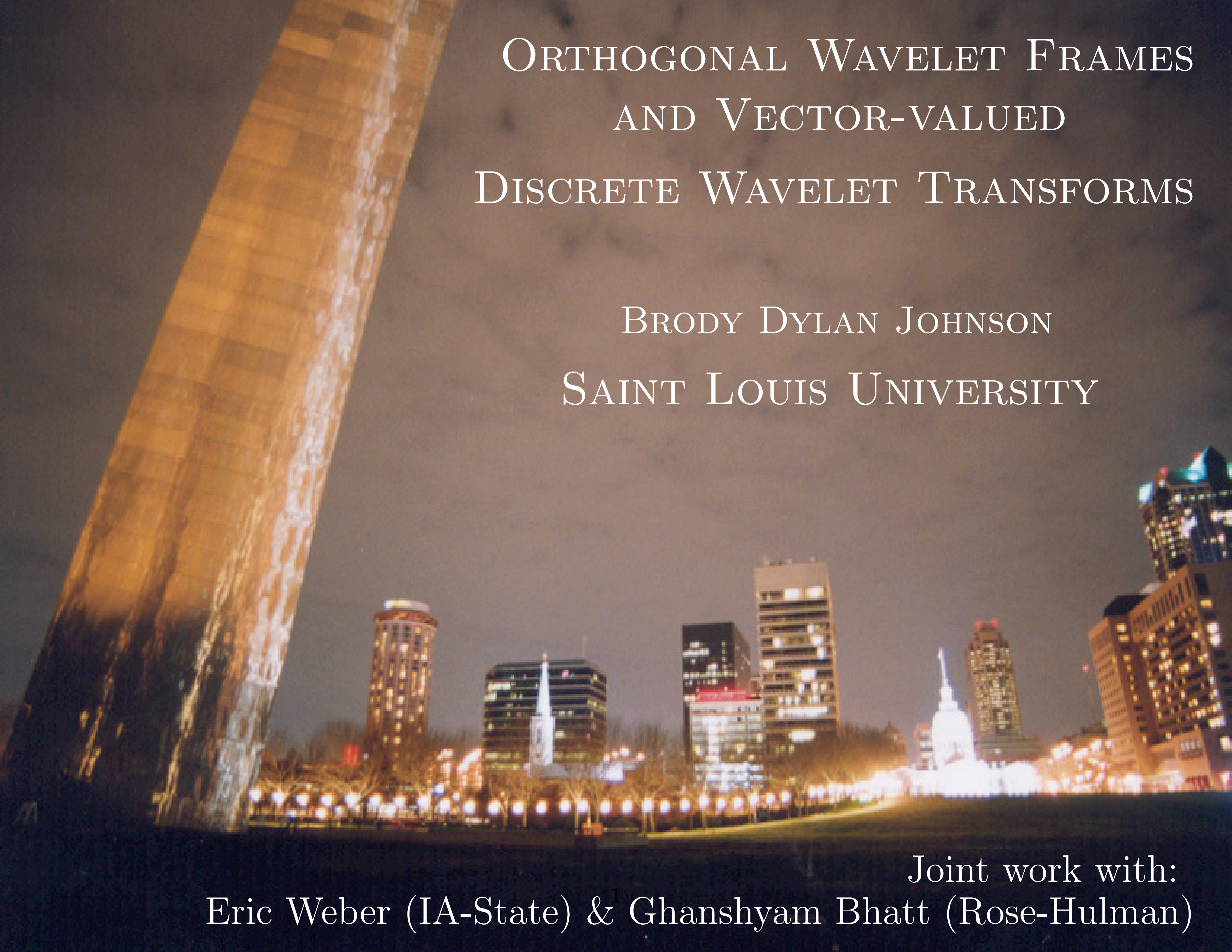
Orthogonal Wavelet Frames and Vector-valued Discrete Wavelet Transforms

Brody Dylan Johnson

Saint Louis University

Joint work with:
Eric Weber (IA-State) & Ghanshyam Bhatt (Rose-Hulman)



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Motivation:

- There are many situations in which correlated multichannel data occurs naturally, e.g., color images, stereo audio, etc.
- One can always apply a standard wavelet transform to each channel, but this fails to take advantage of any correlation between the channels.
- The primary goal of this work is to develop a vector-valued discrete wavelet transform (VDWT) allowing for simultaneous processing of multichannel data.
- By using orthogonal wavelet frames for each channel, one can actually sum the "high-pass" components of the associated DWTs in hopes of achieving a more efficient representation of the signal.

Preliminaries: (1 of 2)

- Fourier transform: $f \in L^1 \cap L^2(\mathbb{R})$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx.$$

- Translation operator: $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$Tf(x) = f(x-1).$$

- Dilation operator: $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$Df(x) = \sqrt{2} f(2x).$$

- Affine systems: Given $\Psi = \{\psi_1, \dots \psi_r\} \subset L^2(\mathbb{R})$

$$X(\Psi) = \left\{ D^j T^k \psi_\ell : j, k \in \mathbb{Z}, 1 \le \ell \le r \right\}.$$

Preliminaries: (2 of 2)

- Frame: $X := \{x_j\}_{j\in J} \subset \mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $0 < C_1 \le C_2 < \infty$ such that for all $x \in \mathbb{H}$,

$$C_1\|x\|^2 \le \sum_{j\in J} |\langle x, x_j\rangle|^2 \le C_2\|x\|^2.$$

  Parseval frames occur when one may choose $C_1 = C_2 = 1$.

- Analysis operator: $\Theta_X : \mathbb{H} \to \ell^2(J)$ given by

$$\Theta_X x = \{\langle x, x_j\rangle\}_{j\in J}.$$

- Synthesis operator: $\Theta_X^* : \ell^2(J) \to \mathbb{H}$ given by

$$\Theta_X^*\{c_j\}_{j\in J} = \sum_{j\in J} c_j x_j.$$

Orthogonal Frames: (1 of 2)

- Orthogonality: Let $X = \{x_j\}_{j\in J}$ and $Y = \{y_j\}_{j\in J}$ be Bessel sequences, then X and Y are *orthogonal* if

$$\Theta_Y^*\Theta_X = \sum_{j\in J}\langle\cdot, x_j\rangle y_j = 0.$$

- Reconstruction: If X and Y are orthogonal Parseval frames then for all $f_1, f_2 \in \mathbb{H}$,

$$\Theta_Y^*(\Theta_X f_1 + \Theta_Y f_2) = \Theta_Y^*\Theta_Y f_2 = f_2.$$

- In order that X and Y are pairwise orthogonal, each Parseval frame must provide a redundant representation of $\mathbb{H}$, e.g., notice that

$$x = \sum_{j\in J}\langle x, x_j + y_j\rangle x_j = \sum_{j\in J}\langle x, x_j\rangle x_j.$$

Orthogonal Frames: (2 of 2)

- Application to multiple channels: Signal $f = f_1 \oplus \cdots \oplus f_N$.
  - ⋆ Start with pairwise orthogonal Parseval frames: $X_1, \ldots, X_N$.
  - ⋆ Apply Θ_{X_k} to f_k and sum the result:

$$f \mapsto \Theta_X f := \sum_{k=1}^{N} \Theta_{X_k} f_k.$$

  - ⋆ Recover f_{k_0} by applying $\Theta^*_{X_{k_0}}$ to $\Theta_X f$:

$$\Theta^*_{X_{k_0}} \Theta_X f = \sum_{k=1}^{N} \Theta^*_{X_{k_0}} \Theta_{X_k} f_k = \Theta^*_{X_{k_0}} \Theta_{X_{k_0}} f_{k_0} = f_{k_0}.$$

- The multichannel analysis operator, $\Theta_X : \bigoplus_{k=1}^{N} \mathbb{H} \to \ell^2(J)$, processes the components of f simultaneously.

Orthogonal Wavelet Frames: (1 of 6)

- Characterizing dual wavelet frames: (Ron and Shen '97)

Theorem 1. *Suppose* $\{\psi_1, \ldots, \psi_r\}$ *and* $\{\eta_1, \ldots, \eta_r\}$ *generate wavelet frames in* $L^2(\mathbb{R})$*. The frames are dual if and only if*

$$\sum_{k=1}^{r} \sum_{j \in \mathbb{Z}} \hat{\psi}_k(2^j \xi) \overline{\hat{\eta}_k(2^j \xi)} = 1, \qquad a.e.\ \xi \in \mathbb{R},$$

and for every $q \in \mathbb{Z} \setminus 2\mathbb{Z}$,

$$\sum_{k=1}^{r} \sum_{j=0}^{\infty} \hat{\psi}_k(2^j \xi) \overline{\hat{\eta}_k(2^j (\xi + q))} = 0, \qquad a.e.\ \xi \in \mathbb{R}.$$

In particular, $\{\psi_1, \ldots, \psi_r\}$ *generates a Parseval wavelet frame if the two equations hold for* $\eta_k = \psi_k$*.*

Orthogonal Wavelet Frames: (2 of 6)

- Characterizing orthogonal wavelet frames: (Weber '04)

Theorem 2. *Suppose* $\{\psi_1, \ldots, \psi_r\}$ *and* $\{\eta_1, \ldots, \eta_r\}$ *generate affine Bessel sequences in* $L^2(\mathbb{R})$, *then they are orthogonal if and only if*

$$\sum_{k=1}^{r} \sum_{j \in \mathbb{Z}} \hat{\psi}_k(2^j \xi) \overline{\hat{\eta}_k(2^j \xi)} = 0, \qquad a.e.\ \xi \in \mathbb{R},$$

and for every $q \in \mathbb{Z} \setminus 2\mathbb{Z}$,

$$\sum_{k=1}^{r} \sum_{j=0}^{\infty} \hat{\psi}_k(2^j \xi) \overline{\hat{\eta}_k(2^j (\xi + q))} = 0, \qquad a.e.\ \xi \in \mathbb{R}.$$

Orthogonal Wavelet Frames: (3 of 6)

- Construction of wavelet frames from a scaling function and filters:
  - ⋆ Let $\varphi \in L^2(\mathbb{R})$ be a refinable function, with low pass filter $m(\xi)$, satisfying:
    1. $\lim_{\xi \to 0} \hat{\varphi}(\xi) = 1$;
    2. $\sum_{l \in \mathbb{Z}} |\hat{\varphi}(\xi + l)|^2 \in L^\infty(\mathbb{R})$.
  - ⋆ Let $m_1(\xi), \ldots, m_r(\xi) \in L^\infty([0,1))$ and define

$$M(\xi) = \begin{pmatrix} m(\xi) & m(\xi + 1/2) \\ m_1(\xi) & m_1(\xi + 1/2) \\ \vdots & \vdots \\ m_r(\xi) & m_r(\xi + 1/2) \end{pmatrix} \quad \tilde{M}(\xi) = \begin{pmatrix} m_1(\xi) & m_1(\xi + 1/2) \\ \vdots & \vdots \\ m_r(\xi) & m_r(\xi + 1/2) \end{pmatrix}.$$

Orthogonal Wavelet Frames: (4 of 6)

- Unitary Extension Principle: (Daubechies, B. Han, Ron, and Shen '03)

Theorem 3. *Suppose* $\varphi \in L^2(\mathbb{R})$ *is a refinable function as described above. Let* $m_1(\xi), \ldots, m_r(\xi) \in L^\infty([0,1))$ *such that the matrix* $M(\xi)$ *satisfies*

$$M^*(\xi)M(\xi) = I_2, \quad a.e.\ \xi \in \mathbb{R}.$$

Then, the affine system generated by $\{\psi_1, \ldots, \psi_r\}$*, where*

$$\hat{\psi}_k(2\xi) = m_k(\xi)\hat{\varphi}(\xi), \quad k = 1, \ldots, r,$$

is a Parseval wavelet frame.

Orthogonal Wavelet Frames: (5 of 6)

Theorem 4 (Bhatt, J–, Weber '06). *Let* $\varphi \in L^2(\mathbb{R})$ *be a refinable function as described above. Let* $\mathcal{M} = \{m_0(\xi), m_1(\xi), \ldots, m_r(\xi)\}$ *and* $\mathcal{N} = \{n_0(\xi), n_1(\xi), \ldots, n_r(\xi)\}$ *be filter banks with* $m_0 = n_0 = m$*. Suppose that the following matrix equations hold:*

1. $M^*(\xi)M(\xi) = I_2$ *for almost every* ξ*;*

2. $N^*(\xi)N(\xi) = I_2$ *for almost every* ξ*;*

3. $\widetilde{M}^*(\xi)\widetilde{N}(\xi) = 0$ *for almost every* ξ*.*

Let $\hat{\psi}_k(2\xi) = m_k(\xi)\hat{\varphi}(\xi)$ *and* $\hat{\eta}_k(2\xi) = n_k(\xi)\hat{\varphi}(\xi)$, $1 \leq k \leq r$*. Then* $\{\psi_1, \ldots, \psi_r\}$ *and* $\{\eta_1, \ldots, \eta_r\}$ *generate orthogonal Parseval wavelet frames.*

Orthogonal Wavelet Frames: (6 of 6)

Theorem 5 (Bhatt, J–, Weber '06). *Suppose* $K(\xi)$ *is a 1/2-periodic* $r \times r$ *matrix which is unitary for a.e.* ξ*; let* $K_j(\xi)$ *denote the* j*-th column. Suppose* m_0 *and* m_1 *are low and high pass filters, respectively, for an orthonormal wavelet basis with scaling function* φ*. For* $j = 1, \ldots, r$*, define new filters via*

$$\begin{pmatrix} n_1^j(\xi) \\ \vdots \\ n_r^j(\xi) \end{pmatrix} = K_j(\xi) m_1(\xi).$$

Then, for $j = 1, \ldots, r$*, the affine systems generated by* $\{\psi_l^j : l = 1, \ldots, r\}$ *obtained via*

$$\hat{\psi}_l^j(2\xi) = n_l^j(\xi)\hat{\varphi}(\xi) \tag{1}$$

are Parseval frames and are pairwise orthogonal.

Discrete Implementation of OWFs: (1 of 4)

- Begin with scaling function φ and wavelet ψ for an orthonormal wavelet (filters $m(\xi)$ and $n(\xi)$, respectively).
- Choose unitary matrix $K(\xi)$ and construct orthogonal wavelet frames as in Theorem 5.
- Analysis/Reconstruction of discrete data is accomplished using the associated filter banks.
- This leads to a notion of orthogonal filter banks that will be applied to the high-pass filters.
- Filter banks: $\mathcal{M} = \{m_0, m_1, \ldots, m_r\}$, $\mathcal{N} = \{n_0, n_1, \ldots, n_r\}$.

Discrete Implementation of OWFs: (2 of 4)

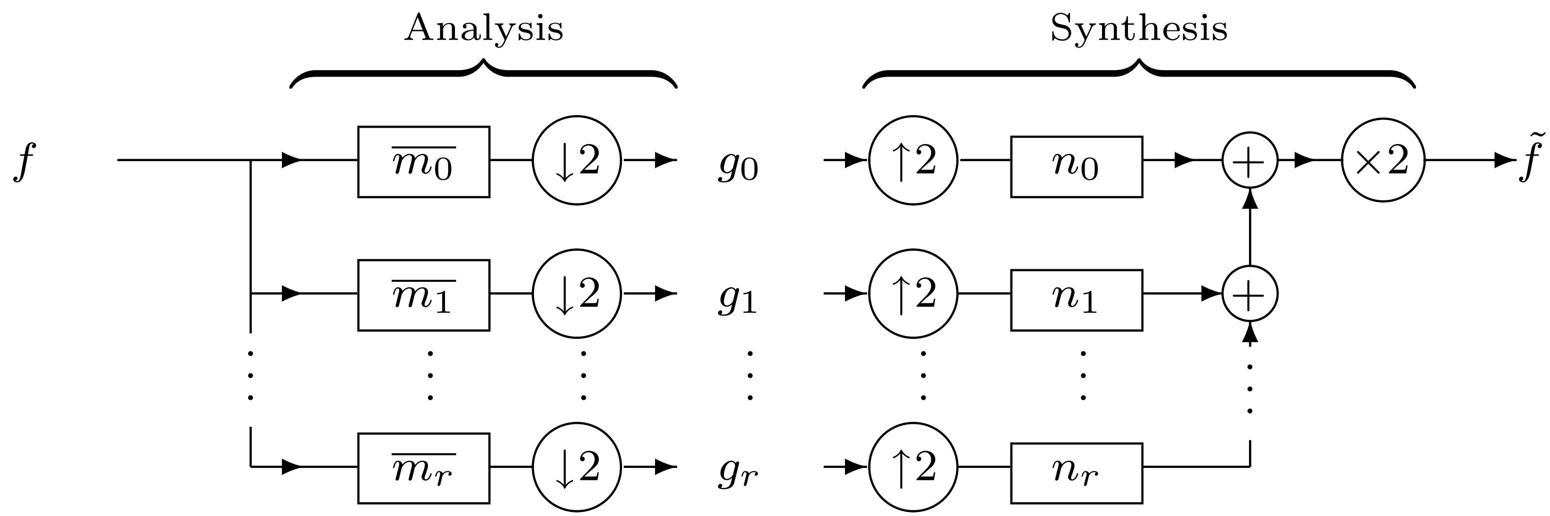


Figure 1: Filter bank block diagram.

$$\hat{\tilde{f}}(\xi) = \sum_{\ell=0}^{r} \left[\overline{m_\ell(\xi)} n_\ell(\xi) \hat{f}(\xi) + \overline{m_\ell(\xi + 1/2)} n_\ell(\xi) \hat{f}(\xi + 1/2) \right]$$

Discrete Implementation of OWFs: (3 of 4)

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal* if, for any input vector, the composition of the analysis stage of $\mathcal{M}$ with the synthesis stage of $\mathcal{N}$ yields 0, i.e. for any input $f \in \ell^2(\mathbb{Z})$, $\tilde{f} = 0$.
- When $m_0 = n_0 = m$ is a low-pass filter it is impossible for $\mathcal{M}$ and $\mathcal{N}$ to be orthogonal. (assuming the remaining filters are high-pass)
- The high-pass portions of the filter banks are orthogonal if and only if $\tilde{M}^*(\xi)\tilde{N}(\xi) = 0$. (as for OWFs.)
- Each filter bank has the *perfect reconstruction property* if and only if $M^*(\xi)M(\xi) = N^*(\xi)N(\xi) = I_2$. (as for OWFs)

Discrete Implementation of OWFs: (4 of 4)

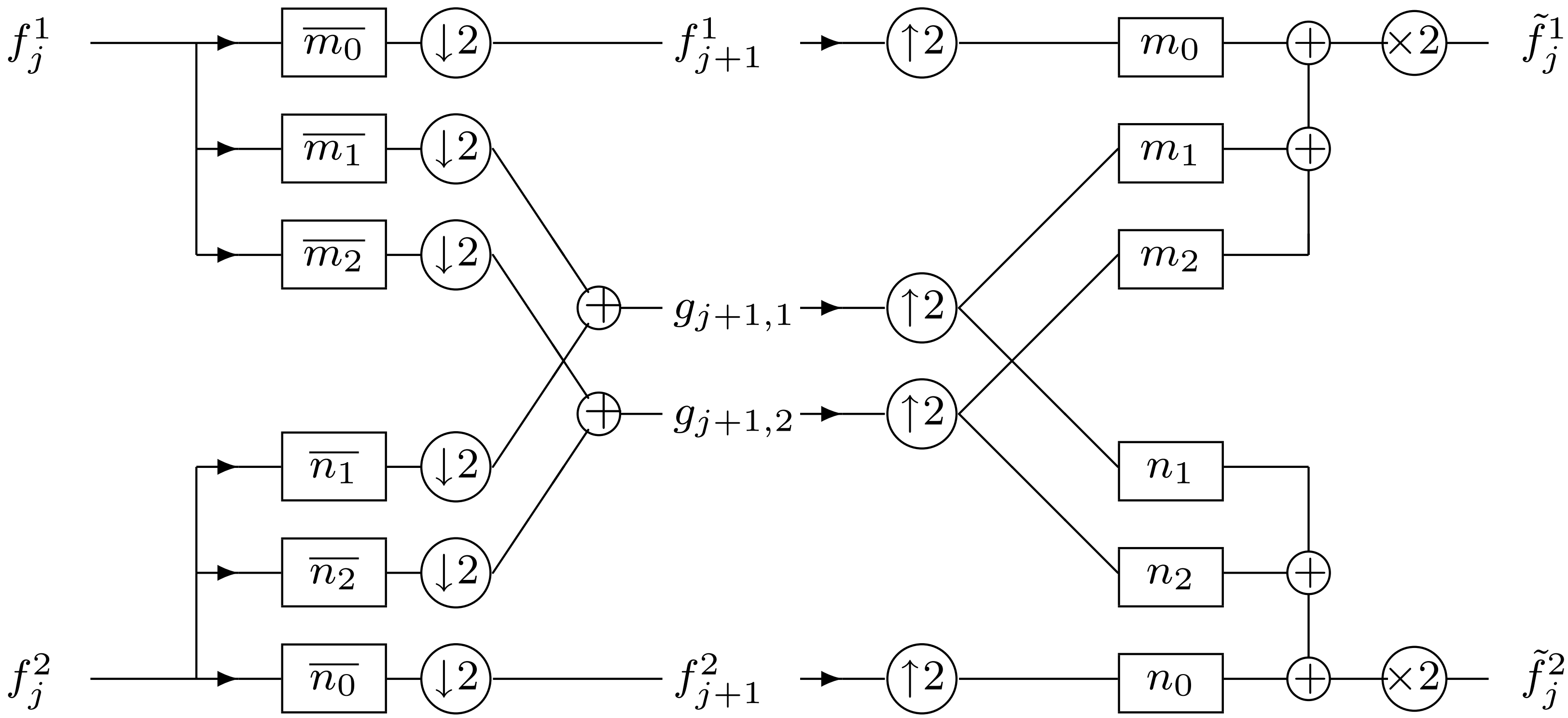


Figure 2: Two-Channel VDWT as in Stereo Audio Context.

Image Compression:

- Hard thresholding was applied over four scales of the associated discrete wavelet transform. Recall that with hard thresholding only the coefficients greater than a chosen threshold $T > 0$ are kept for reconstruction. (No quantizing/encoding is done here.)
- The benefit and cost of thresholding are quantified by:

$$\text{Compression Factor} := \frac{\text{Total \# of pixels} \times 3}{\text{\# of coefficients} \geq \text{threshold}},$$

$$\text{SNR} \;:= 20\log_{10}\left(\frac{\|\text{Original}\|_2}{\|\text{Original} - \text{Reconstruction}\|_2}\right).$$

A higher SNR corresponds to a smaller $\|\cdot\|_2$ error.

Preliminary Results:

Picture	Method	Threshhold	Comp. Ratio	SNR
Lena	D4, none	15	9.36	30.64
Lena	D4, scalar	15	10.96	30.93
Lena	D4, poly.	15	9.88	30.64
Lena	D4, none	50	28.77	26.14
Lena	D4, scalar	50	34.62	26.75
Lena	D4, poly.	50	30.58	26.37
Pepper	D4, none	15	10.71	31.41
Pepper	D4, scalar	15	12.14	32.06
Pepper	D4, poly.	15	10.83	31.66

Table 1: Image compression using orthogonal wavelet frames.

Example:

Original 512×512 Lena image.

Example:

Ordinary DWT: Reconstructed Image.

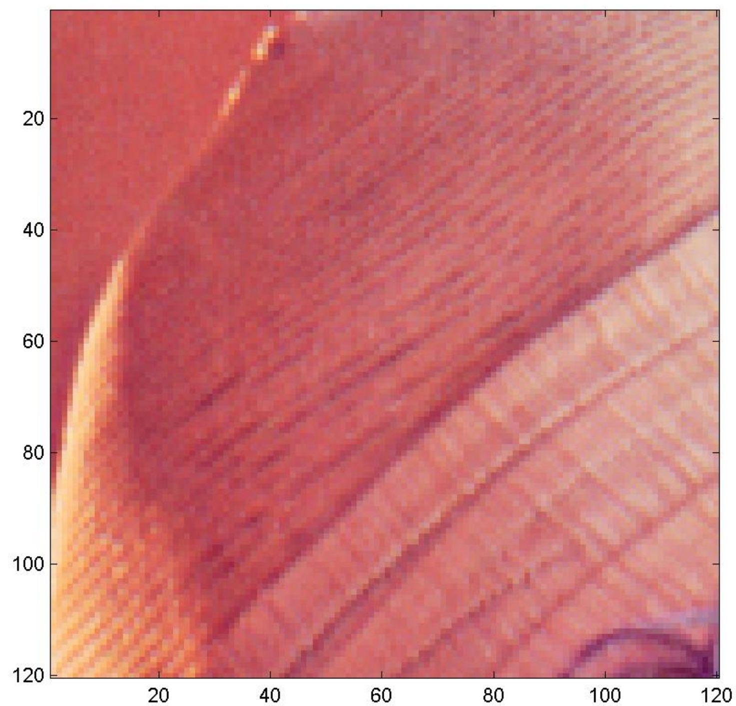
D4 filters, Threshold=15: C.R. ≈ 9.36 & SNR ≈ 30.64.

Example:

VDWT: Reconstructed Image.

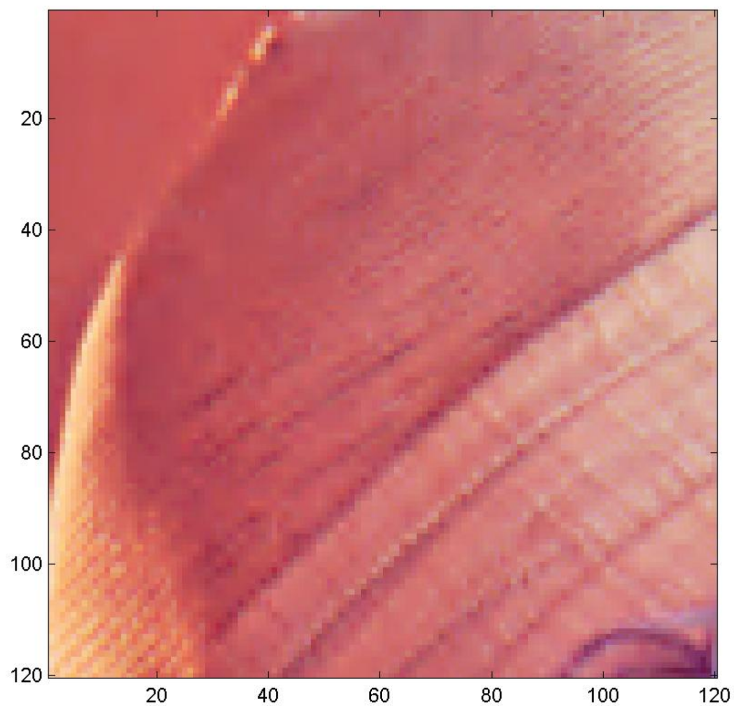
D4 filters, Threshold=15: C.R. ≈ 10.96 & SNR ≈ 30.93.

Example:



120×120 section of the Lena image.

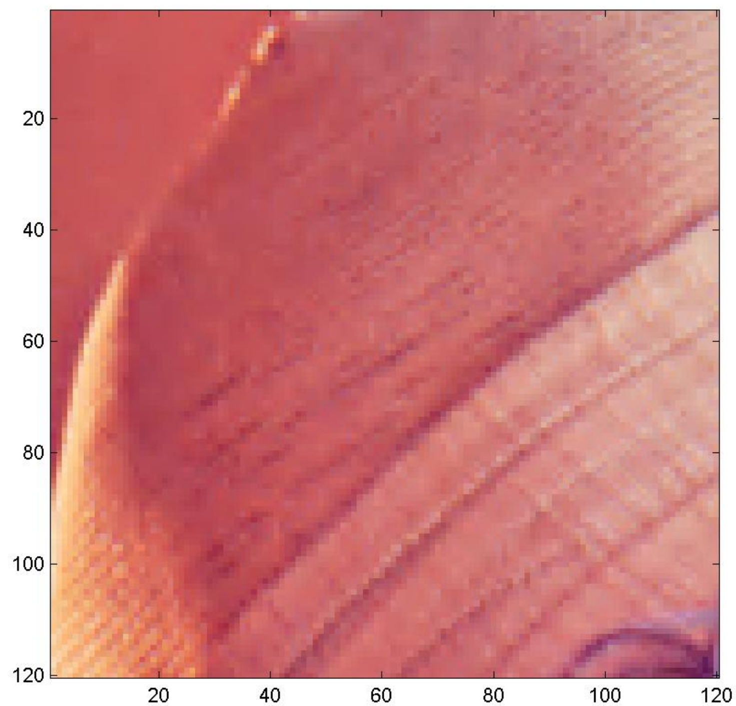
Example:



Ordinary DWT: Reconstructed Image.

D4 filters, Threshold=15: C.R. ≈ 9.36 & SNR ≈ 30.64.

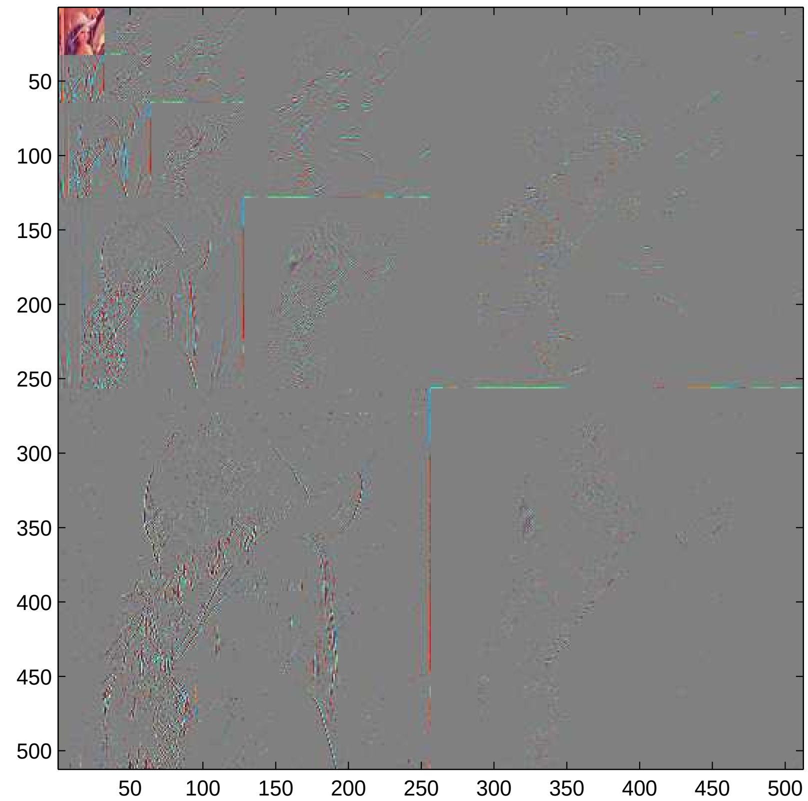
Example:



VDWT: Reconstructed Image.

D4 filters, Threshold=15: C.R. ≈ 10.96 & SNR ≈ 30.93.

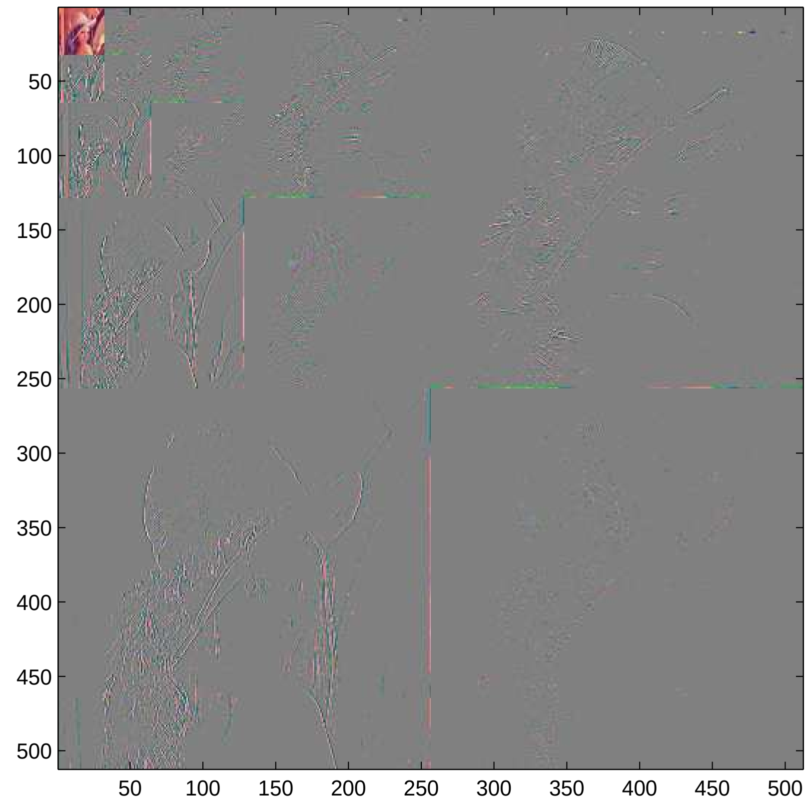
Example:



Ordinary DWT: After thresholding.

D4 filters, Threshold=15: C.R. $\approx$ 9.36 & SNR $\approx$ 30.64.

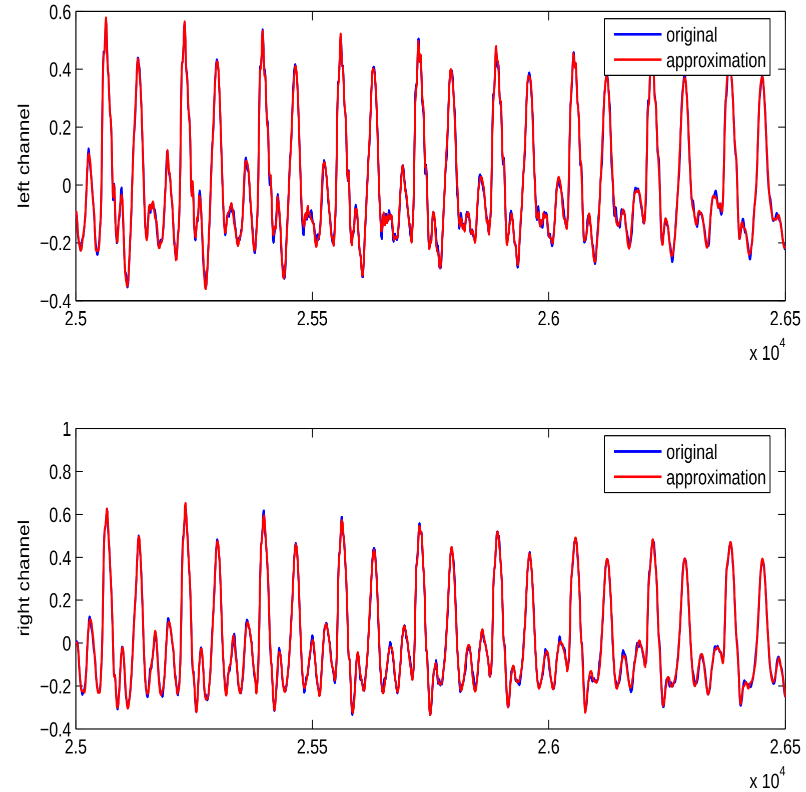
Example:



Scalar Orthogonalization VDWT: After thresholding.

D4 filters, Threshold=15: C.R. $\approx$ 10.96 & SNR $\approx$ 30.93.

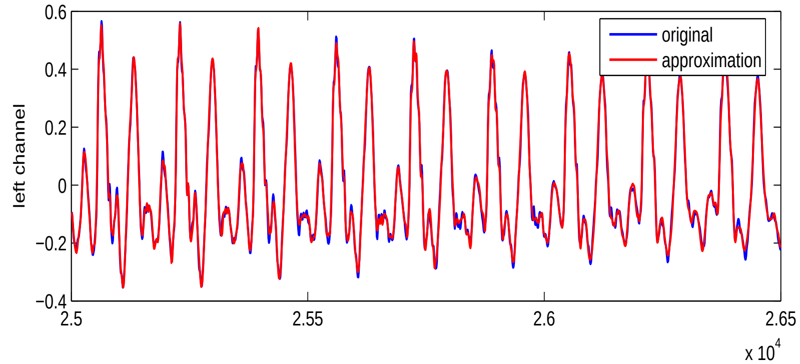
Audio Example:

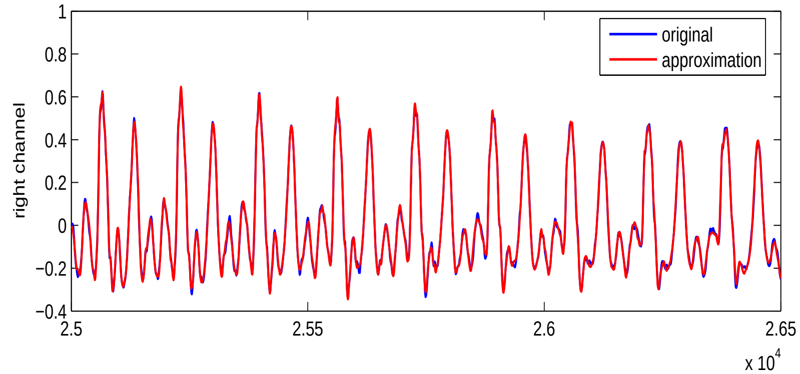


Ordinary DWT: Comparison of Left/Right Channels.

Shannon filters (2000 coeff.), Threshold=0.0175: C.R. $\approx$ 10.90 & SNR $\approx$ 21.05.

Audio Example:





VDWT: Comparison of Left/Right Channels.

Shannon filters (2000 coeff.), Threshold=0.0175: C.R. $\approx$ 11.71 & SNR $\approx$ 21.06.

Conclusion:

- Orthogonal wavelet frames and the VDWT may provide a viable means for dealing with multichannel data.
- Future work:
  - ⋆ consideration of quantization/encoding issues
  - ⋆ optimization of the choice of unitary in construction of OWFs