Quincunx wavelets on $\mathbb{T}^2$

Kenneth R. Hoover and Brody Dylan Johnson

Abstract This article examines a notion of finite-dimensional wavelet systems on $\mathbb{T}^2$, which employ a dilation operation induced by the Quincunx matrix. A theory of multiresolution analysis (MRA) is presented which includes the characterization and construction of MRA scaling functions in terms of low-pass filters. Orthonormal wavelet systems are constructed for any given MRA. Two general examples, based upon the classical Shannon and Haar wavelets, are presented and the approximation properties of the associated systems are studied.

Introduction

This work examines finite-dimensional systems of functions on the torus, $\mathbb{T}^2$, which employ the basic tenets of wavelet theory: dilation and translation. The present study follows a similar analysis on the circle [2], $\mathbb{T}$, where dilation of $f \in L^2(\mathbb{T})$ was accomplished by a dyadic downsampling of the Fourier transform, i.e., $\widehat{Df}(k) = \hat{f}(2k)$, $k \in \mathbb{Z}$. An obvious extension to $L^2(\mathbb{T}^2)$ would involve downsampling of the Fourier transform by $2I_2$; however, this choice fails to utilize the freedom provided by the move from one to two dimensions. Instead, the dilation operation considered here will be achieved through downsampling by a 2×2 matrix, A, satisfying

- A has integer entries;
- A has eigenvalues with modulus strictly greater than 1;
- A has determinant 2.

The first requirement is necessary for the downsampling operation $\widehat{Df}(k) = \hat{f}(Ak)$, $k \in \mathbb{Z}^2$, to be well defined. The second condition ensures that repeated dilation of a function $f \in L^2(\mathbb{T}^2)$ will tend to a constant function in $L^2(\mathbb{T}^2)$. Finally, the third condition specifies that A should have minimal determinant. Indeed, if λ_1 and λ_2

Kenneth R. Hoover, California State University Stanislaus, e-mail: khoover@csustan.edu · Brody Dylan Johnson, Saint Louis University, e-mail: brody@slu.edu

are the eigenvalues of A, then $|\det A| = |\lambda_1 \lambda_2|$ is an integer greater than 1. It is not difficult to see that if $\det A = 2$, then the trace of A will also be 2 under the above assumptions. A certain amount of the analysis will be independent of a specific choice for A. Nevertheless, A will hereafter denote the *Quincunx* dilation matrix,

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This dilation is the composition of a rotation by $\frac{\pi}{4}$ with multiplication by $\sqrt{2}$ and, consequently, facilitates a natural geometric intuition. This discussion has focused on the role that A will play in the creation of a dilation operator. In the next section, the role played by A for translation will also be discussed.

1 Preliminaries

As mentioned in the previous section, the matrix A plays two roles in the proceeding theory, one dealing with dilation and another related to translation. In dilation, the Fourier transform of a function $f \in L^2(\mathbb{T}^2)$ will be downsampled over the subgroup $A\mathbb{Z}^2$ of $\mathbb{Z}^2$. Translation will be considered over a discrete subgroup of $\mathbb{T}^2$ formed as a quotient of $A^{-j}\mathbb{Z}^2$ by $\mathbb{Z}^2$, where $j > 0$ determines the *scale* or *resolution* of the translations being considered.

For a fixed integer $j > 0$, define the *lattice of order* 2^j *generated by A*, Γ_j, as the collection of 2^j distinct coset representatives of $A^{-j}\mathbb{Z}^2/\mathbb{Z}^2$. It will be assumed that each element of Γ_j belongs to the rectangle $[0,1) \times [0,1)$. In the next section, a notion of shift-invariant spaces will be introduced that consists of functions in $L^2(\mathbb{T}^2)$ which are invariant under translation by elements of Γ_j.

Recall that the dilation operation induced by A downsamples the Fourier transform of $f \in L^2(\mathbb{T}^2)$ by A. This operation will be best understood through the quotient groups $\mathbb{Z}^2/B^j\mathbb{Z}^2$, where $B = A^*$. Consequently, define the *dual lattice of order* 2^j *(*$j > 0$*) generated by A*, Γ_j^*, as the collection of 2^j distinct coset representatives of $\mathbb{Z}^2/B^j\mathbb{Z}^2$ determined by the intersection $B^jR \cap \mathbb{Z}^2$, where $R = (-\frac{1}{2}, \frac{1}{2}] \times (-\frac{1}{2}, \frac{1}{2}]$. Because B has integer entries it follows that $B^jR \subseteq B^{j+1}R$, so Γ_j^* is a natural subset of Γ_{j+1}^*.

The following lemma summarizes several elementary, but useful facts about the matrices A and B as well as the lattices Γ_j and Γ_j^*.

Lemma 1. *Let A, B as above and let* $j \geq 2$ *be an integer.*

1. $\Gamma_j = \{A^{-1}\alpha + \alpha' : \alpha \in \Gamma_{j-1}, \alpha' \in \Gamma_1\}$.
2. $\Gamma_j^* = \{B\beta + \beta' : \beta \in \Gamma_{j-1}^*, \beta' \in \Gamma_1^*\}$.
3. $AB^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
4. $A^kB^\ell = B^\ell A^k$, $k, \ell \in \mathbb{Z}$.

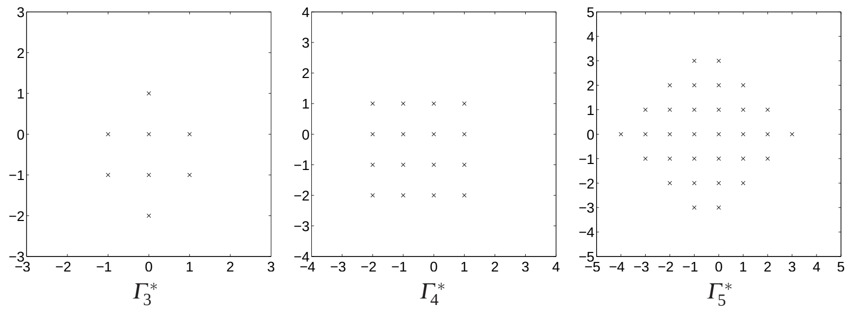


Fig. 1 The dual lattices Γ_3^*, Γ_4^*, and Γ_5^*.

Another important feature of the lattices, Γ_j^*, is their behavior under multiplication by A. Hence, let $d_j : \Gamma_j^* \to \Gamma_j^*$ be the mapping defined by $\alpha \mapsto A\alpha$. For $1 \le k \le j-1$, let $\mathscr{B}_k$ denote the kernel of d_j^k, i.e.,

$$\mathscr{B}_k = \{\beta \in \Gamma_j^* : d_j^k(\beta) = 0\}.$$

The following proposition provides two useful characterizations of $\mathscr{B}_k$.

Proposition 1. *Let $j \in \mathbb{N}$, $j \ge 2$ and $1 \le k \le j-1$.*

1. *$\mathscr{B}_k = B^{j-k}\Gamma_j^*$, i.e., d_j^k is a 2^k-to-1 mapping.*
2. *$\mathscr{B}_k = \left\{ \sum_{\ell=1}^{k} b_\ell A^{j-\ell}\beta_1 : b_\ell \in \{0,1\} \right\}$, where β_1 is the nonzero element of Γ_1^*.*

Proof. To demonstrate the first claim, let $\beta \in \Gamma_j^*$ and assume that $d_j^k(\beta) = 0$. Hence, $A^k\beta \in B^j\mathbb{Z}^2$ or $A^kB^{-k}B^k\beta \in B^j\mathbb{Z}^2$. Now, since powers of A and B commute and AB^{-1} is a rotation, it follows that $B^k\beta \in B^j\mathbb{Z}^2$ and thus $\beta \in B^{j-k}\mathbb{Z}^2$. Likewise if $\beta \in B^{j-k}\mathbb{Z}^2$, then $A^k\beta \in B^j\mathbb{Z}^2$ and, thus, $d_j^k(\beta) = 0$.

To prove the second claim observe first that

$$A^k \sum_{\ell=1}^{k} b_\ell A^{j-\ell}\beta_1 = \sum_{\ell=1}^{k} b_\ell A^{k-\ell}A^j\beta_1 = 0,$$

because $A^{k-\ell}$ has integer entries and $A^j\beta_1 \in B^j\mathbb{Z}^2$. To see that the 2^k elements are unique, assume that

$$\sum_{\ell=1}^{k} b_\ell A^{j-\ell}\beta_1 = \sum_{\ell=1}^{k} b'_\ell A^{j-\ell}\beta_1,$$

or, equivalently, that

$$\sum_{\ell=1}^{k-1} (b_\ell - b'_\ell)A^{j-\ell}\beta_1 = (b'_k - b_k)A^{j-k}\beta_1.$$

Thus the left- and right-hand quantities lie in the intersection of $A^{j-1}\Gamma_j^*$ and $A^{j-k}\Gamma_j^*$. However, $A^{j-k}\beta_1 \notin A^{j-1}\Gamma_j^*$, so it follows each expression must equal zero, i.e., $b_k = b_k'$. This argument may be repeated to show that $b_\ell = b_\ell'$, $1 \le \ell \le k$. □

2 Shift-invariant spaces

This section introduces a notion of shift-invariant spaces for $L^2(\mathbb{T}^2)$ which make use of the lattices Γ_j, $j > 0$. The *translation operator generated by* $\alpha \in \Gamma_j$ will be denoted $T_\alpha : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ and is defined by

$$T_\alpha f(x) = f(x-\alpha), \quad x \in \mathbb{T}^2.$$

A shift-invariant space in this context will consist of a closed subspace V of $L^2(\mathbb{T}^2)$ with the property that $f \in V$ if and only if $T_\alpha f \in V$ for all $\alpha \in \Gamma_j$. Of course, if V is invariant under shifts in Γ_j, then V is also invariant under shifts in Γ_k, $1 \le k \le j$. This work shall focus attention on shift-invariant spaces generated by the Γ_j-translates of a single function.

Definition 1. Let $\phi \in L^2(\mathbb{T}^2)$. The *principal A-shift-invariant space of order* 2^j *generated by* ϕ, denoted $V_j(\phi)$, is the finite-dimensional subspace of $L^2(\mathbb{T}^2)$ spanned by the collection

$$X_j(\phi) = \{T_\alpha \phi : \alpha \in \Gamma_j\}. \tag{1}$$

A function in $V_j(\phi)$ is simply a linear combination of the Γ_j-translates of ϕ, which motivates the following definition. Let $\ell(\Gamma_j)$ denote the space of complex valued functions on Γ_j, with an analogous meaning for $\ell(\Gamma_j^*)$. Define $e_{j,\alpha} \in \ell(\Gamma_j^*)$, $j > 0$, $\alpha \in \Gamma_j$, by

$$e_{j,\alpha}(\beta) = \exp\left(2\pi i \langle \alpha, \beta \rangle\right), \quad \beta \in \Gamma_j^*.$$

Lemma 2. *The collection* $\{2^{-\frac{j}{2}} e_{j,\alpha}\}_{\alpha \in \Gamma_j}$ *is an orthonormal basis for* $\ell(\Gamma_j^*)$.

Proof. Given $\alpha', \alpha'' \in \Gamma_j$, the inner product of $e_{j,\alpha'}$ with $e_{j,\alpha''}$ can be expressed as

$$\langle e_{j,\alpha'}, e_{j,\alpha''} \rangle = \sum_{\beta \in \Gamma_j^*} \exp\left(2\pi i \langle \alpha, \beta \rangle\right),$$

where $\alpha = \alpha' - \alpha''$. If $\alpha = 0$ then the inner product is 2^j. However, if $\alpha \neq 0$, then there exists $\beta' \in \Gamma_j^*$ such that $\langle \alpha, \beta' \rangle \notin \mathbb{Z}$, in which case $\exp\left(2\pi i \langle \alpha, \beta' \rangle\right) \neq 1$. Since $\Gamma_j^* + \beta' \equiv \Gamma_j^*$, this leads to

$$\exp\left(2\pi i \langle \alpha, \beta' \rangle\right) \sum_{\beta \in \Gamma_j^*} \exp\left(2\pi i \langle \alpha, \beta \rangle\right) = \sum_{\beta \in \Gamma_j^*} \exp\left(2\pi i \langle \alpha, \beta \rangle\right),$$

which forces the sum, and hence the inner product, to be zero. □

Recall that the Fourier transform of $f \in L^2(\mathbb{T}^2)$ is given by

$$\hat{f}(k) = \int_{\mathbb{T}^2} f(x) \exp\left(-2\pi i \langle x, k\rangle\right) dx, \quad k \in \mathbb{Z}^2.$$

Therefore, for $\alpha \in \Gamma_j$, $\widehat{T_\alpha f}(k) = \exp\left(-2\pi i \langle \alpha, k\rangle\right) \hat{f}(k)$, $k \in \mathbb{Z}^2$. The following definition adapts the familiar bracket product ([1, 3]) to the present context.

Definition 2. Let $f, g \in L^2(\mathbb{T}^2)$. The *$A$-bracket product of f and g of order* 2^j is the element of $\ell(\Gamma_j^*)$ defined by

$$[\hat{f}, \hat{g}]_{A^j}(\beta) = 2^j \sum_{k \in B^j \mathbb{Z}^2} \hat{f}(\beta + k) \overline{\hat{g}(\beta + k)}, \quad \beta \in \Gamma_j^*.$$

The bracket product so defined captures information about the inner products of f with the Γ_j-translates of g and can be effectively used to determine the frame properties of both principal and finitely-generated shift-invariant spaces. The following proposition, however, focuses on a characterization of orthonormal systems of Γ_j-translates.

Proposition 2. *Let $f, g \in L^2(\mathbb{T}^2)$ and fix $\alpha \in \Gamma_j$. Then,*

$$\langle T_\alpha f, g\rangle = 2^{-j} \langle [\hat{f}, \hat{g}]_{A^j}, e_{j,\alpha}\rangle.$$

In particular, $\langle T_\alpha f, g\rangle = \delta_{\alpha,0}$, $\alpha \in \Gamma_j$, if and only if $[\hat{f}, \hat{g}]_{A^j}(\beta) = 1$, $\beta \in \Gamma_j^*$.

Proof.

$$\begin{aligned}
\langle T_\alpha f, g\rangle &= \sum_{k \in \mathbb{Z}^2} \hat{f}(k) \overline{\hat{g}(k)} e^{-2\pi i \langle \alpha, k\rangle} \\
&= \sum_{\beta \in \Gamma_j^*} \sum_{k \in B^j \mathbb{Z}^2} \hat{f}(\beta + k) \overline{\hat{g}(\beta + k)} e^{-2\pi i \langle \alpha, \beta\rangle} \\
&= 2^{-j} \sum_{\beta \in \Gamma_j^*} [\hat{f}, \hat{g}]_{A^j}(\beta) \overline{e_{j,\alpha}(\beta)}.
\end{aligned}$$

□

The next step will be to incorporate dilation with the shift-invariant spaces examined here for the creation of multiresolution analyses.

3 *A*-refinable functions and multiresolution analysis

The goal of this section is to formulate a theory of multiresolution analysis on the torus making use of dilation by A and translations in Γ_j. The *dilation operator on* $L^2(\mathbb{T}^2)$ *induced by* A will be denoted $D : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ and is defined by

$$\widehat{Df}(k) = \hat{f}(Ak), \quad k \in \mathbb{Z}^2.$$

It follows that $DT_\alpha f = T_{B\alpha} Df$ for all $\alpha \in \Gamma_j$.

Definition 3. A function $\phi \in L^2(\mathbb{T}^2)$ is *A-refinable of order* 2^j if there exists a mask $c \in \ell(\Gamma_j)$ such that

$$D\phi = \sum_{\alpha \in \Gamma_j} c(\alpha)\, T_\alpha \phi. \tag{2}$$

If ϕ is refinable of order 2^j, then it follows that

$$\hat{\phi}(Ak) = \sum_{\alpha \in \Gamma_j} c(\alpha) \widehat{T_\alpha \phi}(k) = \sum_{\alpha \in \Gamma_j} c(\alpha) \overline{e_{j,\alpha}(k)} \hat{\phi}(k) = m(k)\hat{\phi}(k), \quad k \in \mathbb{Z}^2, \tag{3}$$

where $m = \sum_{\alpha \in \Gamma_j} c(\alpha)\overline{e_{j,\alpha}(\cdot)}$ is called the *filter* associated to ϕ. Note that each $k \in \mathbb{Z}^2$ belongs to the coset $\beta + B^j\mathbb{Z}^2$ of a unique element $\beta \in \Gamma_j^*$, i.e., $m \in \ell(\Gamma_j^*)$. The following lemma shows that the dilates of refinable functions are also refinable and provides a relationship between their filters.

Lemma 3. *If ϕ is refinable of order 2^j with filter $m \in \ell(\Gamma_j^*)$, then $D\phi$ is refinable of order 2^{j-1} with filter $m(A\cdot) \in \ell(\Gamma_{j-1}^*)$.*

Proof. Applying D to (2) and using the fact that $DT_\alpha = T_{B\alpha}D$ one finds that

$$D^2\phi = \sum_{\alpha \in \Gamma_j} c(\alpha)\, T_{B\alpha} D\phi.$$

This can be interpreted as a refinement equation for $D\phi$ of order 2^{j-1}, although the sum on the right includes duplicate representations of the elements of Γ_{j-1}. A straight-forward calculation shows that the above equation is equivalent to

$$D^2\phi = \sum_{\alpha \in \Gamma_{j-1}} \left(\sum_{\alpha' \in \Gamma_1} c(B^{-1}\alpha + \alpha') \right) T_\alpha D\phi.$$

In the Fourier domain this can be rewritten as $\widehat{D^2\phi} = \tilde{m}\widehat{D\phi}$, where $\tilde{m} \in \ell(\Gamma_{j-1}^*)$ is given by

$$\tilde{m}(\beta) = \sum_{\substack{\alpha \in \Gamma_{j-1} \\ \alpha' \in \Gamma_1}} c(B^{-1}\alpha + \alpha')\overline{e_{j-1,\alpha}(\beta)} = \sum_{\alpha \in \Gamma_j} c(\alpha)\overline{e_{j,\alpha}(A\beta)} = m(A\beta),$$

where $\beta \in \Gamma_{j-1}^*$. Recall that m is the filter associated to ϕ. □

As in [2], the usual notion of multiresolution analysis requires minor modifications for the torus.

Definition 4. A *multiresolution analysis (MRA) of order* 2^j ($j \in \mathbb{N}$) is a collection of closed subspaces of $L^2(\mathbb{T}^2)$, $\{V_k\}_{k=0}^{j}$, satisfying

1. For $1 \leq k \leq j$, $V_{k-1} \subseteq V_k$;
2. For $1 \leq k \leq j$, $f \in V_k$ if and only if $Df \in V_{k-1}$;
3. V_0 is the subspace of constant functions;
4. There exists a *scaling function* $\varphi \in V_j$ such that $X_k(2^{\frac{j-k}{2}} D^{j-k}\varphi)$ is an orthonormal basis for V_k, $0 \leq k \leq j$.

If φ is a scaling function for an MRA, then it is necessarily refinable and the filter associated to it by (3) is called a *low-pass filter* for φ (usually denoted by m_0). Moreover, the spaces V_k, $0 \leq k \leq j$, take the form $V_k(D^{j-k}\varphi)$ since $X_k(2^{\frac{j-k}{2}} D^{j-k}\varphi)$ is an orthonormal basis for V_k. The main results of this section follow. The first characterizes those refinable functions which are scaling functions for an MRA, while the second guarantees the existence of a scaling function given a suitable candidate filter, $m_0 \in \ell(\Gamma_j^*)$.

Theorem 1. *Suppose that $\varphi \in L^2(\mathbb{T}^2)$ is refinable of order 2^j ($j \in \mathbb{N}$) with $\hat{\varphi}(0) \neq 0$. Then φ is the scaling function of an MRA of order 2^j if and only if*

$$|m_0(\beta)|^2 + |m_0(\beta + B^{j-1}\beta_1)|^2 = 1, \quad \beta \in \Gamma_{j-1}^*, \tag{4}$$

and

$$[\hat{\varphi}, \hat{\varphi}]_{A^j}(\beta) = 1, \quad \beta \in \Gamma_j^*, \tag{5}$$

where β_1 is the nonzero element of Γ_1^*.

Proof. Assume that $\beta \in \Gamma_{j-1}^*$, then

$$\begin{aligned}
[2^{\frac{1}{2}}\widehat{D\varphi}, 2^{\frac{1}{2}}\widehat{D\varphi}]_{A^{j-1}}(\beta) &= 2^j \sum_{k \in B^{j-1}\mathbb{Z}^2} |\widehat{D\varphi}(\beta + k)|^2 \\
&= 2^j \sum_{k \in B^{j-1}\mathbb{Z}^2} |\hat{\varphi}(\beta + k)|^2 \, |m_0(\beta + k)|^2 \\
&= 2^j \sum_{\substack{k \in B^j\mathbb{Z}^2 \\ \beta' \in \Gamma_1^*}} |\hat{\varphi}(\beta + B^{j-1}\beta' + k)|^2 \, |m_0(\beta + B^{j-1}\beta')|^2.
\end{aligned}$$

Assume that (4) and (5) hold. It follows from the above calculation that

$$[2^{\frac{1}{2}}\widehat{D\varphi}, 2^{\frac{1}{2}}\widehat{D\varphi}]_{A^{j-1}}(\beta) = 1, \quad \beta \in \Gamma_{j-1}^*,$$

so $X_{j-1}(2^{\frac{1}{2}} D\varphi)$ is an orthonormal basis for its span. Moreover, Lemma 3 guarantees that the low-pass filter for $D\varphi$ satisfies (4), so, inductively, it follows that $X_k(2^{\frac{j-k}{2}} D^{j-k}\varphi)$ is an orthonormal basis for its span, $V_k(D^{j-k}\varphi)$, for each $0 \leq k \leq j$, guaranteeing MRA property 4. Properties 1 and 2 follow from the refinability of φ. Consider $2^{\frac{j}{2}} D^j \varphi$, which is refinable of order 2^0 with low-pass filter $m = 2^{\frac{j}{2}} m_0(A^j \cdot)$. Since Γ_0^* is the trivial quotient group, m is constant. In the Fourier domain, this means $\widehat{D^j\varphi}(k) = \hat{\varphi}(A^j k) = 2^{\frac{j}{2}} \hat{\varphi}(k)$. Recall that $\varphi \in L^2(\mathbb{T}^2)$, so this relation forces

$\widehat{D^j\varphi}(k) = 0$ unless $k = 0$, i.e., $V_0(\varphi)$ is the space of constant functions, justifying MRA property 3. (The fact that $\hat{\varphi}(0) \neq 0$ has also been used here.)

Conversely, assume that φ is the scaling function for an MRA. The orthonormality of $X_j(\varphi)$ is equivalent to (5). But $X_{j-1}(2^{\frac{1}{2}}D\varphi)$ is also an orthormal collection and the calculation made at the beginning of the proof thus shows that

$$1 = [2^{\frac{1}{2}}\widehat{D\varphi}, 2^{\frac{1}{2}}\widehat{D\varphi}]_{A^{j-1}}(\beta) = |m_0(\beta)|^2 + |m_0(\beta + B^{j-1}\beta_1)|^2, \quad \beta \in \Gamma^*_{j-1},$$

where β_1 is the nonzero element of Γ_1^*. Hence, (4) must hold, completing the proof. $\square$

The next theorem shows that the filter equation (4) is sufficient for the existence of a scaling function φ, provided that the candidate filter additionally satisfies $m(0) = 1$. A detailed discussion of examples will be postponed to Section 5, but it is fairly easy to come up with filters satisfying these requirements, e.g., define m_0 by

$$m_0(\beta) = \begin{cases} 1 & \beta \in \Gamma^*_{j-1} \\ 0 & \text{otherwise,} \end{cases} \qquad \beta \in \Gamma_j^*.$$

The validity of this choice follows from Property 2 of Lemma 1, which implies $B^{j-1}\mathbb{Z}^2 = B^j\mathbb{Z}^2 \cup (B^{j-1}\beta_1 + B^j\mathbb{Z}^2)$. By the definition of Γ^*_{j-1} it follows that

$$\mathbb{Z}^2 = \bigcup_{\substack{\beta\in\Gamma^*_{j-1} \\ b\in\{0,1\}}} (\beta + bB^{j-1}\beta_1 + B^j\mathbb{Z}^2),$$

which shows that Γ^*_{j-1} and $\Gamma^*_{j-1} + B^{j-1}\beta_1$ form a partition of Γ_j^*.

Theorem 2. *Fix $j > 0$ and let $m_0 \in \ell(\Gamma_j^*)$ be a candidate low-pass filter satisfying* (4) *and $m_0(0) = 1$. Then m_0 is the low-pass filter of a trigonometric polynomial scaling function of order 2^j.*

Proof. The proof will rest upon justification of a specific definition for an associated scaling function. The refinability will be accomplished by defining certain Fourier coefficients outside the lattice $A\mathbb{Z}^2$ (which by Property 3 of Lemma 1 is identical to $B\mathbb{Z}^2$) and extending using (2). Hence, let $\mathscr{B} = \Gamma_j^* \cap (B\mathbb{Z}^2)^c$ ($\mathscr{B}$ should be regarded as a subset of $\mathbb{Z}^2$) and define $\varphi \in L^2(\mathbb{T}^2)$ as follows:

1. Let $\hat{\varphi}(0) = 2^{-\frac{j}{2}}$.
2. For $\beta \in \mathscr{B}$, let $\hat{\varphi}(\beta) = 2^{-\frac{j}{2}}$.
3. For $\beta \in \mathscr{B}$ and $1 \le k \le j-1$, let

$$\hat{\varphi}(A^k\beta) = \hat{\varphi}(\beta)\prod_{\ell=0}^{k-1} m_0(A^\ell\beta).$$

4. The remaining Fourier coefficients will be zero.

It is clear from the above definition that φ has finitely many nonzero Fourier coefficients, i.e., φ is a trigonometric polynomial. The refinability of φ with respect to the filter m_0 is inherent in the construction, provided that the *strands* defined in Step 3 terminate, i.e., there must exist k, with $1 \leq k \leq j-1$, such that $m_0(A^k\beta) = 0$. Proposition 1 implies that there are precisely two elements of Γ_j^* such that $A\beta = 0$, namely, 0 and $B^{j-1}\beta_1$. By definition, $A^j\beta = 0$ for all $\beta \in \Gamma_j^*$, so if $\beta \neq 0$ then the previous observation implies that $A^k\beta = B^{j-1}\beta_1$ for $1 \leq k \leq j-1$. Since (4) forces $m_0(B^{j-1}\beta_1) = 0$, this completes the proof of refinability. Therefore, in light of Theorem 1 it suffices to demonstrate (5), which will be accomplished through the following three steps.

1. It is a direct consequence of the definition above that (5) holds for each $\beta \in \mathscr{B} \cup \{0\}$. Hence, it remains only to demonstrate (5) for $\beta \in A\Gamma_j^* \setminus \{0\}$, a collection of $2^{j-1} - 1$ elements.
2. Proposition 1 explains that the mapping $\beta \mapsto A\beta$ is two-to-one, so the image of $\mathscr{B}$ under multiplication by A^k has cardinality 2^{j-1-k}. Because $\mathscr{B} \subseteq \Gamma_j^* \setminus A\Gamma_j^*$, $A^{k_2}\mathscr{B}$ is disjoint from $A^{k_1}\mathscr{B}$ when $k_1 < k_2$. Considering $1 \leq k \leq j-1$ as in the construction above, the total number of unique elements of Γ_j^* belonging to $\{A^k\mathscr{B}\}_{k=1}^{j-1}$ is $2^{j-2} + 2^{j-3} + \cdots + 2^1 + 1 = 2^{j-1} - 1$. None of these elements may belong to $\mathscr{B} \cup \{0\}$, so they are precisely the elements of $A\Gamma_j^* \setminus \{0\}$.
3. Let $\beta \in A\Gamma_j^* \setminus \{0\}$. Then $\beta = A^k(\beta' + \mathscr{B}_k)$ where $1 \leq k \leq j-1$ and $\beta' \in \mathscr{B}$. Moreover, using the fact that $\hat{\varphi}(\beta') = 2^{-\frac{j}{2}}$ and Proposition 1,

$$\begin{aligned}
[\hat{\varphi},\hat{\varphi}]_{A^j}(\beta) &= \sum_{\gamma\in\mathscr{B}_k} \prod_{\ell=0}^{k} |m_0(A^\ell(\beta'+\gamma))|^2 \\
&= |m_0(\beta')|^2 \sum_{\gamma\in\mathscr{B}_{k-1}} \prod_{\ell=0}^{k-1} |m_0(A^\ell(\beta'+\gamma))|^2 \\
&\quad + |m_0(\beta'+B^{j-1}\beta_1)|^2 \sum_{\gamma\in\mathscr{B}_{k-1}} \prod_{\ell=0}^{k-1} |m_0(A^\ell(\beta'+\gamma+B^{j-1}\beta_1))|^2 \\
&= \sum_{\gamma\in\mathscr{B}_{k-1}} \prod_{\ell=0}^{k-1} |m_0(A^\ell(\beta'+\gamma))|^2.
\end{aligned}$$

This eventually reduces to the $k = 1$ case, which equals one by (4).

□

4 MRA Wavelets on the torus

With the MRA theory of Section 3 it is a fairly straightforward task to devise a corresponding theory for MRA wavelets. An MRA of order 2^j consists of spaces

$\{V_k\}_{k=0}^{j}$ with $V_{k-1} \subseteq V_k$, $1 \le k \le j$. In particular, V_j is a 2^j-dimensional subspace of $L^2(\mathbb{T}^2)$ while V_0 is the one-dimensional subspace of constant functions. It is natural, therefore, to seek a wavelet system which provides an orthonormal basis for the orthogonal complement of V_0 in V_j, i.e., $V_j \ominus V_0$.

Definition 5. Let $\{V_k\}_{k=0}^{j}$ be an MRA of order 2^j. A function $\psi \in V_j$ is a *wavelet* for the MRA if the collection

$$\left\{2^{\frac{j-k}{2}} T_\alpha D^{j-(k+1)}\psi : 0 \le k \le j-1,\ \alpha \in \Gamma_k\right\}$$

is an orthonormal basis for $V_j \ominus V_0$.

The following theorem provides a construction of a wavelet for any MRA. The reader is reminded that β_1 denotes the nonzero element of Γ_1^*. Analogously, α_1 will denote the nonzero element of Γ_1.

Theorem 3. *Let φ be the scaling function of an MRA of order 2^j. Define ψ by*

$$\hat{\psi}(k) = m_1(k)\hat{\varphi}(k), \quad k \in \mathbb{Z}^2,$$

where $m_1 \in \ell(\Gamma_j^*)$ is defined by

$$m_1(\beta) = \overline{m_0(\beta + B^{j-1}\beta_1)}\exp\left(2\pi i\langle A^{-(j-1)}\alpha_1, \beta\rangle\right). \tag{6}$$

Then, ψ is an orthonormal wavelet for the MRA.

Proof. The proof will establish an orthogonal decomposition of each space V_k, $1 \le k \le j$. By definition, $V_k = V_k(D^{j-k}\varphi)$, and the desired decomposition will have the form

$$V_k(D^{j-k}\varphi) = V_{k-1}(D^{j-k+1}\varphi) \oplus V_{k-1}(D^{j-k}\psi), \quad 1 \le k \le j.$$

In the wavelet literature the spaces $V_{k-1}(D^{j-k}\psi)$ are often denoted W_k and one has the familiar expression $V_k = V_{k-1} \oplus W_{k-1}$, $1 \le k \le j$. The following calculation demonstrates the orthogonality of W_{j-1} and V_{j-1}. For each $\beta \in \Gamma_{j-1}^*$,

$$\begin{aligned}
[2^{\frac{1}{2}}\widehat{D\varphi}, 2^{\frac{1}{2}}\hat{\psi}]_{A^{j-1}}(\beta) &= 2^j \sum_{k \in B^{j-1}\mathbb{Z}^2} \widehat{D\varphi}(\beta + k)\,\overline{\hat{\psi}(\beta + k)} \\
&= 2^j \sum_{k \in B^{j-1}\mathbb{Z}^2} |\hat{\varphi}(\beta + k)|^2\, m_0(\beta + k)\,\overline{m_1(\beta + k)} \\
&= 2^j \sum_{\substack{k \in B^j\mathbb{Z}^2 \\ \beta' \in \Gamma_1^*}} |\hat{\varphi}(\beta B^{j-1}\beta' + k)|^2\, m_0(\beta + B^{j-1}\beta')\,\overline{m_1(\beta + B^{j-1}\beta')} \\
&= m_0(\beta)\,\overline{m_1(\beta)} + m_0(\beta + B^{j-1}\beta_1)\,\overline{m_1(\beta + B^{j-1}\beta_1)} \\
&= 0,
\end{aligned}$$

based upon (5), (6), and the fact that $\langle \alpha_1, \beta_1\rangle = \frac{1}{2}$. The orthonormality of $X_{j-1}(2^{\frac{1}{2}}\psi)$ relies on a similar calculation, again for $\beta \in \Gamma_{j-1}^*$,

$$
\begin{aligned}
[2^{\frac{1}{2}}\hat{\psi}, 2^{\frac{1}{2}}\hat{\psi}]_{A^{j-1}}(\beta) &= 2^j \sum_{k\in B^{j-1}\mathbb{Z}^2} |\widehat{\psi}(\beta+k)|^2 \\
&= 2^j \sum_{k\in B^{j-1}\mathbb{Z}^2} |\hat{\varphi}(\beta+k)|^2\,|m_1(\beta+k)|^2 \\
&= |m_1(\beta)|^2 + |m_1(\beta+B^{j-1}\beta_1)|^2 \\
&= |m_0(\beta)|^2 + |m_0(\beta+B^{j-1}\beta_1)|^2 \\
&= 1.
\end{aligned}
$$

The remainder of the proof stems from an induction argument. Lemma 3 implies that $2^{\frac{1}{2}}D\varphi$ is refinable with filter $m_0(A\cdot)$, which satisfies the Γ_{j-1}^* equivalent of (4). Moreover, a calculation analogous to that in Lemma 3 shows that $2^{\frac{1}{2}}\widehat{D\psi} = m_1(A\cdot)2^{\frac{1}{2}}\widehat{D\varphi}$, so the above calculations may be repeated at the next lower scale to prove that $X_k(2^{\frac{j-k}{2}}D^{j-(k+1)}\psi)$ is an orthonormal basis for W_k, $0 \le k \le j-1$. The orthogonality of W_{k_1} and W_{k_2} for $k_1 > k_2$ follows in the usual manner, i.e., $W_{k_2} \subseteq V_{k_1}$ which is orthogonal to W_{k_1}. □

The last objective of this section is to examine the approximation provided by the wavelet systems considered in this work. The general approach mirrors that of [2]. If ψ is an orthonormal wavelet, then the system of functions in Definition 5 provides an orthonormal basis for $V_j \ominus V_0$ and together, with the constant function $D^j\varphi$, can be used to approximate any $f \in L^2(\mathbb{T}^2)$. However, this is equivalent to considering the approximation of f by the collection $X_j(\varphi)$. Consider the orthogonal projection $P_j : L^2(\mathbb{T}^2) \to V_j(\varphi)$, given by

$$
P_j f = \sum_{\alpha\in\Gamma_j} \langle f, T_\alpha\varphi\rangle T_\alpha\varphi.
$$

In the Fourier domain this is equivalent to $\widehat{P_j f}(k) = [\hat{f},\hat{\varphi}]_{A^j}(k)\,\hat{\varphi}(k)$, $k \in \mathbb{Z}^2$. For the purpose of this discussion, it suffices to consider f such that $\hat{f}(k) = \delta_{k,r}$, where $r,k \in \mathbb{Z}^2$. In this case,

$$
[\hat{f},\hat{\varphi}]_{A^j}(\beta) = \begin{cases} 2^j\overline{\hat{\varphi}(r)} & r \equiv \beta \bmod B^j\mathbb{Z}^2 \\ 0 & r \not\equiv \beta \bmod B^j\mathbb{Z}^2, \end{cases} \qquad \beta \in \Gamma_j^*,
$$

so that

$$
\widehat{P_j f}(k) = \begin{cases} 2^j\overline{\hat{\varphi}(r)}\,\hat{\varphi}(k) & r \equiv k \bmod B^j\mathbb{Z}^2 \\ 0 & r \not\equiv k \bmod B^j\mathbb{Z}^2, \end{cases} \qquad k \in \mathbb{Z}^2.
$$

The squared error of approximation is thus given by

$$
\begin{aligned}
\|P_j f - f\|^2 &= \sum_{k\in\mathbb{Z}^2} |\widehat{P_j f}(k) - \hat{f}(k)|^2 \\
&= (1-2^j|\hat{\varphi}(r)|^2)^2 + 2^j|\hat{\varphi}(r)|^2 2^j \sum_{\substack{k\in B^j\mathbb{Z}^2 \\ k\neq 0}} |\hat{\varphi}(r+k)|^2
\end{aligned}
$$

$$= (1-2^j|\hat{\varphi}(r)|^2)^2 + 2^j|\hat{\varphi}(r)|^2\,(1-2^j|\hat{\varphi}(r)|^2)$$
$$= 1-2^j|\hat{\varphi}(r)|^2,$$

where fact that $[\hat{\varphi},\hat{\varphi}]_{A^j} \equiv 1$ has been used to simplify the sum in the second term of the second line in this calculation. Define $E_j(k)$ by

$$E_j(k) = \sqrt{1-2^j|\hat{\varphi}(k)|^2}, \quad k \in \mathbb{Z}^2.$$

Evidently, $E_j(k)$ represents the approximation error $\|P_j f - f\|$ when f is a trigonometric monomial with unit Fourier coefficient at $r \in \mathbb{Z}^2$. Observe that $E_j(k) = 0$ when $|\hat{\varphi}(k)| = 2^{-\frac{j}{2}}$.

5 Examples

Since the systems considered in this work are finite-dimensional, proper examples should provide a well-defined MRA at any scale $j \geq 2$, hopefully leading to arbitrarily close approximation of functions in $L^2(\mathbb{T}^2)$. Moreover, given a low-pass filter satisfying $m_0(-\beta) = \overline{m_0(\beta)}$, $\beta \in \Gamma_j^*$ it is natural to expect a corresponding real-valued scaling function. The next proposition describes a modification of the construction in Theorem 2 that serves this purpose.

Proposition 3. *Let $m_0 \in \ell(\Gamma_j^*)$ be a low-pass filter satisfying* (4) *and such that $m_0(0) = 1$ and $m_0(-\beta) = \overline{m_0(\beta)}$, $\beta \in \Gamma_j^*$. Define φ as follows: (where $\beta \in \mathscr{B}$ should be regarded as an element of $\mathbb{Z}^2$)*

1. *Let $\hat{\varphi}(0) = 2^{-\frac{j}{2}}$.*
2. *If $\beta, -\beta \in \mathscr{B}$, let $\hat{\varphi}(\beta) = 2^{-\frac{j}{2}}$ and define*

$$\hat{\varphi}(A^k\beta) = \hat{\varphi}(\beta)\prod_{\ell=0}^{k-1} m_0(A^\ell\beta), \quad 1 \leq k \leq j-1.$$

3. *If $\beta \in \mathscr{B}$, but $-\beta \notin \mathscr{B}$, let $\hat{\varphi}(\pm\beta) = 2^{-\frac{j+1}{2}}$ and define*

$$\hat{\varphi}(\pm A^k\beta) = \hat{\varphi}(\pm\beta)\prod_{\ell=0}^{k-1} m_0(\pm A^\ell\beta), \quad 1 \leq k \leq j-1.$$

4. *The remaining Fourier coefficients will be zero.*

Then φ is refinable with respect to m_0 and is a real-valued scaling function for an MRA of order 2^j.

Proof. The fact that φ is real-valued follows from the fact that the construction leads to the conjugate-symmetry $\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$, $k \in \mathbb{Z}^2$. The m_0-refinability is independent of the values of $\hat{\varphi}$ chosen at points in $\mathscr{B}$ or $-\mathscr{B}$, provided that the refinement equation (2) is respected at the points in the A-orbit of such points. The fact that these orbits result in only finitely many nonzero Fourier coefficients for φ follows in exactly the same manner as in Theorem 2. Moreover, the argument given in the proof of Theorem 2 to demonstrate (5) requires only minor modification to account for the *splitting* of strands between $\pm\beta$ in Step 3, above. □

Recall that the Shannon wavelet on $\mathbb{R}$ is associated with an MRA consisting of band-limited subspaces of $L^2(\mathbb{R})$, with scaling function φ defined by $\hat{\varphi}(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$ with corresponding low-pass filter $m_0(\xi) = \chi_{[-\frac{1}{4},\frac{1}{4}]}$. An ideal analog of the Shannon MRA in this context should correspond to a low-pass filter which is symmetric about the origin and equal to the characteristic function of a set including 0. The following proposition describes a low-pass filter for each scale $j \geq 2$ which essentially captures these properties.

Proposition 4 (Shannon Filter). *Fix $j \geq 2$ and let $S_j = \{\beta \in \Gamma_j^* : \beta, -\beta \in \Gamma_{j-1}^*\}$. The low-pass filter $m_0 \in \ell(\Gamma_j^*)$ defined by*

$$m_0(\beta) = \begin{cases} 1 & \beta \in S_j \\ \frac{1}{\sqrt{2}} & \beta \in \Gamma_{j-1}^* \setminus S_j \\ \sqrt{1-|m_0(\beta - B^{j-1}\beta_1)|^2} & \textit{otherwise}, \end{cases} \quad \beta \in \Gamma_j^*,$$

satisfies (4) *and is symmetric in the sense that $m_0(-\beta) = m_0(\beta)$, $\beta \in \Gamma_j^*$.*

Proof. Recall that Γ_{j-1}^* and $\Gamma_{j-1}^* + B^{j-1}\beta_1$ form a partition of Γ_j^*, justifying the last part of the above definition. Hence, (4) is satisfied by construction.

The symmetry of m_0 requires attention to various cases. If $\beta \in S_j$, then $-\beta \in S_j$ and hence $m_0(\beta) = m_0(-\beta) = 1$. If $\beta \in \Gamma_{j-1}^* \setminus S_j$, then $m_0(\beta) = \frac{1}{\sqrt{2}}$. Moreover, $-\beta \notin \Gamma_{j-1}$ and, therefore, can be written as $-\beta = \beta' + B^{j-1}\beta_1$ for some $\beta' \in \Gamma_{j-1}^* \setminus S_j$. It follows that $m_0(\beta) = m_0(-\beta) = \frac{1}{\sqrt{2}}$. This demonstrates symmetry for all $\beta \in \Gamma_{j-1}^*$ and it now follows from (4) that m_0 is symmetric on all of Γ_j^*. □

Figure 2 depicts the low-pass filter described by Proposition 4 for $j = 5$. Notice that the symmetry requirement, together with (4), makes it necessary to define $m_0(\beta) = m_0(-\beta) = \frac{1}{\sqrt{2}}$ for certain points in Γ_j^*. Figures 3 and 4 depict the scaling function and wavelet corresponding to the Shannon MRA.

The next proposition concerns the approximation of trigonometric polynomials provided by the Shannon MRA of order j.

Proposition 5. *Let φ be the scaling function corresponding to the low-pass filter of Proposition 4 given by Proposition 3. If $j \geq 6 + \log_2 r^2$, then $E_j(k) = 0$ for all $k \in \{k = (k_1, k_2) : \max\{|k_1|, |k_2|\} \leq r\}$.*

Proof. Suppose that $\pm\beta \in \Gamma_{j-1}^*$. Then Proposition 4 guarantees that $m_0(\pm\beta) = 1$. Moreover, if $\pm\beta \in \mathscr{B} \cup \{0\}$ then Proposition 3 implies that $\hat{\varphi}(\beta) = 2^{-\frac{j}{2}}$. Alternatively, if $\pm\beta \notin \mathscr{B} \cup \{0\}$, then $\pm\beta = \pm A^k\beta'$ for some β' such that $\pm\beta' \in \Gamma_{j-1}^*$ with

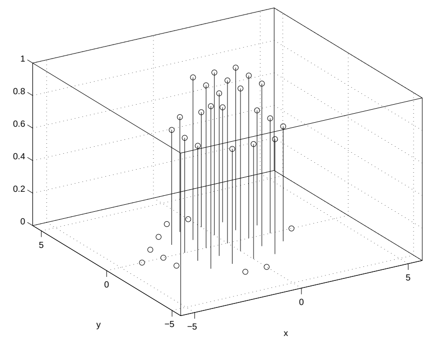
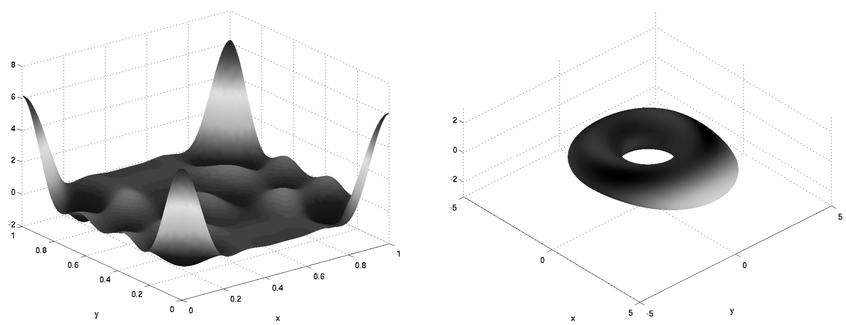
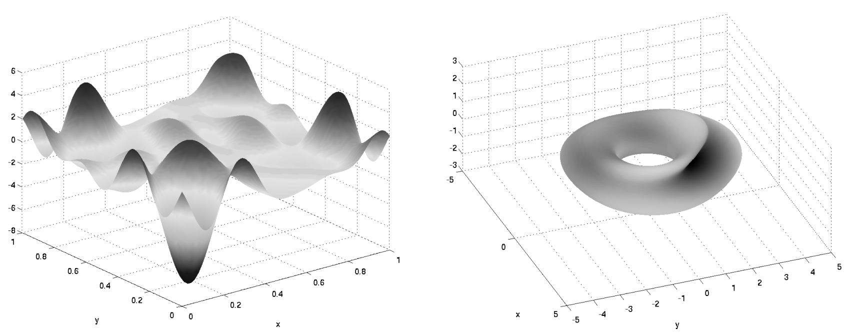


Fig. 2 The low-pass filter m_0 of Proposition 4 with $j = 5$.



(a) (b)

Fig. 3 Graphs of the Shannon scaling function φ for $j = 5$: (a) $\varphi(x,y)$, $(x,y) \in \mathbb{T}^2$ and (b) surface plot of φ as a distortion of the torus.



(a) (b)

Fig. 4 Graphs of the wavelet function ψ corresponding to the Shannon MRA for $j = 5$: (a) $\psi(x,y)$, $(x,y) \in \mathbb{T}^2$ and (b) surface plot of ψ as a distortion of the torus.

$1 \le k \le j-1$. In this latter case, $|\hat{\varphi}(\beta')| = 2^{-\frac{j}{2}}$, while $m_0(A^\ell \beta') = 1$ for $1 \le \ell \le k-1$. The upshot of these observations is that if $\pm\beta \in \Gamma_{j-1}^*$, then $|\hat{\varphi}(\beta)| = 2^{-\frac{j}{2}}$. Therefore, $E_j(k) = 0$ whenever $\pm k \in \Gamma_{j-1}^*$.

Recall that $\Gamma_{j-1}^* = B^{j-1}R \cap \mathbb{Z}^2$, where $R = \left(-\frac{1}{2}, \frac{1}{2}\right] \times \left(-\frac{1}{2}, \frac{1}{2}\right]$. Instead consider $B^{j-1}R' \cap \mathbb{Z}^2$, where $R' = \left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{4}\right]$, so that the set in question has symmetry about the origin. Observe that $B^{j-1}R'$ is a square with side-length $2^{\frac{j-3}{2}}$ centered at the origin and oriented with its corners either on the x,y-axes or along the lines $y = \pm x$. The former situation is the more constraining, but contains all $k = (k_1, k_2) \in \mathbb{Z}^2$ such that $\max\{|k_1|, |k_2|\} \le \frac{1}{2} 2^{\frac{j-3}{2}} 2^{-\frac{1}{2}} = 2^{\frac{j}{2}-3}$. The claimed lower bound on j follows from this last calculation. $\square$

Another example important in the classical theory of MRAs is the Haar MRA. The Haar wavelet on $\mathbb{R}$ is the product of an MRA whose component spaces consist of functions in $L^2(\mathbb{R})$ which are piecewise constant on certain dyadic intervals. The Haar scaling function φ is given by $\varphi = \chi_{[0,1]}$ and the corresponding low-pass filter is given by

$$m_0(\xi) = \frac{1}{2}\left(1 + \exp(-2\pi i \xi)\right).$$

Therefore, a natural counterpart to the Haar MRA should be associated with a conjugate-symmetric low-pass filter corresponding to a refinement involving just two translates from Γ_j. Moreover, assuming the first translate is zero, the nonzero translate should be as close to zero as possible. The following proposition describes a low-pass filter which meets these requirements.

Proposition 6 (Haar Filter). *Fix $j \ge 2$. Define $m_0 \in \ell(\Gamma_j^*)$ by*

$$m_0(\beta) = \frac{1}{2}\Big(1 + \exp\left(-2\pi i \langle A^{-(j-1)}\alpha_1, \beta\rangle\right)\Big),$$

where α_1 is the nonzero element of Γ_1. Then m_0 satisfies (4) with $m_0(0) = 1$ and is conjugate-symmetric, i.e., $m_0(-\beta) = \overline{m_0(\beta)}$, $\beta \in \Gamma_j^*$.

Proof. It is routine to verify that m_0 so defined is $B^j\mathbb{Z}^2$-periodic and conjugate-symmetric with $m_0(0) = 1$. Observe that the filter may also be expressed as

$$m_0(\beta) = \cos\left(\pi \langle A^{-(j-1)}\alpha_1, \beta\rangle\right) \exp\left(-\pi i \langle A^{-(j-1)}\alpha_1, \beta\rangle\right).$$

Hence, the filter identity (4) follows from the calculation,

$$\begin{aligned}
&|m_0(\beta)|^2 + |m_0(\beta + B^{j-1}\beta)|^2 \\
&\quad = \cos^2\left(\pi\langle A^{-(j-1)}\alpha_1, \beta\rangle\right) + \cos^2\left(\pi\langle A^{-(j-1)}\alpha_1, \beta + B^{j-1}\beta_1\rangle\right) \\
&\quad = \cos^2\left(\pi\langle A^{-(j-1)}\alpha_1, \beta\rangle\right) + \cos^2\left(\pi\langle A^{-(j-1)}\alpha_1, \beta\rangle + \frac{\pi}{2}\right) \\
&\quad = \cos^2\left(\pi\langle A^{-(j-1)}\alpha_1, \beta\rangle\right) + \sin^2\left(\pi\langle A^{-(j-1)}\alpha_1, \beta\rangle\right) \\
&\quad = 1.
\end{aligned}$$

□

The final result of the section provides a somewhat coarse approximation result for the Haar MRA.

Proposition 7. *Let φ be the scaling function corresponding to the low-pass filter of Proposition 6 given by Proposition 3. Then for any $r \in \mathbb{Z}^2$,*

$$\lim_{j\to\infty} E_j(r) = 0.$$

Proof. Fix $r \in \mathbb{Z}^2$ and let J be the smallest positive integer such that $r \in B^J R \cap \mathbb{Z}^2$ (R as in the definition of Γ_j^*), so that $r = A^k \beta'$ for some $\beta' \in \mathscr{B}$ and $k \le J$. For sufficiently large j, both β' and $-\beta'$ will belong to $\mathscr{B}$, so without loss of generality it may be assumed that $|\hat{\varphi}(\beta)| = 2^{\frac{j}{2}}$. The construction of φ implies that

$$|\hat{\varphi}(r)| = 2^{-\frac{j}{2}} \left| \prod_{\ell=0}^{k-1} m_0(A^\ell \beta') \right|$$

where

$$|m_0(A^\ell \beta')| = \cos\left(\pi \langle A^{-(j-1)}\alpha_1, A^\ell \beta'\rangle\right) = \cos\left(\pi\langle \alpha_1, (AB^{-1})^{j-1} A^{\ell+1-j}\beta'\rangle\right).$$

Notice that AB^{-1} is norm-preserving and $A^{\ell+1-j}\beta' \to 0$ as $j \to \infty$, which means that the terms in the above product tend to 1 as $j \to \infty$. Hence,

$$\lim_{j\to\infty} |\hat{\varphi}(r)| = 2^{-\frac{j}{2}}$$

and $\lim_{j\to\infty} E_j(r) = 0$, concluding the proof. □

Figure 5 depicts the modulus of the low-pass filter described by Proposition 6 for $j = 5$, while Figures 6 and 7 depict the corresponding scaling function and wavelet for the Haar MRA.

References

1. C. de Boor, R. DeVore, and A. Ron, *The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$*, J. Funct. Anal., **119**(1) (1995), 37–78.
2. B. D. Johnson, *A finite-dimensional approach to wavelet systems on the circle*, (2008), submitted.
3. A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$*, Canad. J. Math., **47** (1995), 1051-1094.

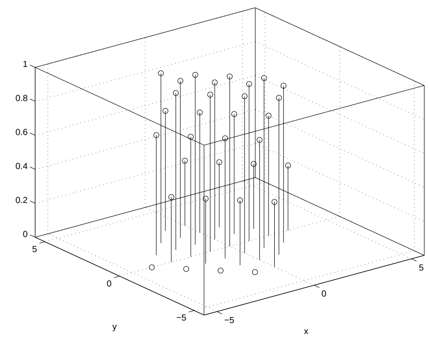
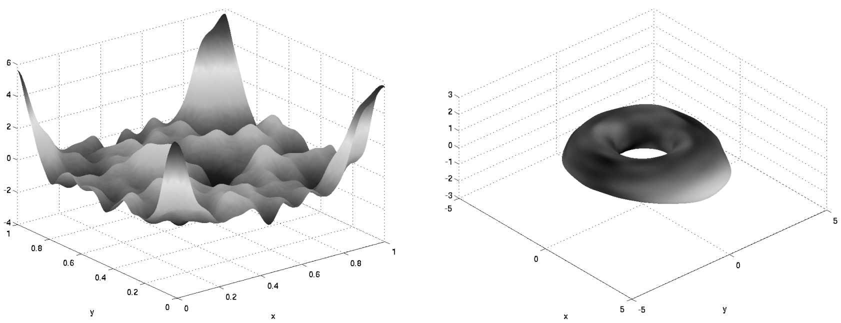
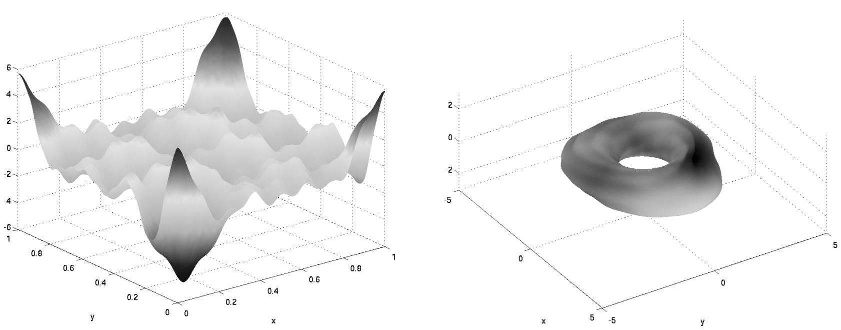


Fig. 5 The modulus of the low-pass filter m_0 of Proposition 6 with $j = 5$.



(a) (b)

Fig. 6 Graphs of the Haar scaling function φ for $j = 5$: (a) $\varphi(x,y)$, $(x,y) \in \mathbb{T}^2$ and (b) surface plot of φ as a distortion of the torus.



(a) (b)

Fig. 7 Graphs of the wavelet function ψ corresponding to the Haar MRA for $j = 5$: (a) $\psi(x,y)$, $(x,y) \in \mathbb{T}^2$ and (b) surface plot of ψ as a distortion of the torus.