end{aligned}$$

If this last expression is written in matrix form after dividing by $\hat{p}(0)$, one obtains equation (1) for $q := p/\hat{p}(0)$. The zero set of q is the same as that of p and thus coincides with S.

Finally, suppose that q is any solution of (1) satisfying $\hat{q}(0) = 1$ and $\hat{q}(m) = 0$ for $m > s$. The fact that x is s-sparse implies that $q \cdot x$ is s-sparse and (1) forces $\widehat{q \cdot x}(m) = 0$ for $0 \leq m \leq s-1$. Corollary 3 implies that $q \cdot x \equiv 0$ and thus $\operatorname{supp}(x) \subseteq \{j \in [N] : q(j) = 0\}$, as claimed. Moreover, because $\hat{q}(m) = 0$ for $m > s$, it follows that $\operatorname{card}(\{j \in [N] : q(j) = 0\} \leq s$. ☐

Acknowledgement

The second author would like to express his gratitude to Professor Guido Weiss for the guidance he provided as an advisor, teacher, and friend.

References

[1] Foucart, S., and Rauhut, H., *A Mathematical Introduction to Compressive Sensing*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2013.

[2] Martucci, S., *Symmetric Convolution and the Discrete Sine and Cosine Transforms*, IEEE Trans. Signal Process., **42**(5) (1994), 1038–1051.

[3] Muir, T., *A Treatise on the Theory of Determinants*, Revised and enlarged by William H. Metzler, Dover Publications, New York, 1960.

Department of Mathematics and Statistics, Saint Louis University, 220 North Grand Blvd., St. Louis, MO 63103

E-mail address: benbarros50@gmail.com

Department of Mathematics and Statistics, Saint Louis University, 220 North Grand Blvd., St. Louis, MO 63103

E-mail address: brody.johnson@slu.edu