An Introduction to Frames

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Overview

This goal of this talk is to introduce students to an area of mathematics called frame theory, which draws heavily on linear algebra and finds application in many real-world settings. In essence, frames provide a means for storing the numeric data found in digital signals, such as those originating from images, audio, and video. Frames can be designed for a variety of uses, e.g., data compression, noise reduction, frequency analysis, etc.
Outline

- Introduction
- Elements of Linear Algebra
- Frame Fundamentals
- A Simple Tight Frame
- A Bigger Example
- Intuition for 2-D signals
- Show & Tell
Audio

Audio was recorded and stored in a continuous, or analog, format. The earliest versions imprinted the signal in wax that could be retraced afterwards to recover the recorded sound. Modern versions of analog recording imprint the signal on magnetic tape (as shown below).

Figure: An 8mm tape reel from the early 1970’s.
The Olden Days

IMAGES
Early photography made use of silver compounds that would undergo a chemical reaction when exposed to light. The image was then captured on a copper plate. This later developed into modern film photography where the silver compounds are bonded to a plastic sheet (as shown below).

Figure: An 35mm negative from the early 2000’s.
A digital audio signal consists of a discrete sequence of numbers and a *sample rate*. The sample rate describes how many digital samples are taken from the analog signal in a given period of time.

**Figure:** 1000 of the 8,087,552 samples of the digitized version of the 1970’s tape recording. This corresponds to about 0.0227 seconds.
Images
A digital image typically consists of a rectangular array of numbers. The numeric values in the array describe the intensity of light in the image at the corresponding location. For grayscale images the intensity ranges from 0 (black) to 255 (white), while color images combine three separate intensity values (red, green, and blue channels).

Figure: Digitized version of previously shown film negative.
**Sampling**

One obtains a discrete signal from a continuous one by a *sampling* procedure. In the case of digital recording and digital photography the sampling is typically performed by combining special hardware and software.

\[ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{20} \end{pmatrix} \]

**Figure:** Representations of one- and two-dimensional sampling.
**Sampling**

One obtains a discrete signal from a continuous one by a *sampling* procedure. In the case of digital recording and digital photography the sampling is typically performed by combining special hardware and software.

![Figure: Representations of one- and two-dimensional sampling.](image)

In either case, one ends up with a *vector* representation of the original signal. This allows us to use our knowledge of Linear Algebra.
**VECTORS**

A *vector* has the form \( x = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k} \).
Vector Spaces (Dimension 3)

- **Vectors**
  A vector has the form \( x = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k} \).

- **Inner Products**
  The inner product of two vectors \( x \) and \( y \) is given by

  \[
  \langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 = \|x\| \|y\| \cos \theta,
  \]

  where \( \|x\|^2 = x_1^2 + x_2^2 + x_3^2 \) and \( \theta \) is the angle between \( x \) and \( y \).
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  \]
  where \( \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2 \) and \( \theta \) is the angle between \( \mathbf{x} \) and \( \mathbf{y} \).

- **Basis**
  A basis consists of three linearly independent vectors \( \{u, v, w\} \), where
  linearly independent means: the only solution of \( c_1 u + c_2 v + c_3 w = 0 \) is \( c_1 = c_2 = c_3 = 0 \).
Given a basis \( \{u, v, w\} \), how can one find the coefficients of a given vector \( x \)?

I.e., what values \( c_1, c_2, c_3 \) achieve

\[
x = c_1 u + c_2 v + c_3 w?
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\[
x = c_1 u + c_2 v + c_3 w?
\]

Notice that one can write the above as a matrix equation:

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
u_1 & v_1 & w_1 \\
u_2 & v_2 & w_3 \\
u_3 & v_3 & w_3
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}.
\]

Hence, one can find the coefficients by inverting the \( 3 \times 3 \) matrix. Note that this is possible because the vectors are linearly independent, implying that the determinant is nonzero.
The basis is *orthonormal* if it satisfies

\[ u \cdot v = v \cdot w = w \cdot u = 0 \]  (orthogonality)

and

\[ u \cdot u = v \cdot v = w \cdot w = 1. \]  (unit length)
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Orthogonality allows us to find the coefficients using the inner product:

\[ x \cdot u = c_1 u \cdot u + c_2 v \cdot u + c_3 w \cdot u = c_1. \]

\[ x \cdot v = c_1 u \cdot v + c_2 v \cdot v + c_3 w \cdot v = c_2. \]

\[ x \cdot w = c_1 u \cdot w + c_2 v \cdot w + c_3 w \cdot w = c_3. \]
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\[ x \cdot w = c_1 u \cdot w + c_2 v \cdot w + c_3 w \cdot w = c_3. \]

Recall that the inner product \( x \cdot y \) can also be written \( \langle x, y \rangle \).
The quantity $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 = \langle x, x \rangle$ is commonly referred to as the squared length of the vector $x$. However, in many applications it is reasonable to consider this quantity as a measure of the energy in the signal $x$. 

This shows that the inner products "capture" the energy of the signal.
Elements of Linear Algebra

Vector Spaces (Dimension 3)

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If \{u, v, w\} is an orthonormal basis, then

\[
\|x\|^2 = \langle x, x \rangle \\
= \langle c_1 u + c_2 v + c_3 w, c_1 u + c_2 v + c_3 w \rangle \\
= c_1^2 \langle u, u \rangle + c_2^2 \langle v, v \rangle + c_3^2 \langle w, w \rangle + \text{(zero terms)} \\
= c_1^2 + c_2^2 + c_3^2 \\
= \langle x, u \rangle^2 + \langle x, v \rangle^2 + \langle x, w \rangle^2.
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The quantity $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 = \langle x, x \rangle$ is commonly referred to as the squared length of the vector $x$. However, in many applications it is reasonable to consider this quantity as a measure of the energy in the signal $x$.

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**Vectors:**

\[ x = (x_1 \ x_2 \ \cdots \ x_n) \]
Vector Spaces (Arbitrary Dimension)

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  \[ \langle x, y \rangle = x_1y_1 + \cdots + x_ny_n = \sum_{k=1}^{n} x_ky_k \]
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- **Bases:**
  As in the case of dimension 3, bases consist of \( n \) linearly independent vectors \( \{v_1, \ldots, v_n\} \) and matrix inversion can be used to determine the coefficients in a basis expansion of the form
  \[ x = \sum_{k=1}^{n} c_k v_k. \]
Definition of a Frame

Consider the following alternative to a basis, which focuses on the idea of capturing the energy of a signal through inner products.

Definition

A collection of vectors $e_1, e_2, \ldots, e_m$ is a frame for an $n$-dimensional vector space if there exist $0 < A \leq B < \infty$ such that for all vectors $x$,

$$A \|x\|_2^2 \leq m \sum_{k=1}^m \langle x, e_k \rangle_2^2 \leq B \|x\|_2^2.$$ 

The numbers $\langle x, e_k \rangle$ will be referred to as frame coefficients.
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Okay, but so what? How can one know that this will be useful?
Question #1:

Does this definition include bases?

Yes. Notice that for an orthonormal basis one has

\[ \| x \|_2^2 = \sum_{k=1}^{n} \langle x, e_k \rangle^2, \]

which means the vectors form a frame with \( A = B = 1 \).

One can also prove the following result.

Proposition (see [1]):

A collection of vectors is a frame for an \( n \)-dimensional vector space if and only if it contains a basis.
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Question #2:

How does one use a frame to represent a signal?
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Recall that if the basis is orthonormal and we write $x = c_1 e_1 + \cdots + c_m e_m$, then taking inner products with $e_j$ on both sides gives

$$
\langle x, e_j \rangle = \sum_{k=1}^{m} c_k \langle e_k, e_j \rangle = c_j \langle e_j, e_j \rangle = c_j.
$$
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In other words,

\[
x = \sum_{k=1}^{m} \langle x, e_k \rangle e_k
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coefficient vector
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In other words,

$$x = \sum_{k=1}^{m} \langle x, e_k \rangle \begin{bmatrix} e_k \end{bmatrix} \text{ coefficient vector}.$$  

This motivates a slightly different question.
How does one recover $x$ from the frame coefficients $\{\langle x, e_k \rangle\}_{k=1}^m$?
How does one recover \( x \) from the frame coefficients \( \{ \langle x, e_k \rangle \}_{k=1}^{m} \)?

Well, we can try the same solution. Define the frame operator \( S \) by

\[
Sx = \sum_{k=1}^{m} \langle x, e_k \rangle e_k.
\]
Question #2: (modified)

How does one recover $x$ from the frame coefficients $\{\langle x, e_k \rangle\}_{k=1}^m$?

Well, we can try the same solution. Define the frame operator $S$ by

$$Sx = \sum_{k=1}^m \langle x, e_k \rangle e_k.$$

If $Sx = x$, then it follows that $A = B = 1$ because

$$\|x\|^2 = \langle x, x \rangle = \langle Sx, x \rangle = \left\langle \sum_{k=1}^m \langle x, e_k \rangle e_k, x \right\rangle = \sum_{k=1}^m \langle x, e_k \rangle^2.$$

Thus, when $A \neq B$ recovery of $x$ cannot be this easy.
Question #2: (modified)

However, using a little bit of advanced linear algebra one can prove the following theorem about recovery from frame coefficients.
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**Theorem (Frame Algorithm – see [1])**

Given a frame \( \{e_k\}_{k=1}^m \) one may recover \( x \) from its frame coefficients as follows. Define \( x_0 = 0 \) and

\[
x_j = x_{j-1} + \frac{2}{A + B} S(x - x_{j-1}).
\]

Then, \( \|x - x_j\| \leq \left( \frac{B-A}{B+A} \right)^j \|x\| \).
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Then, \( \|x - x_j\| \leq \left( \frac{B-A}{B+A} \right)^j \|x\| \).

Notice that we only need the coefficients \( \langle x, e_k \rangle \) to compute \( Sx \).
Some Remarks

- The Frame Algorithm converges geometrically (error is reduced by the same factor with each iteration) and in the case that $A = B$, convergence is immediate. Frames where $A = B$ are called *tight frames*. 

It is also possible to construct a *dual frame* consisting of vectors $\{\tilde{e}_k\}_{m=1}^M$ so that for all signals $x$, one has $x = \sum_{k=1}^M \langle x, e_k \rangle \tilde{e}_k$.

One can even find the $\tilde{e}_k$ vectors using the Frame Algorithm, since $\tilde{e}_k = S^{-1}e_k$, or, $e_k = S\tilde{e}_k$. 

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Define \( \{e_1, e_2, e_3\} \) by

\[
e_1 = (1, 0) \quad e_2 = \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \quad e_3 = \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right).
\]
A Simple Tight Frame

A tight-frame for $\mathbb{R}^2$

To see that the collection is a frame, let $x = (x_1, x_2)$.

$$\sum_{k=1}^{3} \langle x, e_k \rangle^2 = \langle x, e_1 \rangle^2 + \langle x, e_2 \rangle^2 + \langle x, e_3 \rangle^2$$

$$= x_1^2 + \left( -\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \right)^2 + \left( -\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2 \right)^2$$

$$= x_1^2 + \frac{1}{4}x_1^2 - \frac{\sqrt{3}}{2}x_1x_2 + \frac{3}{4}x_2^2 + \frac{1}{4}x_1^2 + \frac{\sqrt{3}}{2}x_1x_2 + \frac{3}{4}x_2^2$$

$$= \frac{3}{2}(x_1^2 + x_2^2)$$

$$= \frac{3}{2} \|x\|^2.$$
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= x_1^2 + \left( -\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \right)^2 + \left( -\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2 \right)^2
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\[
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\]

\[
= \frac{3}{2}(x_1^2 + x_2^2)
\]

\[
= \frac{3}{2} \|x\|^2.
\]

So, it’s actually a tight frame.
A Bigger Example

A Random Frame of $\mathbb{R}^{10}$ of Size 25

The first 15 vectors:

-0.29 0.24 -0.18 0.33 0.61 -0.11 0.95 -0.29 0.70 -0.85 -0.91 -0.47 -0.81 0.64 -0.66
-0.62 -0.51 -0.43 0.45 0.66 -0.27 -0.56 -0.90 -0.58 -0.60 0.20 1.00 -0.97 -0.47 0.08
-0.02 0.17 -0.21 -0.44 -0.67 -0.39 0.41 0.51 -0.09 -0.90 0.90 -0.58 -0.42 0.51 0.25
-0.18 0.01 0.01 -0.48 -0.21 0.70 0.04 0.79 -0.84 0.13 -0.42 -0.00 0.63 0.32 0.37
-0.07 -0.07 0.44 0.42 0.04 0.52 0.87 -0.43 0.70 -0.76 0.78 -0.42 0.97 -0.57 0.35
0.22 0.08 -0.39 0.57 0.44 0.90 0.43 -0.50 0.12 0.04 -0.80 0.35 -0.97 0.20 0.75
-0.86 0.88 -0.78 0.97 0.14 0.12 -0.54 0.87 -0.36 -0.77 -0.87 0.92 0.64 0.21 -0.97
-0.37 -0.32 -0.11 -0.05 -0.08 -0.97 -0.10 -0.74 -0.25 0.54 -0.53 0.53 0.24 0.32 -0.38
0.22 -0.20 -0.07 0.81 -0.11 0.19 -0.66 0.88 0.74 -0.25 0.87 0.33 0.12 -0.63 0.56
-0.65 -0.38 -0.97 -0.10 -0.82 0.63 0.94 0.40 -0.26 0.65 -0.87 -0.74 -0.51 0.27 -0.39

The last 10 vectors:

0.85 0.02 0.90 0.30 -0.32 0.06 0.94 0.04 -0.20 0.33
0.36 -0.85 0.66 0.51 -0.07 -0.64 -0.95 0.79 -0.28 -0.73
-0.85 -0.61 0.84 0.33 0.83 0.00 0.74 0.88 -0.43 -0.96
-0.86 -0.24 -0.77 0.77 -0.54 -0.16 -0.95 -0.33 0.74 -0.48
-0.98 -0.45 0.62 -0.46 0.72 0.32 0.04 -0.13 0.25 -0.77
-0.55 0.54 0.82 -0.16 0.31 0.35 -0.62 -0.06 -0.52 -0.86
0.03 -0.37 -0.69 -0.57 0.78 0.91 0.43 -0.70 0.96 0.71
-0.08 0.28 -0.76 -0.93 -0.02 -0.62 -0.50 -0.73 0.28 -0.64
0.41 0.97 0.53 -0.84 0.99 -0.78 0.87 0.06 -0.54 -0.94
0.16 0.01 0.44 0.70 -0.25 0.13 -0.73 0.45 0.36 0.47
A frame of $\mathbb{R}^{10}$ of size 25

Let $F$ be the $10 \times 25$ matrix having these vectors as its columns.

$$F = \begin{pmatrix} e_1 & e_2 & \cdots & e_{25} \end{pmatrix}.$$
A frame of $\mathbb{R}^{10}$ of size 25

- Let $F$ be the $10 \times 25$ matrix having these vectors as its columns.

$$F = \begin{pmatrix} e_1 & e_2 & \cdots & e_{25} \end{pmatrix}.$$

- To compute the frame coefficients one multiplies the transpose by a given vector:

$$F^T x = \begin{pmatrix} - & e_1 & - \\ - & e_2 & - \\ \vdots & \vdots & \vdots \\ - & e_{25} & - \end{pmatrix} \begin{pmatrix} x \\ \end{pmatrix} = \begin{pmatrix} \langle x, e_1 \rangle \\ \langle x, e_2 \rangle \\ \vdots \\ \langle x, e_{25} \rangle \end{pmatrix}.$$
A frame of $\mathbb{R}^{10}$ of size 25

The frame operator can also be computed using $F$. In fact, $S = FF^T$ and is shown below: (to two decimal places)

\[
\begin{bmatrix}
8.18 & 0.30 & 0.83 & -3.62 & -0.68 & 1.36 & 0.70 & -1.75 & -1.04 & 1.94 \\
0.30 & 9.67 & 0.86 & -0.88 & -0.69 & 2.38 & -1.75 & 0.62 & -0.00 & -0.21 \\
0.83 & 0.86 & 8.79 & -0.74 & 3.20 & 0.41 & -2.32 & -2.85 & 2.20 & -0.07 \\
-3.62 & -0.88 & -0.74 & 7.05 & -0.35 & -0.18 & 0.87 & 0.66 & -3.39 & 2.17 \\
-0.68 & -0.69 & 3.20 & -0.35 & 7.89 & 1.34 & -0.66 & -1.27 & 2.33 & -1.99 \\
1.36 & 2.38 & 0.41 & -0.18 & 1.34 & 7.20 & -1.59 & -0.27 & 1.04 & 1.21 \\
0.70 & -1.75 & -2.32 & 0.87 & -0.66 & -1.59 & 12.34 & 1.05 & -0.55 & -0.26 \\
-1.75 & 0.62 & -2.85 & 0.66 & -1.27 & -0.27 & 1.05 & 6.16 & -0.75 & -1.10 \\
-1.04 & -0.00 & 2.20 & -3.39 & 2.33 & 1.04 & -0.55 & -0.75 & 9.83 & -3.82 \\
1.94 & -0.21 & -0.07 & 2.17 & -1.99 & 1.21 & -0.26 & -1.10 & -3.82 & 7.78
\end{bmatrix}
\]

The minimum and maximum eigenvalues of this matrix (Matlab) actually determine the frame bounds.

1.725  2.785  3.777  5.413  6.72  8.69  11.00  11.94  14.32  18.50
The movie below shows 20 iterations of the frame algorithm.
Implementing the Frame Algorithm

The movie below shows 20 iterations of the frame algorithm.

The convergence is slow because $\frac{B - A}{B + A} \approx 0.8294$. 
A Bigger Example

Another frame of $\mathbb{R}^{10}$ of size 25

The first 15 vectors:

| 1.00 | 0.97 | 0.87 | 0.72 | 0.52 | 0.28 | 0.03 | -0.22 | -0.46 | -0.67 | -0.84 | -0.95 | -1.00 | -0.98 | -0.90 |
| 1.00 | 0.87 | 0.52 | 0.03 | -0.46 | -0.84 | -1.00 | -0.90 | -0.57 | -0.10 | 0.40 | 0.80 | 0.99 | 0.93 | 0.62 |
| 1.00 | 0.72 | 0.03 | -0.67 | -1.00 | -0.76 | -0.10 | 0.62 | 0.99 | 0.80 | 0.16 | -0.57 | -0.98 | -0.84 | -0.22 |
| 1.00 | 0.52 | -0.46 | -1.00 | -0.57 | 0.40 | 0.99 | 0.62 | -0.35 | -0.98 | -0.67 | 0.28 | 0.97 | 0.72 | -0.22 |
| 1.00 | 0.28 | -0.84 | -0.76 | 0.40 | 0.99 | 0.16 | -0.90 | -0.67 | 0.52 | 0.97 | 0.03 | -0.95 | -0.57 | 0.62 |
| 1.00 | 0.03 | -1.00 | -0.10 | 0.99 | 0.16 | -0.98 | -0.22 | 0.97 | 0.28 | -0.95 | -0.35 | 0.93 | 0.40 | -0.90 |
| 1.00 | -0.22 | -0.90 | 0.62 | 0.62 | -0.90 | -0.22 | 1.00 | -0.22 | -0.90 | 0.62 | 0.62 | -0.90 | -0.22 | 1.00 |
| 1.00 | -0.46 | -0.57 | 0.99 | -0.35 | -0.67 | 0.97 | -0.22 | -0.76 | 0.93 | -0.10 | -0.84 | 0.87 | 0.03 | -0.90 |
| 1.00 | -0.67 | -0.10 | 0.80 | -0.98 | 0.52 | 0.28 | -0.90 | 0.93 | -0.35 | -0.46 | 0.97 | -0.84 | 0.16 | 0.62 |
| 1.00 | -0.84 | 0.40 | 0.16 | -0.67 | 0.97 | -0.95 | 0.62 | -0.10 | -0.46 | 0.87 | -1.00 | 0.80 | -0.35 | -0.22 |

The last 10 vectors:

| -0.76 | -0.57 | -0.35 | -0.10 | 0.16 | 0.40 | 0.62 | 0.80 | 0.93 | 0.99 |
| 0.16 | -0.35 | -0.76 | -0.98 | -0.95 | -0.67 | -0.22 | 0.28 | 0.72 | 0.97 |
| 0.52 | 0.97 | 0.87 | 0.28 | -0.46 | -0.95 | -0.90 | -0.35 | 0.40 | 0.93 |
| -0.95 | -0.76 | 0.16 | 0.93 | 0.80 | -0.10 | -0.90 | -0.84 | 0.03 | 0.87 |
| 0.93 | -0.10 | -0.98 | -0.46 | 0.72 | 0.87 | -0.22 | -1.00 | -0.35 | 0.80 |
| -0.46 | 0.87 | 0.52 | -0.84 | -0.57 | 0.80 | 0.62 | -0.76 | -0.67 | 0.72 |
| -0.22 | -0.90 | 0.62 | 0.62 | -0.90 | -0.22 | 1.00 | -0.22 | -0.90 | 0.62 |
| 0.80 | 0.16 | -0.95 | 0.72 | 0.28 | -0.98 | 0.62 | 0.40 | -1.00 | 0.52 |
| -1.00 | 0.72 | 0.03 | -0.76 | 0.99 | -0.57 | -0.22 | 0.87 | -0.95 | 0.40 |
| 0.72 | -0.98 | 0.93 | -0.57 | 0.03 | 0.52 | -0.90 | 0.99 | -0.76 | 0.28 |
Implementing the Frame Algorithm

The movie below shows 20 iterations of the frame algorithm.
The movie below shows 20 iterations of the frame algorithm.

The convergence is faster because \( \frac{B - A}{B + A} \approx 0.1723 \). (This frame was not chosen randomly.)
Vector Spaces for 2-D Signals

The following representation is used for two-dimensional signals, e.g., images.
Vector Spaces for 2-D Signals

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- **Vectors:**

\[
x = \begin{pmatrix}
x_{11} & \cdots & x_{1n} \\
\vdots & \ddots & \vdots \\
x_{m1} & \cdots & x_{mn}
\end{pmatrix}
\]
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- **Inner Product:**
  \[
  \langle x, y \rangle = \sum_{j=1}^{m} \sum_{k=1}^{n} x_{j,k} y_{j,k}
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- **Vectors:**
  \[ x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \]

- **Inner Product:**
  \[ \langle x, y \rangle = \sum_{j=1}^{m} \sum_{k=1}^{n} x_{j,k} y_{j,k} \]

- **Basis:** A collection \( \{ u_{j,k} : 1 \leq j \leq m, 1 \leq k \leq n \} \) is a basis provided that the only solution of
  \[ \sum_{j=1}^{m} \sum_{k=1}^{n} c_{j,k} u_{j,k} = 0 \]
  is the trivial solution \( c_{j,k} = 0, 1 \leq j \leq m, 1 \leq k \leq n \).
3 × 3 Image Space

An example signal:

\[ x = \begin{pmatrix} 0.95 & 0.49 & 0.45 \\ 0.23 & 0.89 & 0.02 \\ 0.61 & 0.76 & 0.82 \end{pmatrix} \]
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\[ x = \begin{pmatrix}
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Below we see an image and its wavelet basis coefficients. (no redundancy)
Below we see an image and its wavelet basis coefficients. (no redundancy)
Wavelet Basis

Below we see an image and its wavelet basis coefficients. (no redundancy)
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The gray pixels correspond to coefficients close to zero. Black and white pixels correspond to - and + coefficients, respectively.
By ignoring small coefficients one can “compress” the image.
Wavelet Basis – Compression

By ignoring small coefficients one can “compress” the image.

Original

Reconstructed
By ignoring small coefficients one can “compress” the image.

The compression ratio here is 33.8 with a mean-squared error of 2.66.
Below we see an image and its wavelet frame coefficients (highly redundant).
Below we see an image and its wavelet frame coefficients (highly redundant).
Below we see an image and its wavelet frame coefficients (highly redundant).

Each component of the coefficient image has the same size as the original.
Wavelet Frame – Denoising

By ignoring small coefficients one can remove noise from the image. The greater redundancy helps preserve the original signal features.

Noisy
Wavelet Frame – Denoising

By ignoring small coefficients one can remove noise from the image. The greater redundancy helps preserve the original signal features.

Noisy

Reconstructed
Wavelet Frame – Denoising

By ignoring small coefficients one can remove noise from the image. The greater redundancy helps preserve the original signal features.

As the level of the noise increases denoising will begin to affect important signal features, resulting in a blurring of the image.
The JPEG image standard originally made use of a basis whose elements are described by discrete cosine functions. (Newer JPEG standards use wavelets.)

Original
Discrete Cosine Transform (DCT) – Basis

The JPEG image standard originally made use of a basis whose elements are described by discrete cosine functions. (Newer JPEG standards use wavelets.)
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The periodic nature of the basis elements lends itself to representation of signals with repeating patterns.
The rise of computers has opened new avenues of research in mathematics and other fields. Frame theory is just one example.
Concluding Remarks

- The rise of computers has opened new avenues of research in mathematics and other fields. Frame theory is just one example.
- Frame theory is very accessible to students with some exposure to linear algebra. A little analysis doesn’t hurt.

Computational platforms like Matlab are great for working with real data (like audio or images). There is also an open-source Matlab emulator called Octave:

http://www.octave.org/
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THANK YOU!
B.D. Johnson,

*Frames in* \( \mathbb{R}^n \), expository notes,