Frames in $\mathbb{R}^n$

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Abstract

These notes provide an introduction to the theory of frames for $\mathbb{R}^n$. We begin by recalling some basic facts about the inner product on $\mathbb{R}^n$. We then introduce the notion of a frame for $\mathbb{R}^n$ and show how frames may be used to reconstruct a vector from its inner products with the elements of the frame. We show that a finite collection of vectors in $\mathbb{R}^n$ is a frame if and only if it contains a basis. We conclude by considering an iterative method by which a vector can be approximated from its inner products with the frame elements.

For general treatments of frames the reader is referred to [1], [2], [3], and [4]. We now recall some important facts about $\mathbb{R}^n$.

Definition 1. Let $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

(i) $\vec{x} \cdot \vec{y} = \langle \vec{x}, \vec{y} \rangle := \sum_{k=1}^{n} x_k y_k$ (inner product),

(ii) $\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle} = \left(\sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}$ (length of a vector).

Lemma 1. *Let $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then*

(i) $\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$ *(triangle inequality),*

(ii) $|\langle \vec{x}, \vec{y} \rangle| \le \|\vec{x}\| \, \|\vec{y}\|$ *(Cauchy-Schwarz inequality).*

Proof: We begin with the Cauchy-Schwarz inequality. If $\vec{y} = 0$ the result holds, so assume $\vec{y} \neq 0$. We have for $t \in \mathbb{R}$

$$0 \leq \|\vec{x} - t\vec{y}\|^2 = \langle \vec{x} - t\vec{y}, \vec{x} - t\vec{y} \rangle = \|\vec{x}\|^2 - 2t\langle \vec{x}, \vec{y} \rangle + t^2\|\vec{y}\|^2.$$

Choosing $t = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}$ we then have

$$\|\vec{x}\|^2 - 2\frac{(\langle \vec{x}, \vec{y} \rangle)^2}{\|\vec{y}\|^2} + \frac{(\langle \vec{x}, \vec{y} \rangle)^2}{\|\vec{y}\|^2} \geq 0,$$

which implies $\|\vec{x}\|^2 \, \|\vec{y}\|^2 \geq (\langle \vec{x}, \vec{y} \rangle)^2$. This proves (ii).

Now for (i), we observe that

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2 + 2\|\vec{x}\| \, \|\vec{y}\| - 2\|\vec{x}\| \, \|\vec{y}\| \\ &\leq (\|\vec{x}\| + \|\vec{y}\|)^2. \quad \square \end{aligned}$$

Definition 2. Let $V = \{\vec{v}_1, \ldots, \vec{v}_N\} \subset \mathbb{R}^n$. V is a *frame* for $\mathbb{R}^n$ if and only if there exists $A, B > 0$ such that for each $\vec{x} \in \mathbb{R}^n$ we have

$$A\|\vec{x}\|^2 \leq \sum_{k=1}^{N} \left| \langle \vec{x}, \vec{v}_k \rangle \right|^2 \leq B\|\vec{x}\|^2. \tag{1}$$

Proposition 2. *Suppose $V = \{\vec{v}_1, \ldots, \vec{v}_N\} \subset \mathbb{R}^n$ contains a basis for $\mathbb{R}^n$, then there exists $U = \{\vec{u}_1, \ldots, \vec{u}_N\} \subset \mathbb{R}^n$ (not necessarily unique) such that for all $\vec{x} \in \mathbb{R}^n$*

$$\vec{x} = \sum_{k=1}^{N} \langle \vec{x}, \vec{u}_k \rangle \vec{v}_k. \tag{2}$$

Proof: Without loss of generality we may assume that $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for $\mathbb{R}^n$. Thus, the $n \times n$ matrix

$$\tilde{F} := \begin{pmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \cdots & v_{n,n} \end{pmatrix}$$

is invertible and the equation $\tilde{F}\vec{c} = \vec{x}$ has the solution $\vec{c} = \tilde{F}^{-1}\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$. Using the notation $\vec{c} = (c_1, \ldots, c_n)$ we see that

$$\vec{x} = \sum_{k=1}^{n} c_k \vec{v}_k.$$

For $1 \le k \le n$, let $\vec{u}_k = (\tilde{F}^{-1}_{k,1}, \dots, \tilde{F}^{-1}_{k,n})$, i.e. $\vec{u}_k$ is the k^{th} row of $\tilde{F}^{-1}$. Notice that $c_k = \langle \vec{x}, \vec{u}_k \rangle$ and setting $\vec{u}_k = 0$ for $n+1 \le k \le N$ we have shown that

$$\vec{x} = \sum_{k=1}^{N} \langle \vec{x}, \vec{u}_k \rangle \vec{v}_k. \quad \square$$

The vectors $\vec{u}_k$, $1 \le k \le N$ in Proposition 2 are said to be *dual* to the vectors $\vec{v}_k$, $1 \le k \le N$.

Theorem 3. *Let $V = \{\vec{v}_1, \dots, \vec{v}_N\} \subset \mathbb{R}^n$. V is a frame if and only if V contains a basis for $\mathbb{R}^n$.*

Proof: ($\Rightarrow$) Suppose by way of contradiction that V does not contain a basis. Then the rank of

$$F = \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{N,1} & \cdots & v_{N,n} \end{pmatrix}$$

is strictly less than n. Thus, $F^T F$ has rank strictly less than n (see [5]) and is, therefore, not invertible. This implies the existence of $\vec{x} \ne 0$ such that $F^T F \vec{x} = 0$ and

$$0 = \langle F^T F \vec{x}, \vec{x} \rangle = \sum_{k=1}^{N} |\langle \vec{x}, \vec{v}_k \rangle|^2,$$

a contradiction of the frame equation (1).

($\Leftarrow$) We begin by demonstrating the upper bound. Let $\vec{x} \in \mathbb{R}^n$ and observe that by the Cauchy-Schwarz inequality we have

$$\sum_{k=1}^{n} |\langle \vec{x}, \vec{v}_k \rangle|^2 \le \sum_{k=1}^{N} \|\vec{x}\|^2 \, \|\vec{v}_k\|^2 = \Big(\sum_{k=1}^{N} \|\vec{v}_k\|^2 \Big) \|\vec{x}\|^2.$$

Letting $B := \sum_{k=1}^{N} \|\vec{v}_k\|^2$ we have the right inequality of (1).

Let $\vec{u}_k$, $1 \le k \le N$, be as guaranteed by Proposition 2. By repeating the argument above we obtain the inequality

$$\sum_{k=1}^{n} |\langle \vec{x}, \vec{u}_k \rangle|^2 \le \tilde{B} \|\vec{x}\|^2, \tag{3}$$

for all $\vec{x} \in \mathbb{R}^n$ and where $\tilde{B} := \sum_{k=1}^{N} \|\vec{u}_k\|^2$. By taking inner products with $\vec{x}$ on each side of (2) we obtain

$$\begin{aligned}
\|\vec{x}\|^2 &= \sum_{k=1}^{N} \langle \vec{x}, \vec{u}_k \rangle \langle \vec{x}, \vec{v}_k \rangle \\
&\leq \Big(\sum_{k=1}^{N} |\langle \vec{x}, \vec{u}_k \rangle|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{N} |\langle \vec{x}, \vec{v}_k \rangle|^2 \Big)^{\frac{1}{2}} \\
&\leq \sqrt{\tilde{B}} \|\vec{x}\| \Big(\sum_{k=1}^{N} |\langle \vec{x}, \vec{v}_k \rangle|^2 \Big)^{\frac{1}{2}},
\end{aligned}$$

from which we conclude

$$\frac{1}{\tilde{B}} \|\vec{x}\|^2 \leq \sum_{k=1}^{N} |\langle \vec{x}, \vec{v}_k \rangle|^2.$$

This shows that V is a frame. $\square$

Theorem 3 says that every frame for $\mathbb{R}^n$ contains a basis. The "extra" vectors in the frame, i.e. those not needed for the basis, make the representation redundant. The redundancy of frames is important in many applications, e.g. noise reduction.

Example 1. Suppose $V = \{\vec{v}_1, \dots, \vec{v}_n\} \subset \mathbb{R}^n$ comprises an orthonormal basis for $\mathbb{R}^n$. By considering the relationship between the inner products of the basis vectors and the matrix FF^T, where F is as above, we see that $FF^T = F^T F = I_{n \times n}$. This means for each $x \in \mathbb{R}^n$,

$$\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \langle F^T F \vec{x}, \vec{x} \rangle = \sum_{k=1}^{n} |\langle \vec{x}, \vec{v}_k \rangle|^2.$$

We conclude that any orthonormal basis for $\mathbb{R}^n$ is a frame with $A = B = 1$. This is an example of a *tight frame* with constant one.

Proposition 2 provides one method of reconstructing a vector from knowledge of its inner products with elements of a frame, but the dual frame constructed in the proof fails to take advantage of the redundancy inherent in frames. The following iterative algorithm for frame inversion is better in this sense.

Theorem 4 (Frame algorithm [2]). *Let $\{\vec{v}_k\}_{k=1}^{N}$ be a frame for $\mathbb{R}^n$ with frame bounds A and B. Given $\vec{x} \in \mathbb{R}^n$, define $\vec{x}_0 = \vec{0}$ and for $j \geq 1$*

$$\vec{x}_j = \vec{x}_{j-1} + \frac{2}{A+B} F^T F (\vec{x} - \vec{x}_{j-1}), \tag{4}$$

where F is as above. Then $\vec{x}_j \to \vec{x}$ as $j \to \infty$ in $\mathbb{R}^n$ and

$$\|\vec{x} - \vec{x}_j\| \leq \Big(\frac{B-A}{B+A}\Big)^j \|\vec{x}\|. \tag{5}$$

Proof: Using the frame equation (1) we obtain for each $\vec{x} \in \mathbb{R}^n$ the inequalities

$$-\frac{B-A}{B+A}\|\vec{x}\|^2 \leq \Big\langle \big(I_{n\times n} - \frac{2F^TF}{B+A}\big)\vec{x}, \vec{x} \Big\rangle \leq \frac{B-A}{B+A}\|\vec{x}\|^2.$$

Letting $U = I_{n\times n} - \frac{2}{A+B}F^TF$ we see that U is also self-adjoint (symmetric) and, hence, $\|U\| \leq \frac{B-A}{B+A} < 1$. Now by (4) we have

$$\begin{aligned} \vec{x} - \vec{x}_j &= \vec{x} - \vec{x}_{j-1} - \frac{2}{A+B}F^TF(\vec{x} - \vec{x}_{j-1}) \\ &= \Big(I_{n\times n} - \frac{2}{A+B}F^TF\Big)(\vec{x} - \vec{x}_{j-1}) \\ &= \Big(I_{n\times n} - \frac{2}{A+B}F^TF\Big)^j(\vec{x} - \vec{x}_0) \\ &= \Big(I_{n\times n} - \frac{2}{A+B}F^TF\Big)^j\vec{x}, \end{aligned}$$

from which we conclude

$$\|\vec{x} - \vec{x}_n\| \leq \Big(\frac{B-A}{B+A}\Big)^j \|\vec{x}\|.$$

This establishes both the convergence and the error estimate (5). $\square$

References

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