An Introduction to Frames

Brody Dylan Johnson

St. Louis University

12 October 2010

Overview

This goal of this talk is to introduce students to an area of mathematics called *frame theory*, which draws heavily on linear algebra and finds application in many real-world settings. In essence, frames provide a means for storing the numeric data found in digital signals, such as those originating from images, audio, and video. Frames can be designed for a variety of uses, e.g., data compression, noise reduction, frequency analysis, etc.

- Introduction
- Elements of Linear Algebra
- Frame Fundamentals
- A Simple Tight Frame
- A Bigger Example
- Intuition for 2-D signals
- Show & Tell

The Olden Days

AUDIO

Audio was recorded and stored in a continuous, or *analog*, format. The earliest versions imprinted the signal in wax that could be retraced afterwards to recover the recorded sound. Modern versions of analog recording imprint the signal on magnetic tape (as shown below).

Figure: An 8mm tape reel from the early 1970's.

The Olden Days

Images

Early photography made use of silver compounds that would undergo a chemical reaction when exposed to light. The image was then captured on a copper plate. This later developed into modern film photography where the silver compounds are bonded to a plastic sheet (as shown below).

Figure: An 35mm negative from the early 2000's.

The Digital Revolution

AUDIO

A digital audio signal consists of a discrete sequence of numbers and a *sample rate*. The sample rate describes how many digital samples are taken from the analog signal in a given period of time.

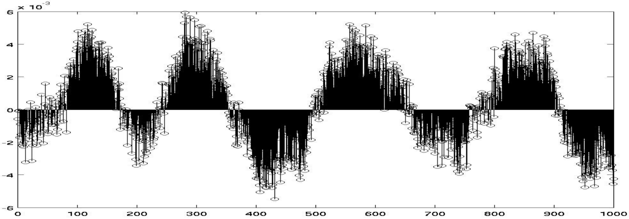


Figure: 1000 of the 8,087,552 samples of the digitized version of the 1970's tape recording. This corresponds to about 0.0227 seconds.

The Digital Revolution

IMAGES

A digital image typically consists of a rectangular array of numbers. The numeric values in the array describe the intensity of light in the image at the corresponding location. For grayscale images the intensity ranges from 0 (black) to 255 (white), while color images combine three separate intensity values (red, green, and blue channels).

Figure: Digitized version of previously shown film negative.

The Digital Revolution

SAMPLING

One obtains a discrete signal from a continuous one by a *sampling* procedure. In the case of digital recording and digital photography the sampling is typically performed by combining special hardware and software.

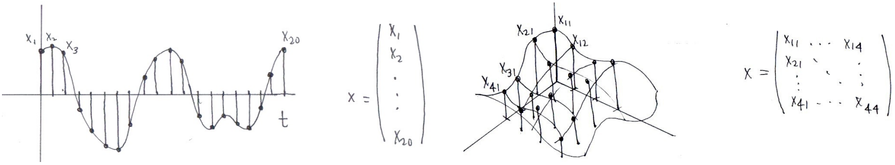


Figure: Representations of one- and two-dimensional sampling.

The Digital Revolution

SAMPLING

One obtains a discrete signal from a continuous one by a *sampling* procedure. In the case of digital recording and digital photography the sampling is typically performed by combining special hardware and software.

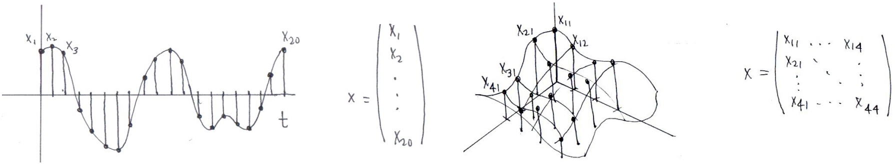


Figure: Representations of one- and two-dimensional sampling.

In either case, one ends up with a *vector* representation of the original signal. This allows us to use our knowledge of Linear Algebra.

Vector Spaces (Dimension 3)

- VECTORS
  A *vector* has the form $x = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$.

Vector Spaces (Dimension 3)

- VECTORS
  A *vector* has the form $x = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$.
- INNER PRODUCTS
  The *inner product* of two vectors x and y is given by

$$\langle x, y\rangle = x \cdot y = x_1y_1 + x_2y_2 + x_3y_3 = \|x\| \, \|y\| \cos\theta,$$

where $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$ and θ is the angle between x and y.

Vector Spaces (Dimension 3)

- VECTORS
  A *vector* has the form $x = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$.
- INNER PRODUCTS
  The *inner product* of two vectors x and y is given by

  $$\langle x, y\rangle = x \cdot y = x_1y_1 + x_2y_2 + x_3y_3 = \|x\|\,\|y\| \cos\theta,$$

  where $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$ and θ is the angle between x and y.
- BASIS
  A *basis* consists of three *linearly independent* vectors $\{u, v, w\}$, where linearly independent means: the only solution of $c_1u + c_2v + c_3w = 0$ is $c_1 = c_2 = c_3 = 0$.

Vector Spaces (Dimension 3)

Given a basis $\{u, v, w\}$, how can one find the coefficients of a given vector x? I.e., what values c_1, c_2, c_3 achieve

$$x = c_1 u + c_2 v + c_3 w?$$

Vector Spaces (Dimension 3)

Given a basis $\{u, v, w\}$, how can one find the coefficients of a given vector x? I.e., what values c_1, c_2, c_3 achieve

$$x = c_1 u + c_2 v + c_3 w?$$

Notice that one can write the above as a matrix equation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_3 \\ u_3 & v_3 & w_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Hence, one can find the coefficients by inverting the 3×3 matrix. Note that this is possible because the vectors are linearly independent, implying that the determinant is nonzero.

Vector Spaces (Dimension 3)

The basis is *orthonormal* if it satisfies

$$u \cdot v = v \cdot w = w \cdot u = 0 \quad \text{(orthogonality)}$$

and

$$u \cdot u = v \cdot v = w \cdot w = 1. \quad \text{(unit length)}$$

Vector Spaces (Dimension 3)

The basis is *orthonormal* if it satisfies

$$u \cdot v = v \cdot w = w \cdot u = 0 \quad \text{(orthogonality)}$$

and

$$u \cdot u = v \cdot v = w \cdot w = 1. \quad \text{(unit length)}$$

Orthogonality allows us to find the coefficients using the inner product:

$$x \cdot u = c_1 u \cdot u + c_2 v \cdot u + c_3 w \cdot u = c_1.$$

$$x \cdot v = c_1 u \cdot v + c_2 v \cdot v + c_3 w \cdot v = c_2.$$

$$x \cdot w = c_1 u \cdot w + c_2 v \cdot w + c_3 w \cdot w = c_3.$$

Vector Spaces (Dimension 3)

The basis is *orthonormal* if it satisfies

$$u \cdot v = v \cdot w = w \cdot u = 0 \quad \text{(orthogonality)}$$

and

$$u \cdot u = v \cdot v = w \cdot w = 1. \quad \text{(unit length)}$$

Orthogonality allows us to find the coefficients using the inner product:

$$x \cdot u = c_1 u \cdot u + c_2 v \cdot u + c_3 w \cdot u = c_1.$$

$$x \cdot v = c_1 u \cdot v + c_2 v \cdot v + c_3 w \cdot v = c_2.$$

$$x \cdot w = c_1 u \cdot w + c_2 v \cdot w + c_3 w \cdot w = c_3.$$

Recall that the inner product $x \cdot y$ can also be written $\langle x, y \rangle$.

Vector Spaces (Dimension 3)

The quantity $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 = \langle x, x \rangle$ is commonly referred to as the squared *length* of the vector x. However, in many applications it is reasonable to consider this quantity as a measure of the *energy* in the signal x.

Vector Spaces (Dimension 3)

The quantity $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 = \langle x, x \rangle$ is commonly referred to as the squared *length* of the vector x. However, in many applications it is reasonable to consider this quantity as a measure of the *energy* in the signal x.

If $\{u, v, w\}$ is an orthonormal basis, then

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle \\ &= \langle c_1 u + c_2 v + c_3 w, c_1 u + c_2 v + c_3 w \rangle \\ &= c_1^2 \langle u, u \rangle + c_2^2 \langle v, v \rangle + c_3^2 \langle w, w \rangle + \text{(zero terms)} \\ &= c_1^2 + c_2^2 + c_3^2 \\ &= \langle x, u \rangle^2 + \langle x, v \rangle^2 + \langle x, w \rangle^2. \end{aligned}$$

Vector Spaces (Dimension 3)

The quantity $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 = \langle x, x\rangle$ is commonly referred to as the squared *length* of the vector x. However, in many applications it is reasonable to consider this quantity as a measure of the *energy* in the signal x.

If $\{u, v, w\}$ is an orthonormal basis, then

$$\begin{aligned}\|x\|^2 &= \langle x, x\rangle \\ &= \langle c_1 u + c_2 v + c_3 w, c_1 u + c_2 v + c_3 w\rangle \\ &= c_1^2\langle u, u\rangle + c_2^2\langle v, v\rangle + c_3^2\langle w, w\rangle + \text{(zero terms)} \\ &= c_1^2 + c_2^2 + c_3^2 \\ &= \langle x, u\rangle^2 + \langle x, v\rangle^2 + \langle x, w\rangle^2.\end{aligned}$$

This shows that the inner products “capture” the energy of the signal.

Vector Spaces (Arbitrary Dimension)

- VECTORS:

$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

Vector Spaces (Arbitrary Dimension)

- VECTORS:

$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

- INNER PRODUCT:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k$$

Vector Spaces (Arbitrary Dimension)

- VECTORS:

$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

- INNER PRODUCT:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k$$

- BASES:
  As in the case of dimension 3, bases consist of n linearly independent vectors $\{v_1, \ldots, v_n\}$ and matrix inversion can be used to determine the coefficients in a basis expansion of the form

$$x = \sum_{k=1}^{n} c_k v_k.$$

Definition of a Frame

Consider the following alternative to a basis, which focuses on the idea of capturing the energy of a signal through inner products.

Definition of a Frame

Consider the following alternative to a basis, which focuses on the idea of capturing the energy of a signal through inner products.

Definition

A collection of vectors $e_1, e_2, \ldots, e_m$ is a *frame* for an n-dimensional vector space if there exist $0 < A \leq B < \infty$ such that for all vectors x,

$$A\|x\|^2 \leq \sum_{k=1}^{m} \langle x, e_k \rangle^2 \leq B\|x\|^2.$$

The numbers $\langle x, e_k \rangle$ will be referred to as *frame coefficients*.

Definition of a Frame

Consider the following alternative to a basis, which focuses on the idea of capturing the energy of a signal through inner products.

Definition

A collection of vectors $e_1, e_2, \ldots, e_m$ is a *frame* for an n-dimensional vector space if there exist $0 < A \leq B < \infty$ such that for all vectors x,

$$A\|x\|^2 \leq \sum_{k=1}^{m} \langle x, e_k \rangle^2 \leq B\|x\|^2.$$

The numbers $\langle x, e_k \rangle$ will be referred to as *frame coefficients*.

Okay, but so what? How can one know that this will be useful?

Question #1:

Does this definition include bases?

Question #1:

Does this definition include bases?

Yes. Notice that for an orthonormal basis one has

$$\|x\|^2 = \sum_{k=1}^{n} \langle x, e_k \rangle^2,$$

which means the vectors form a frame with $A = B = 1$.

Question #1:

Does this definition include bases?

Yes. Notice that for an orthonormal basis one has

$$\|x\|^2 = \sum_{k=1}^{n} \langle x, e_k \rangle^2,$$

which means the vectors form a frame with $A = B = 1$.

One can also prove the following result.

Proposition (see [1])

A collection of vectors is a frame for an n-dimensional vector space if and only if it contains a basis.

Question #2:

How does one use a frame to represent a signal?

Question #2:

How does one use a frame to represent a signal?

Recall that if the basis is orthonormal and we write $x = c_1 e_1 + \cdots + c_m e_m$, then taking inner products with e_j on both sides gives

$$\langle x, e_j \rangle = \sum_{k=1}^{m} c_k \langle e_k, e_j \rangle = c_j \langle e_j, e_j \rangle = c_j.$$

Question #2:

How does one use a frame to represent a signal?

Recall that if the basis is orthonormal and we write $x = c_1 e_1 + \cdots + c_m e_m$, then taking inner products with e_j on both sides gives

$$\langle x, e_j \rangle = \sum_{k=1}^{m} c_k \langle e_k, e_j \rangle = c_j \langle e_j, e_j \rangle = c_j.$$

In other words,

$$x = \sum_{k=1}^{m} \underbrace{\langle x, e_k \rangle}_{\text{coefficient}} \overbrace{e_k}^{\text{vector}}$$

Question #2:

How does one use a frame to represent a signal?

Recall that if the basis is orthonormal and we write $x = c_1 e_1 + \cdots + c_m e_m$, then taking inner products with e_j on both sides gives

$$\langle x, e_j \rangle = \sum_{k=1}^{m} c_k \langle e_k, e_j \rangle = c_j \langle e_j, e_j \rangle = c_j.$$

In other words,

$$x = \sum_{k=1}^{m} \underbrace{\langle x, e_k \rangle}_{\text{coefficient}} \overbrace{e_k}^{\text{vector}}$$

This motivates a slightly different question.

Question #2: (modified)

How does one recover x from the frame coefficients $\{\langle x, e_k \rangle\}_{k=1}^{m}$?

Question #2: (modified)

How does one recover x from the frame coefficients $\{\langle x, e_k\rangle\}_{k=1}^{m}$?

Well, we can try the same solution. Define the *frame operator* S by

$$Sx = \sum_{k=1}^{m} \langle x, e_k\rangle \, e_k.$$

Question #2: (modified)

How does one recover x from the frame coefficients $\{\langle x, e_k \rangle\}_{k=1}^{m}$?

Well, we can try the same solution. Define the *frame operator* S by

$$Sx = \sum_{k=1}^{m} \langle x, e_k \rangle \, e_k.$$

If $Sx = x$, then it follows that $A = B = 1$ because

$$\|x\|^2 = \langle x, x \rangle = \langle Sx, x \rangle = \left\langle \sum_{k=1}^{m} \langle x, e_k \rangle e_k, x \right\rangle = \sum_{k=1}^{m} \langle x, e_k \rangle^2.$$

Thus, when $A \neq B$ recovery of x cannot be this easy.

Question #2: (modified)

However, using a little bit of advanced linear algebra one can prove the following theorem about recovery from frame coefficients.

Question #2: (modified)

However, using a little bit of advanced linear algebra one can prove the following theorem about recovery from frame coefficients.

Theorem (Frame Algorithm – see [1])

Given a frame $\{e_k\}_{k=1}^m$ one may recover x from its frame coefficients as follows. Define $x_0 = 0$ and

$$x_j = x_{j-1} + \frac{2}{A+B} S(x - x_{j-1}).$$

Then, $\|x - x_j\| \leq \left(\frac{B-A}{B+A}\right)^j \|x\|$.

Question #2: (modified)

However, using a little bit of advanced linear algebra one can prove the following theorem about recovery from frame coefficients.

Theorem (Frame Algorithm – see [1])

Given a frame $\{e_k\}_{k=1}^m$ *one may recover* x *from its frame coefficients as follows. Define* $x_0 = 0$ *and*

$$x_j = x_{j-1} + \frac{2}{A+B} S(x - x_{j-1}).$$

Then, $\|x - x_j\| \leq \left(\frac{B-A}{B+A}\right)^j \|x\|$.

Notice that we only need the coefficients $\langle x, e_k \rangle$ to compute Sx.

Some Remarks

- The Frame Algorithm converges geometrically (error is reduced by the same factor with each iteration) and in the case that $A = B$, convergence is immediate. Frames where $A = B$ are called *tight frames*.

Some Remarks

- The Frame Algorithm converges geometrically (error is reduced by the same factor with each iteration) and in the case that $A = B$, convergence is immediate. Frames where $A = B$ are called *tight frames*.
- It is also possible to construct a *dual frame* consisting of vectors $\{\tilde{e}_k\}_{k=1}^m$ so that for all signals x one has

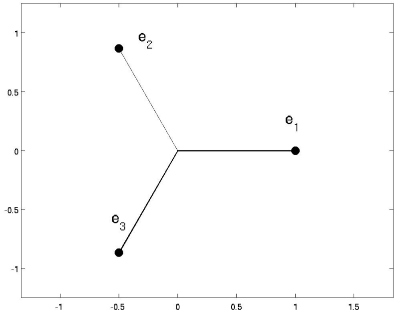
$$x = \sum_{k=1}^{m} \langle x, e_k \rangle \tilde{e}_k.$$

One can even find the $\tilde{e}_k$ vectors using the Frame Algorithm, since $\tilde{e}_k = S^{-1} e_k$, or, $e_k = S\tilde{e}_k$.

A tight-frame for $\mathbb{R}^2$

Define $\{e_1, e_2, e_3\}$ by

$$e_1 = (1,0) \quad e_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \quad e_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$



A tight-frame for $\mathbb{R}^2$

To see that the collection is a frame, let $x = (x_1, x_2)$.

$$\begin{aligned}
\sum_{k=1}^{3} \langle x, e_k \rangle^2 &= \langle x, e_1 \rangle^2 + \langle x, e_2 \rangle^2 + \langle x, e_3 \rangle^2 \\
&= x_1^2 + \left(-\frac{1}{2} x_1 + \frac{\sqrt{3}}{2} x_2 \right)^2 + \left(-\frac{1}{2} x_1 - \frac{\sqrt{3}}{2} x_2 \right)^2 \\
&= x_1^2 + \frac{1}{4} x_1^2 - \frac{\sqrt{3}}{2} x_1 x_2 + \frac{3}{4} x_2^2 + \frac{1}{4} x_1^2 + \frac{\sqrt{3}}{2} x_1 x_2 + \frac{3}{4} x_2^2 \\
&= \frac{3}{2} (x_1^2 + x_2^2) \\
&= \frac{3}{2} \|x\|^2.
\end{aligned}$$

A tight-frame for $\mathbb{R}^2$

To see that the collection is a frame, let $x = (x_1, x_2)$.

$$\begin{aligned}\sum_{k=1}^{3} \langle x, e_k \rangle^2 &= \langle x, e_1 \rangle^2 + \langle x, e_2 \rangle^2 + \langle x, e_3 \rangle^2 \\ &= x_1^2 + \left(-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \right)^2 + \left(-\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2 \right)^2 \\ &= x_1^2 + \frac{1}{4}x_1^2 - \frac{\sqrt{3}}{2}x_1x_2 + \frac{3}{4}x_2^2 + \frac{1}{4}x_1^2 + \frac{\sqrt{3}}{2}x_1x_2 + \frac{3}{4}x_2^2 \\ &= \frac{3}{2}(x_1^2 + x_2^2) \\ &= \frac{3}{2}\|x\|^2.\end{aligned}$$

So, it's actually a tight frame.

A Random Frame of $\mathbb{R}^{10}$ of Size 25

The first 15 vectors:

-0.29	0.24	-0.18	0.33	0.61	-0.11	0.95	-0.29	0.70	-0.85	-0.91	-0.47	-0.81	0.64	-0.66
-0.62	-0.51	-0.43	0.45	0.66	-0.27	-0.56	-0.90	-0.58	-0.60	0.20	1.00	-0.97	-0.47	0.08
-0.02	0.17	-0.21	-0.44	-0.67	-0.39	0.41	0.51	-0.09	-0.90	0.90	-0.58	-0.42	0.51	0.25
-0.18	0.01	0.01	-0.48	-0.21	0.70	0.04	0.79	-0.84	0.13	-0.42	-0.00	0.63	0.32	0.37
-0.07	-0.07	0.44	0.42	0.04	0.52	0.87	-0.43	0.70	-0.76	0.78	-0.42	0.97	-0.57	0.35
0.22	0.08	-0.39	0.57	0.44	0.90	0.43	-0.50	0.12	0.04	-0.80	0.35	-0.97	0.20	0.75
-0.86	0.88	-0.78	0.97	0.14	0.12	-0.54	0.87	-0.36	-0.77	-0.87	0.92	0.64	0.21	-0.97
-0.37	-0.32	-0.11	-0.05	-0.08	-0.97	-0.10	-0.74	-0.25	0.54	-0.53	0.53	0.24	0.32	-0.38
0.22	-0.20	-0.07	0.81	-0.11	0.19	-0.66	0.88	0.74	-0.25	0.87	0.33	0.12	-0.63	0.56
-0.65	-0.38	-0.97	-0.10	-0.82	0.63	0.94	0.40	-0.26	0.65	-0.87	-0.74	-0.51	0.27	-0.39

The last 10 vectors:

0.85	0.02	0.90	0.30	-0.32	0.06	0.94	0.04	-0.20	0.33
0.36	-0.85	0.66	0.51	-0.07	-0.64	-0.95	0.79	-0.28	-0.73
-0.85	-0.61	0.84	0.33	0.83	0.00	0.74	0.88	-0.43	-0.96
-0.86	-0.24	-0.77	0.77	-0.54	-0.16	-0.95	-0.33	0.74	-0.48
-0.98	-0.45	0.62	-0.46	0.72	0.32	0.04	-0.13	0.25	-0.77
-0.55	0.54	0.82	-0.16	0.31	0.35	-0.62	-0.06	-0.52	-0.86
0.03	-0.37	-0.69	-0.57	0.78	0.91	0.43	-0.70	0.96	0.71
-0.08	0.28	-0.76	-0.93	-0.02	-0.62	-0.50	-0.73	0.28	-0.64
0.41	0.97	0.53	-0.84	0.99	-0.78	0.87	0.06	-0.54	-0.94
0.16	0.01	0.44	0.70	-0.25	0.13	-0.73	0.45	0.36	0.47

A frame of $\mathbb{R}^{10}$ of size 25

- Let F be the 10×25 matrix having these vectors as its columns.

$$F = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_{25} \\ | & | & & | \end{pmatrix}.$$

A frame of $\mathbb{R}^{10}$ of size 25

- Let F be the 10×25 matrix having these vectors as its columns.

$$F = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_{25} \\ | & | & & | \end{pmatrix}.$$

- To compute the frame coefficients one multiplies the transpose by a given vector:

$$F^T x = \begin{pmatrix} - & e_1 & - \\ - & e_2 & - \\ & \vdots & \\ - & e_{25} & - \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix} = \begin{pmatrix} \langle x, e_1 \rangle \\ \langle x, e_2 \rangle \\ \vdots \\ \langle x, e_{25} \rangle \end{pmatrix}.$$

A frame of $\mathbb{R}^{10}$ of size 25

The frame operator can also be computed using F. In fact, $S = FF^T$ and is shown below: (to two decimal places)

8.18	0.30	0.83	-3.62	-0.68	1.36	0.70	-1.75	-1.04	1.94
0.30	9.67	0.86	-0.88	-0.69	2.38	-1.75	0.62	-0.00	-0.21
0.83	0.86	8.79	-0.74	3.20	0.41	-2.32	-2.85	2.20	-0.07
-3.62	-0.88	-0.74	7.05	-0.35	-0.18	0.87	0.66	-3.39	2.17
-0.68	-0.69	3.20	-0.35	7.89	1.34	-0.66	-1.27	2.33	-1.99
1.36	2.38	0.41	-0.18	1.34	7.20	-1.59	-0.27	1.04	1.21
0.70	-1.75	-2.32	0.87	-0.66	-1.59	12.34	1.05	-0.55	-0.26
-1.75	0.62	-2.85	0.66	-1.27	-0.27	1.05	6.16	-0.75	-1.10
-1.04	-0.00	2.20	-3.39	2.33	1.04	-0.55	-0.75	9.83	-3.82
1.94	-0.21	-0.07	2.17	-1.99	1.21	-0.26	-1.10	-3.82	7.78

The minimum and maximum eigenvalues of this matrix (Matlab) actually determine the frame bounds.

`1.725 2.785 3.777 5.413 6.72 8.69 11.00 11.94 14.32 18.50`

Implementing the Frame Algorithm

The movie below shows 20 iterations of the frame algorithm.

Implementing the Frame Algorithm

The movie below shows 20 iterations of the frame algorithm.

The convergence is slow because $\dfrac{B-A}{B+A} \approx 0.8294$.

Another frame of $\mathbb{R}^{10}$ of size 25

The first 15 vectors:

1.00	0.97	0.87	0.72	0.52	0.28	0.03	-0.22	-0.46	-0.67	-0.84	-0.95	-1.00	-0.98	-0.90
1.00	0.87	0.52	0.03	-0.46	-0.84	-1.00	-0.90	-0.57	-0.10	0.40	0.80	0.99	0.93	0.62
1.00	0.72	0.03	-0.67	-1.00	-0.76	-0.10	0.62	0.99	0.80	0.16	-0.57	-0.98	-0.84	-0.22
1.00	0.52	-0.46	-1.00	-0.57	0.40	0.99	0.62	-0.35	-0.98	-0.67	0.28	0.97	0.72	-0.22
1.00	0.28	-0.84	-0.76	0.40	0.99	0.16	-0.90	-0.67	0.52	0.97	0.03	-0.95	-0.57	0.62
1.00	0.03	-1.00	-0.10	0.99	0.16	-0.98	-0.22	0.97	0.28	-0.95	-0.35	0.93	0.40	-0.90
1.00	-0.22	-0.90	0.62	0.62	-0.90	-0.22	1.00	-0.22	-0.90	0.62	0.62	-0.90	-0.22	1.00
1.00	-0.46	-0.57	0.99	-0.35	-0.67	0.97	-0.22	-0.76	0.93	-0.10	-0.84	0.87	0.03	-0.90
1.00	-0.67	-0.10	0.80	-0.98	0.52	0.28	-0.90	0.93	-0.35	-0.46	0.97	-0.84	0.16	0.62
1.00	-0.84	0.40	0.16	-0.67	0.97	-0.95	0.62	-0.10	-0.46	0.87	-1.00	0.80	-0.35	-0.22

The last 10 vectors:

-0.76	-0.57	-0.35	-0.10	0.16	0.40	0.62	0.80	0.93	0.99
0.16	-0.35	-0.76	-0.98	-0.95	-0.67	-0.22	0.28	0.72	0.97
0.52	0.97	0.87	0.28	-0.46	-0.95	-0.90	-0.35	0.40	0.93
-0.95	-0.76	0.16	0.93	0.80	-0.10	-0.90	-0.84	0.03	0.87
0.93	-0.10	-0.98	-0.46	0.72	0.87	-0.22	-1.00	-0.35	0.80
-0.46	0.87	0.52	-0.84	-0.57	0.80	0.62	-0.76	-0.67	0.72
-0.22	-0.90	0.62	0.62	-0.90	-0.22	1.00	-0.22	-0.90	0.62
0.80	0.16	-0.95	0.72	0.28	-0.98	0.62	0.40	-1.00	0.52
-1.00	0.72	0.03	-0.76	0.99	-0.57	-0.22	0.87	-0.95	0.40
0.72	-0.98	0.93	-0.57	0.03	0.52	-0.90	0.99	-0.76	0.28

Implementing the Frame Algorithm

The movie below shows 20 iterations of the frame algorithm.

Implementing the Frame Algorithm

The movie below shows 20 iterations of the frame algorithm.

The convergence is faster because $\dfrac{B-A}{B+A} \approx 0.1723$. (This frame was not chosen randomly.)

Vector Spaces for 2-D Signals

The following representation is used for two-dimensional signals, e.g., images.

Vector Spaces for 2-D Signals

The following representation is used for two-dimensional signals, e.g., images.

- Vectors:

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

Vector Spaces for 2-D Signals

The following representation is used for two-dimensional signals, e.g., images.

- Vectors:

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

- Inner Product:

$$\langle x, y \rangle = \sum_{j=1}^{m} \sum_{k=1}^{n} x_{j,k}\, y_{j,k}$$

Vector Spaces for 2-D Signals

The following representation is used for two-dimensional signals, e.g., images.

- Vectors:

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

- Inner Product:

$$\langle x, y \rangle = \sum_{j=1}^{m} \sum_{k=1}^{n} x_{j,k}\, y_{j,k}$$

- Basis: A collection $\{u_{j,k} : 1 \leq j \leq m, 1 \leq k \leq n\}$ is a basis provided that the only solution of

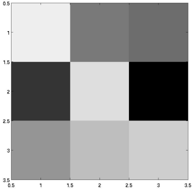
$$\sum_{j=1}^{m} \sum_{k=1}^{n} c_{j,k}\, u_{j,k} = 0$$

is the trivial solution $c_{j,k} = 0$, $1 \leq j \leq m$, $1 \leq k \leq n$.

3 × 3 Image Space

- An example signal:

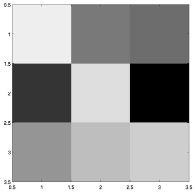
$$x = \begin{pmatrix} 0.95 & 0.49 & 0.45 \\ 0.23 & 0.89 & 0.02 \\ 0.61 & 0.76 & 0.82 \end{pmatrix}$$



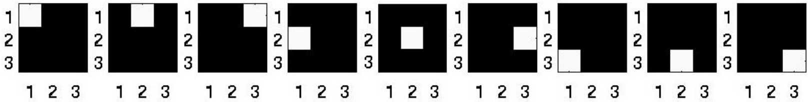
3 × 3 Image Space

- An example signal:

$$x = \begin{pmatrix} 0.95 & 0.49 & 0.45 \\ 0.23 & 0.89 & 0.02 \\ 0.61 & 0.76 & 0.82 \end{pmatrix}$$



- An orthonormal basis:



Wavelet Basis

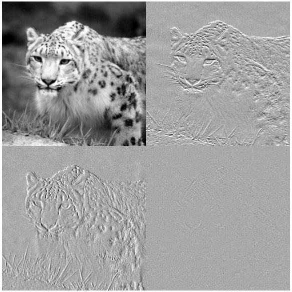
Below we see an image and its wavelet basis coefficients. (no redundancy)

Original

Wavelet Basis

Below we see an image and its wavelet basis coefficients. (no redundancy)

Original

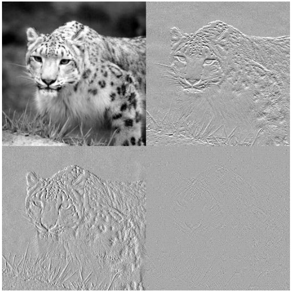


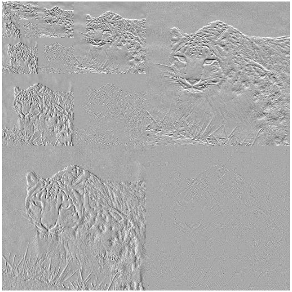
Single Scale

Wavelet Basis

Below we see an image and its wavelet basis coefficients. (no redundancy)


Original

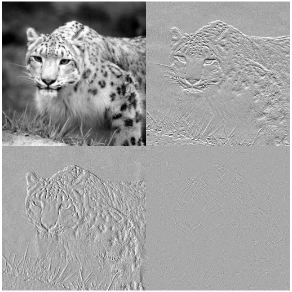

Single Scale


Full

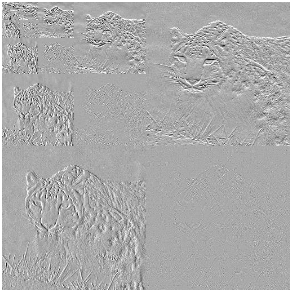
Wavelet Basis

Below we see an image and its wavelet basis coefficients. (no redundancy)

Original



Single Scale



Full

The gray pixels correspond to coefficients close to zero. Black and white pixels correspond to - and + coefficients, respectively.

Wavelet Basis – Compression

By ignoring small coefficients one can "compress" the image.

Original

Wavelet Basis – Compression

By ignoring small coefficients one can “compress” the image.

Original

Reconstructed

Wavelet Basis – Compression

By ignoring small coefficients one can “compress” the image.

Original

Reconstructed

The compression ratio here is 33.8 with a mean-squared error of 2.66.

Wavelet Frame – Algorithme À Trous

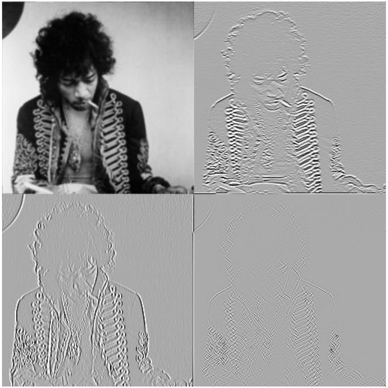
Below we see an image and its wavelet frame coefficients (highly redundant).

Original

Wavelet Frame – Algorithme À Trous

Below we see an image and its wavelet frame coefficients (highly redundant).

Original

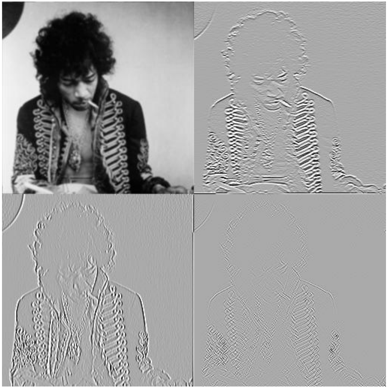


Single Scale

Wavelet Frame – Algorithme À Trous

Below we see an image and its wavelet frame coefficients (highly redundant).


Original


Single Scale

Each component of the coefficient image has the same size as the original.

Wavelet Frame – Denoising

By ignoring small coefficients one can remove noise from the image. The greater redundancy helps preserve the original signal features.

Noisy

Wavelet Frame – Denoising

By ignoring small coefficients one can remove noise from the image. The greater redundancy helps preserve the original signal features.

Noisy

Reconstructed

Wavelet Frame – Denoising

By ignoring small coefficients one can remove noise from the image. The greater redundancy helps preserve the original signal features.

Noisy

Reconstructed

As the level of the noise increases denoising will begin to affect important signal features, resulting in a blurring of the image.

Discrete Cosine Transform (DCT) – Basis

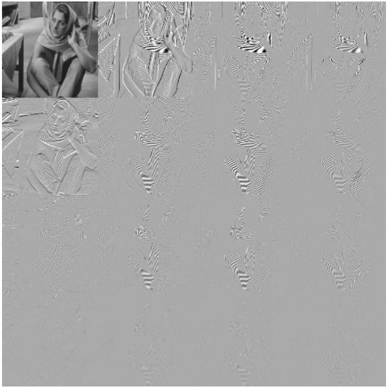
The JPEG image standard originally made use of a basis whose elements are described by discrete cosine functions. (Newer JPEG standards use wavelets.)

Original

Discrete Cosine Transform (DCT) – Basis

The JPEG image standard originally made use of a basis whose elements are described by discrete cosine functions. (Newer JPEG standards use wavelets.)

Original

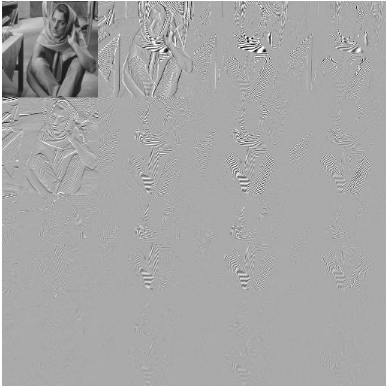


DCT

Discrete Cosine Transform (DCT) – Basis

The JPEG image standard originally made use of a basis whose elements are described by discrete cosine functions. (Newer JPEG standards use wavelets.)

Original



DCT

The periodic nature of the basis elements lends itself to representation of signals with repeating patterns.

Concluding Remarks

- The rise of computers has opened new avenues of research in mathematics and other fields. Frame theory is just one example.

Concluding Remarks

- The rise of computers has opened new avenues of research in mathematics and other fields. Frame theory is just one example.
- Frame theory is very accessible to students with some exposure to linear algebra. A little analysis doesn't hurt.

Concluding Remarks

- The rise of computers has opened new avenues of research in mathematics and other fields. Frame theory is just one example.
- Frame theory is very accessible to students with some exposure to linear algebra. A little analysis doesn't hurt.
- Computational platforms like Matlab are great for working with real data (like audio or images). There is also an open-source Matlab emulator called Octave:

  `http://www.octave.org/`

Concluding Remarks

- The rise of computers has opened new avenues of research in mathematics and other fields. Frame theory is just one example.
- Frame theory is very accessible to students with some exposure to linear algebra. A little analysis doesn't hurt.
- Computational platforms like Matlab are great for working with real data (like audio or images). There is also an open-source Matlab emulator called Octave:

  `http://www.octave.org/`

THANK YOU!

References

B.D. Johnson,
Frames in $\mathbb{R}^n$, expository notes,
http://mathcs.slu.edu/~johnson/public/maths/frames.pdf.