Another look at periodic wavelets

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Overview

The torus (circle) will be denoted $\mathbb{T}$ and identified with $[0, 1)$.

- The term *periodic wavelet* was originally applied to an orthonormal system for $L^2(\mathbb{T})$ obtained through the periodization of an L^1-wavelet on the line. (see, e.g., the texts of Meyer and Daubechies)

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- In 1994, Plonka & Tasche adapt shift-invariant theory to the torus and construct a more general notion of MRA [4]. (No explicit dilation operation between scales.)

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- In 1994, Plonka & Tasche adapt shift-invariant theory to the torus and construct a more general notion of MRA [4]. (No explicit dilation operation between scales.)
- The goal of this talk is to explore another version of wavelet theory on the torus which begins with explicit dilation and translation operators on the torus.

Preliminary Definitions

Definition

An *orthonormal wavelet* on $\mathbb{R}$ is a function $\psi \in L^2(\mathbb{R})$ such that the collection

$$X(\psi) = \{D^j T^k \psi : j, k \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

Here, D and T are the unitary dilation and translation operators, i.e.,

$$Df(x) = 2^{\frac{1}{2}} f(2x) \qquad Tf(x) = f(x-1).$$

Motivation

- Periodization: $P : L^1(\mathbb{R}) \to L^1(\mathbb{T})$

$$Pf(x) = \sum_{k\in\mathbb{Z}} f(x+k), \quad x \in \mathbb{T}.$$

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- Observe that

$$PDf(x) = 2^{-\frac{1}{2}} \sum_{k\in\mathbb{Z}} f(2^{-1}(x+k)) = 2^{-\frac{1}{2}} \left[Pf(2^{-1}x) + Pf(2^{-1}x + 2^{-1})\right].$$

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- The above calculation motivates a notion of dilation for the torus.

Dilation and Translation on $\mathbb{T}$

- Dilation: $\mathsf{D} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$

$$\mathsf{D}f(x) = 2^{-1}\left(f(2^{-1}x) + f(2^{-1}x + 2^{-1})\right), \quad x \in \mathbb{T}.$$

This definition leads to $PDf = \sqrt{2}\mathsf{D}Pf$.

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- Translation: $\mathsf{T}_N : L^2(\mathbb{T}) \to \mathbb{T}$

$$\mathsf{T}_N f(x) = f(x - N^{-1}), \quad x \in \mathbb{T}.$$

  The index N will be referred to as the *order* of the translation operation.

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- The dilation and translation operators satisfy:

$$\mathsf{T}_N^2\mathsf{D} = \mathsf{D}\mathsf{T}_N.$$

  This corresponds to $T^2D^{-1} = D^{-1}T$ on the line.

Fourier Analysis on $\mathbb{T}$, $\mathbb{Z}$, and $\mathbb{Z}_N$

$L^2(\mathbb{T})$:

$$\boxed{\hat{f}(k) = \int_{\mathbb{T}} f(x) e^{-2\pi i k x}\, dx, \quad k \in \mathbb{Z}}$$

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$\ell(\mathbb{Z}_N)$:

$$\mathcal{F}_N f(n) = \hat{f}(n) = \frac{1}{\sqrt{N}} \sum_{k\in\mathbb{Z}_N} f(k) e^{-2\pi i k n/N}, \quad n \in \mathbb{Z}_N$$

Another Look at Dilation on the Torus

Proposition

For $f \in L^2(\mathbb{T})$, $\widehat{\mathsf{D}f}(k) = \hat{f}(2k)$, $k \in \mathbb{Z}$.

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Some Observations:

- Dilation on the torus performs a *downsampling* of the Fourier coefficients.

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- Dilation on the torus is not invertible, hence MRAs on the torus will be one-sided.

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Remark

Some Observations:

- Dilation on the torus performs a *downsampling* of the Fourier coefficients.
- Dilation on the torus is not invertible, hence MRAs on the torus will be one-sided.
- Dilation of a trigonometric polynomial will eventually result in a constant function, i.e., if f is a trigonometric polynomial then $\mathsf{D}^j f = \hat{f}(0)$ for sufficiently large $j \in \mathbb{N}$.

The Bracket Product

Definition

The *bracket product of order* N of two functions $f, g \in L^2(\mathbb{T})$ is the vector $[\hat{f}, \hat{g}]_N \in \ell(\mathbb{Z}_N)$ defined by

$$[\hat{f}, \hat{g}]_N(n) = N \sum_{k \in \mathbb{Z}} \hat{f}(n + kN)\, \overline{\hat{g}(n + kN)}, \quad 0 \leq n \leq N - 1.$$

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Proposition

For all $f, g \in L^2(\mathbb{T})$,

$$\mathcal{F}_N \left(\{\langle f, \mathsf{T}_N^n g\rangle\}_{n=0}^{N-1} \right) = \frac{1}{\sqrt{N}} [\hat{f}, \hat{g}]_N.$$

Shift-Invariant Spaces

Definition

The *principal shift-invariant space of order N* generated by $\phi \in L^2(\mathbb{T})$ is the finite-dimensional space $V_N(\phi) = \text{span} X_N(\phi)$, where

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The collection $X_N(\phi)$ forms an orthonormal basis for $V_N(\Phi)$ if and only if

$$[\hat{\phi}, \hat{\phi}]_N(n) = 1, \quad n \in \mathbb{Z}_N.$$

Refinable functions on $\mathbb{T}$

Definition

A function $\phi \in L^2(\mathbb{T})$ is said to be *refinable of order* N ($N \in \mathbb{N}$) if there exists a *mask* $c \in \ell(\mathbb{Z}_N)$ such that

$$\mathsf{D}\phi = \sum_{n \in \mathbb{Z}_N} c(n) \mathsf{T}_N^n \phi. \tag{1}$$

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Lemma

Suppose that $\phi \in L^2(\mathbb{T})$ *is refinable of order* N*, then there exists* $m \in \ell(\mathbb{Z}_N)$ *such that*

$$\hat{\phi}(2k) = m(k)\hat{\phi}(k), \quad k \in \mathbb{Z}. \tag{2}$$

Multiresolution Analysis

Definition

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iv) There exists a *scaling function* $\varphi \in V_0$ such that $X_{2^{-j}N}(2^{\frac{j}{2}}\mathsf{D}^j\varphi)$ is an orthonormal basis for V_j.

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Remark

Notice that MRA properties i, ii, and iv imply that a scaling function φ is necessarily refinable of order N. Moreover, it follows from MRA properties iii and iv that $\mathsf{D}^J\varphi$ must be constant and nonzero, implying that $\hat{\varphi}(0) \neq 0$.

Some Observations

Remark

If $\varphi \in L^2(\mathbb{T})$ is refinable of order $N = 2^J$ $(J \in \mathbb{N})$ with filter $m_0 \in \ell(\mathbb{Z}_N)$ then

$$\mathsf{D}^{j+1}\varphi = \sum_{n \in \mathbb{Z}_{2^{-j}N}} \left(\sum_{\ell=0}^{2^j-1} c(n + \ell 2^{-j} N) \right) \mathsf{T}^n_{2^{-j}N} \mathsf{D}^j \varphi,$$

i.e., $\mathsf{D}^j\varphi$ is refinable of order $2^{-j}N$.

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i.e., $\mathsf{D}^j\varphi$ is refinable of order $2^{-j}N$.

It is not difficult to show that $m_0(2^j \cdot)$ is a low-pass filter for $\mathsf{D}^j\varphi$.

A Characterization of Scaling Functions

Theorem

Suppose $\varphi \in L^2(\mathbb{T})$ *is a refinable function of order* $N = 2^J$ $(J \in \mathbb{N})$ *with* $\hat{\varphi}(0) \neq 0$. *Then* φ *is the scaling function of an MRA of order* N *if and only if*

$$|m_0(n)|^2 + |m_0(n + 2^{-1}N)|^2 = 1, \quad n \in \mathbb{Z}_N, \tag{3}$$

and

$$[\hat{\varphi}, \hat{\varphi}]_N(n) = 1, \quad n \in \mathbb{Z}_N. \tag{4}$$

Existence of Scaling Functions

Theorem

Fix $N = 2^J$, $J \in \mathbb{N}$. *Suppose* $m_0 \in \ell(\mathbb{Z}_N)$ *satisfies* (3) *with* $m_0(0) = 1$. *Then* m_0 *is the low-pass filter of a trigonometric polynomial scaling function of order* N.

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The construction:

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3. For $-2^{J-2} \leq k \leq 2^{J-2} - 1$ and $1 \leq j \leq J-1$, define $\hat{\varphi}(2^j(2k+1))$ according to (4.2), i.e.,

$$\hat{\varphi}(2^j(2k+1)) = m_0(2^{j-1}(2k+1))\,\hat{\varphi}(2^{j-1}(2k+1)).$$

Borrowing from the Line

Proposition

Suppose $c \in \ell^2(\mathbb{Z})$ is an absolutely summable sequence whose Fourier transform $m = \hat{c}$ satisfies

$$|m(\xi)|^2 + |m(\xi + 2^{-1})|^2 = 1, \quad \xi \in \mathbb{T},$$

If $c_0 \in \ell(\mathbb{Z}_N)$ is defined by

$$c_0(n) = \sqrt{N} \sum_{k \in \mathbb{Z}} c(n + kN), \quad n \in \mathbb{Z}_N,$$

where $N = 2^J$ ($J \in \mathbb{N}$), then $m_0 = \sqrt{N}\widehat{c_0}$ satisfies (3).

The Haar Scaling Function

Example

Fix $N = 8$ and let $c \in \ell(\mathbb{Z}_N)$ be given by $c(0) = c(1) = \frac{1}{2}$ with $c(n) = 0$ for $n \neq 0, 1$. The low-pass filter $m_0 \in \ell(\mathbb{Z}_N)$ is given by

$$m_0(n) = e^{-\pi i n/8} \cos(n\pi/8), \quad n \in \mathbb{Z}_8.$$

The Haar Scaling Function

$$\begin{aligned}
\hat{\varphi}(-3) &= \frac{1}{\sqrt{8}} \longrightarrow \hat{\varphi}(-6) = \hat{\varphi}(-3)m_0(5) \longrightarrow \hat{\varphi}(-12) = \hat{\varphi}(-6)m_0(5)m_0(2), \\
\hat{\varphi}(-1) &= \frac{1}{\sqrt{8}} \longrightarrow \hat{\varphi}(-2) = \hat{\varphi}(-1)m_0(7) \longrightarrow \hat{\varphi}(-4) = \hat{\varphi}(-2)m_0(7)m_0(6), \\
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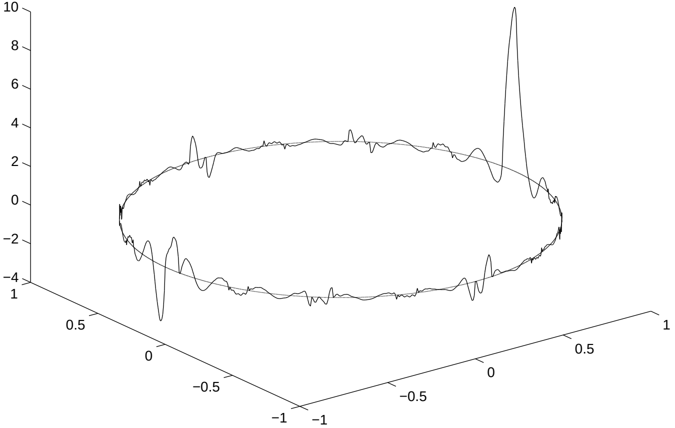
\end{aligned}$$

Each “strand” terminates because $m_0(4) = 0$.

The Haar Scaling Function ($N = 64$)



Orthonormal Wavelets

Definition

Let $\{V_j\}_{j=0}^{J}$ be an MRA of order $N = 2^J$ ($J \in \mathbb{N}$). A function $\psi \in V_0$ is a *wavelet* for the MRA if the collection

$$\{2^{\frac{j}{2}} \mathsf{T}^{n}_{2^{-j}N} \mathsf{D}^{j-1} \psi : 1 \leq j \leq J,\ n \in \mathbb{Z}_{2^{-j}N}\}$$

is an orthonormal basis for $V_0 \ominus V_J$.

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This construction rests on a decomposition of V_j as $V_j = V_{j+1} \oplus W_{j+1}$, $0 \le j \le J-1$, where W_j is of the form

$$W_{j+1} = V_{2^{-(j+1)}N}(\mathsf{D}^j\psi).$$

The High-Pass Filter

Theorem

Suppose that φ is the scaling function of an MRA of order $N = 2^J$ ($J \in \mathbb{N}$) and define $\psi \in V_0$ by

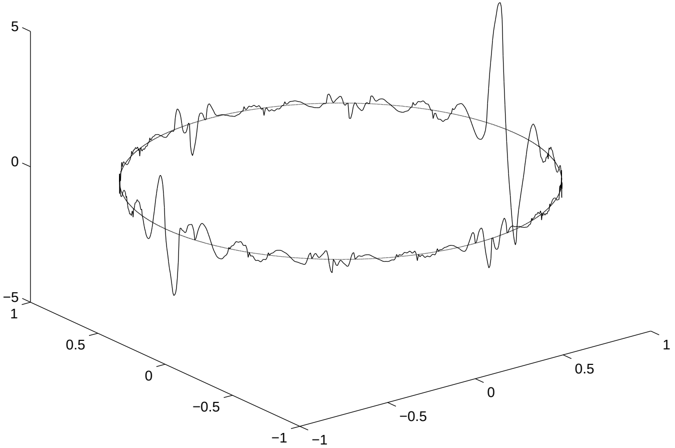
$$\hat{\psi}(k) = m_1(k)\hat{\varphi}(k), \quad k \in \mathbb{Z},$$

where $m_1 \in \ell(\mathbb{Z}_N)$ is chosen as

$$m_1(n) = \overline{m_0(n + 2^{-1}N)}\, e^{-2\pi i n/N}, \quad n \in \mathbb{Z}_N. \tag{5}$$

Then ψ is a wavelet for the MRA.

The Haar Wavelet ($N = 64$)



Approximation in $L^2(\mathbb{T})$

Lemma

Suppose that φ is a scaling function for an MRA of order $N = 2^J$ ($J \in \mathbb{N}$). The orthogonal projection onto $V_0 = V_N(\varphi)$ is described by

$$\widehat{Pf}(k) = [\hat{f}, \hat{\varphi}]_N(k)\hat{\varphi}(k), \quad k \in \mathbb{Z}.$$

Approximation of Trigonometric Monomials

Suppose that f is a trigonometric monomial, i.e., $\hat{f}(k) = \delta_{rk}$ for some $r \in \mathbb{Z}$, then $[\hat{f}, \hat{\varphi}]_N$ is given by

$$[\hat{f}, \hat{\varphi}]_N(n) = \begin{cases} 0, & n \not\equiv r, \\ N\overline{\hat{\varphi}(r)}, & n \equiv r. \end{cases}$$

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The error of approximation is thus given by

$$\|Pf - f\|^2 = \left(N|\hat{\varphi}(r)|^2 - 1\right)^2 + \sum_{k \neq 0} |N\hat{\varphi}(r)\hat{\varphi}(r + kN)|^2,$$

Approximation of Trigonometric Monomials

Because $X_N(\varphi)$ is an orthonormal basis,

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$$1 = [\hat{\varphi}, \hat{\varphi}]_N(r) = N|\hat{\varphi}(r)|^2 + N \sum_{k \neq 0} |\hat{\varphi}(r + kN)|^2.$$

Thus, the approximation error can be rewritten as

$$\|Pf - f\|^2 = 1 - N|\hat{\varphi}(r)|^2.$$

Approximation Error

Definition

The error of approximation, denoted $E_N(k)$, is defined as

$$E_N(k) = \left(1 - N|\hat{\varphi}(k)|^2\right)^{\frac{1}{2}}, \quad k \in \mathbb{Z}.$$

Hence, $E_N(k)$ represents the approximation error $\|Pf - f\|$ where $f = e^{2\pi ikx}$ and P is the orthogonal projection onto $V_N(\varphi)$.

An Approximation Result

If $m(\xi)$ is a continuous function on the torus with $m(0) = 1$ and satisfying (3), one can define a scaling function associated to m of order $N = 2^J$ for each $J > 0$.

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If $m(\xi)$ is a continuous function on the torus with $m(0) = 1$ and satisfying (3), one can define a scaling function associated to m of order $N = 2^J$ for each $J > 0$.

Proposition

Fix $r \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $N = 2^J$ ($J \in \mathbb{N}$) such that $E_N(k) < \varepsilon$ for $|k| < r$, where φ is constructed as above.

The Shannon Scaling Function

Example

Let $m_0 \in \ell(\mathbb{Z}_N)$ be defined by

$$m_0(n) = \begin{cases} 1, & n < \frac{N}{4} \text{ or } n > \frac{3N}{4}, \\ \frac{1}{\sqrt{2}}, & n = \frac{N}{4} \text{ or } n = \frac{3N}{4}, \\ 0, & \text{otherwise}, \end{cases} \qquad n \in \mathbb{Z}_N,$$

where $N = 2^J$ for a natural number $J > 2$.

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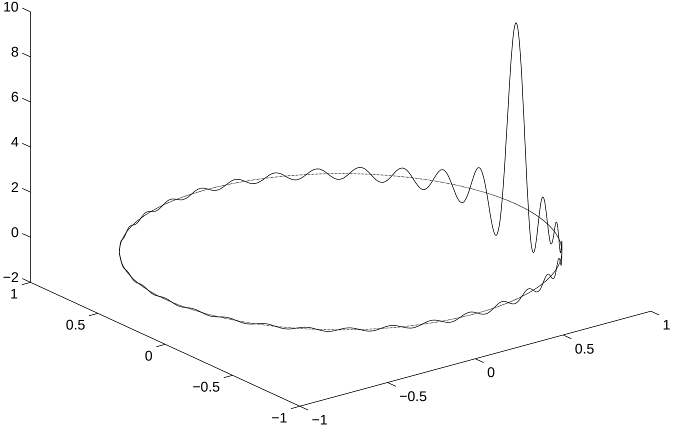
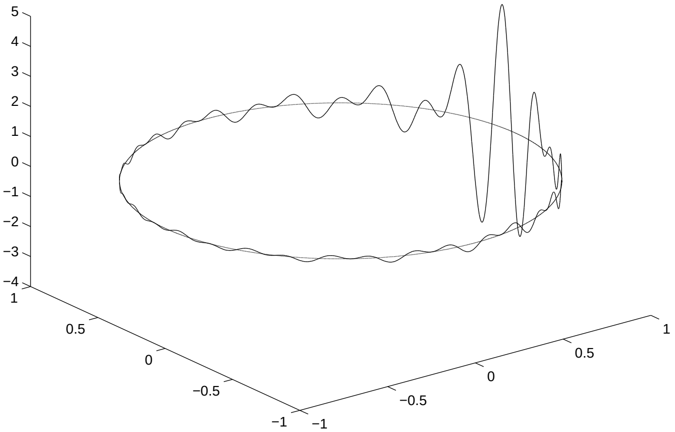
where $N = 2^J$ for a natural number $J > 2$.
If φ is constructed as in Theorem 4.7, then $\hat{\varphi}(k) = \frac{1}{\sqrt{N}}$ whenever $|k| < \frac{N}{2}$.
Hence, $E_N(k)$ is zero for $|k| < \frac{N}{2}$.

The Shannon Scaling Function ($N = 64$)



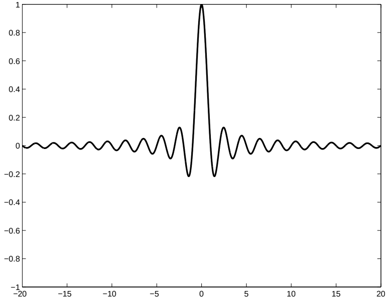
The Shannon Wavelet ($N = 64$)



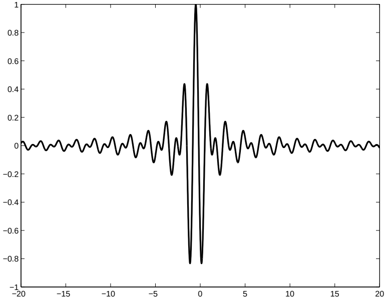
Shannon Wavelet on the Line

Let V_j, $j \in \mathbb{Z}$, be the collection of functions in $L^2(\mathbb{R})$ such that $\hat{f}$ is supported inside $2^j[-\frac{1}{2}, \frac{1}{2}]$. These sets form an MRA associated with

$$\varphi = \frac{\sin(\pi x)}{\pi x} \quad \text{and} \quad \psi = -2\,\frac{\sin(2\pi x) + \cos(\pi x)}{\pi(2x+1)}.$$



φ



ψ

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