Shift-invariant frames and the frame potential

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Abstract:

The recently introduced notion of a frame potential has led to useful characterizations of finite-dimensional tight frames consisting of vectors with prescribed lengths. It is natural to ask whether the frame potential leads to similar characterizations for systems with additional imposed structure. We will describe how such a generalization can be obtained for the class of shift-invariant systems. The fast algorithms associated with convolution make shift-invariant systems advantageous in applications. (joint work with M. Fickus, K. Kornelson, & K. Okoudjou)

What is a frame?

- A collection $X := \{x_j\}_{j\in\mathcal{J}} \subset \mathbb{H}$ is a *frame* for $\mathbb{H}$ if and only if there exist constants $0 < B_1 \leq B_2 < \infty$ such that for each $x \in \mathbb{H}$

$$B_1\|x\|^2 \leq \sum_{j\in\mathcal{J}} \left|\langle x, x_j\rangle\right|^2 \leq B_2\|x\|^2; \tag{1}$$

- X is called a *tight frame* if it is possible that $B_1 = B_2$;
- Frames generalize the notion of an orthonormal basis.
  - ⋆ Frame coefficients uniquely determine elements:

$$\langle x, x_j\rangle = \langle y, x_j\rangle,\ \forall j \in \mathcal{J} \quad \Rightarrow \quad x = y;$$

  - ⋆ However, representation of an element in terms of frame vectors is not, in general, unique:

$$x = \sum_j \alpha_j x_j = \sum_j \beta_j x_j \quad \not\Rightarrow \quad \alpha_j = \beta_j,\ \forall j \in \mathcal{J}.$$

Operators associated to frames:

- Analysis operator: $L_X : \mathbb{H} \to \ell^2(\mathcal{J})$, defined by

$$L_X x = \{\langle x, x_j \rangle\}_{j \in \mathcal{J}};$$

- Synthesis operator: $L_X^* : \ell^2(\mathcal{J}) \to \mathbb{H}$, given by

$$L_X^* y = \sum_{j \in \mathcal{J}} y(j) x_j;$$

- In finite dimensions the matrix representation of L_X^* is

$$L_X^* = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}_{d \times N}.$$

Operators associated to frames:

- Frame operator: $S_X := L_X^* L_X : \mathbb{H} \to \mathbb{H}$, given by:

$$S_X x = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle x_j.$$

- X is a frame with bounds $B_1 \le B_2$ if and only if

$$B_1 I \le S_X \le B_2 I.$$

- Between analysis and synthesis one may perform various processing operations (thresholding, truncating, etc.), but ultimately one wants to recover x (or a good approximation) from $L_X x$.

Reconstruction via frames:

- Dual frame: a collection $Y := \{y_j\}_{j \in \mathcal{J}}$ such that for any $x \in \mathbb{H}$

$$x = L_Y^* L_X x = L_X^* L_Y x;$$

- Canonical dual frame: define $y_j := S^{-1} x_j$ so that

$$L_X^* L_Y x = \sum_{j \in \mathcal{J}} \langle x, S_X^{-1} x_j \rangle x_j = S_X S_X^{-1} x = x;$$

- Frame algorithm (recursive):
  - ⋆ Set $x_0 = 0$;
  - ⋆ For $n \geq 1$, let $x_n = x_{n-1} + \lambda S_X (x - x_{n-1})$;
  - ⋆ If $\lambda := \frac{2}{B_1 + B_2}$ then $x_n \to x$ with $\|x - x_n\| \leq \left(\frac{B_2 - B_1}{B_1 + B_2}\right)^n \|x\|$.

Proof of the frame algorithm:

Let $T := \frac{2}{B_1+B_2}S - I$, so $\|T\|_{\text{op}} \leq \frac{B_2-B_1}{B_1+B_2} < 1$.

Hence,

$$\begin{aligned} x - x_n &= x - x_{n-1} - \frac{2}{B_1 + B_2} S(x - x_{n-1}) \\ &= T(x - x_{n-1}), \end{aligned}$$

so $\|x - x_n\| \leq \left(\frac{B_2-B_1}{B_1+B_2}\right)\|x - x_{n-1}\|$ and one concludes

$$\|x - x_n\| \leq \left(\frac{B_2 - B_1}{B_1 + B_2}\right)^n \|x\|. \quad \square$$

Remark: Rather than inverting S_X, one can use the frame algorithm to approximate the canonical dual frame. (Set $x = S^{-1}x_j$, algorithm requires $S_X x = x_j$.)

Tight frames:

- Tight frames are ideal for applications, since they eliminate the need for a dual frame or use of the frame algorithm.
  - ⋆ If $B_1 = B_2$ then $S = B_1 I$, so an easy dual frame is obtained by defining $y_j = \frac{1}{B_1} x_j$;
  - ⋆ If $B_1 = B_2$ then the frame algorithm converges in a single iteration.
- 2000: Benedetto and Fickus [1] introduced the frame potential,

$$\mathrm{FP}\big(\{x_n\}_{n=1}^N\big) = \sum_{m,n=1}^{N} \big|\langle x_m, x_n \rangle\big|^2,$$

to study tight frames of unit-norm vectors in finite dimensional Hilbert space. ($\mathbb{H}^d \equiv$ d-dimensional Hilbert space)

Motivation behind the frame potential:

- "Frame force" between $x, y \in S^{d-1}$

$$\mathrm{FF}(x, y) = \langle x, y \rangle (x - y).$$

  Notice that mutually orthogonal vectors are in equilibrium.

- Potential betwen $x, y \in S^{d-1}$ (so that $\nabla P = -FF$)

$$\mathrm{P}(x, y) = \frac{1}{2}\left(\left|\langle x, y \rangle\right|^2 - 1\right).$$

- Total potential of a collection is obtained by summing the potential between pairs of points on the sphere:

$$\mathrm{TP}(x, y) = \sum_{m=1}^{N} \sum_{m \neq n} \mathrm{P}(x_m, x_n).$$

Minimizers of the frame potential:

- If $X = \{x_n\}_n \subset S^{d-1}$ is a minimizer of the frame potential then it can be shown that each x_n is an eigenvector of the associated frame operator S_X.

- Minimizer sets may be partitioned via eigenvalues into mutually orthogonal sequences which are Parseval frames for their spans:

$$E_\lambda = \{x_n : Sx_n = \lambda x_n\}.$$

- Benedetto and Fickus used this decomposition together with a perturbation argument to show that local minimizers over $\{x_n\}_{n=1}^N \subset S^{d-1}$ must either comprise an orthonormal set (if underdetermined) or a tight frame (if overdetermined).

Frame potential characterizations:

Theorem 1 (Benedetto and Fickus [1]). Let N, d be positive integers and let $\{x_n\}_{n=1}^N \subset \mathbb{H}^d$ with $\|x_n\| = 1$ for each $1 \leq n \leq N$.

(a) Every local minimizer of $\mathrm{FP}(\{x_n\}_{n=1}^N)$ (under the constraint that $\|x_n\| = 1 \ \forall n$) is also a global minimizer.

(b) If $N \leq d$ the minimum of the frame potential is N and the minimizers are the orthonormal sequences.

(c) If $N \geq d$ the minimum value of the frame potential is N^2/d and the minimizers are the tight frames.

Theorem 1 implies that one may search for tight frames of unit norm vectors using the potential gradient. (also guarantees existence)

Frame potential characterizations:

- 2002: Casazza, Fickus, Kovačević, Leon, & Tremain considered collections of non-uniform norms and again studied the minimizers of the frame potential;
- They found that tight frames do not exist for every sequence of prescribed norms.

Theorem 2 (Casazza et al. [2]). If $X = \{x_n\}_{n=0}^{N-1} \subset \mathbb{H}^d$ is a tight frame with $\|x_n\| = a_n$ and $a_0 \geq a_1 \geq \cdots \geq a_{N-1} > 0$, then

$$da_0^2 \leq \sum_{n=0}^{N-1} a_n^2.$$

This inequality is called the *fundamental frame inequality (FFI)*.

Frame potential characterizations:

- Casazza et al. generalized Theorem 1 as follows.
- Let $\mathcal{A} = \Big\{ \{x_n\}_{n=0}^{N-1} \subset \mathbb{H}^d : \|x_n\| = a_n, 0 \leq n \leq N-1 \Big\}$.

  Theorem 3. If $X = \{x_n\}_{n=0}^{N-1}$ is a minimizer of the frame potential over $\mathcal{A}$ and $\{a_n\}_{n=0}^{N-1}$ satisfies FFI then X is a tight frame for $\mathbb{H}^d$. $(N \geq d)$

- Their work also describes the minimizers of the frame potential when FFI is not satisfied. The vectors of largest norm "push" the remaining smaller vectors into an orthogonal subspace:

$$X = \{x_0\} \perp \{x_1\} \cdots \perp \{x_{n_0-1}\} \perp \{x_n\}_{n=n_0}^{N-1},$$

  where $\{x_n\}_{n=n_0}^{N-1}$ is a tight frame for its span [2].

Question:

Will the frame potential lead to characterizations of tight frames with additional imposed structure? (e.g., shift-invariance)

Shift-invariant systems in $\mathbb{H}^d$:

- Think of $\mathbb{H}^d$ as $\ell(\mathbb{Z}_d)$, where $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$. Let $x \in \ell(\mathbb{Z}_d)$ and suppose that $N \mid d$.
  - ⋆ Translation: $Tx(k) = x(k-1)$, $k \in \mathbb{Z}_d$;
  - ⋆ convolution: $x * y(k) = \sum_{n\in\mathbb{Z}_d} x(n)y(k-n)$;
  - ⋆ Involution: $\tilde{x}(k) = \overline{x(-k)}$;
  - ⋆ Fourier transform: $\mathcal{F}_d x(n) = \hat{x}(n) = \dfrac{1}{\sqrt{d}} \sum_{k\in\mathbb{Z}_d} x(k) e^{-2\pi i \frac{k}{d} n}$;
  - ⋆ Downsampling by N: $\downarrow_N : \ell(\mathbb{Z}_d) \to \ell(\mathbb{Z}_{d/N})$,

$$(\downarrow_N x)(k) = x(Nk).$$

  - ⋆ Upsampling by N: $\uparrow_N : \ell(\mathbb{Z}_{d/N}) \to \ell(\mathbb{Z}_d)$,

$$(\uparrow_N x)(k) = \begin{cases} x(k/N), & N \mid k, \\ 0, & N \nmid k. \end{cases}$$

Shift-invariant systems in $\mathbb{H}^d$:

Definition 1. Let N, d be positive integers with $N \mid d$. Given $\{h_m\}_{m=0}^{M-1} \subset \mathbb{H}^d$, the N *shift-invariant system* generated by $\{h_m\}_m$ is

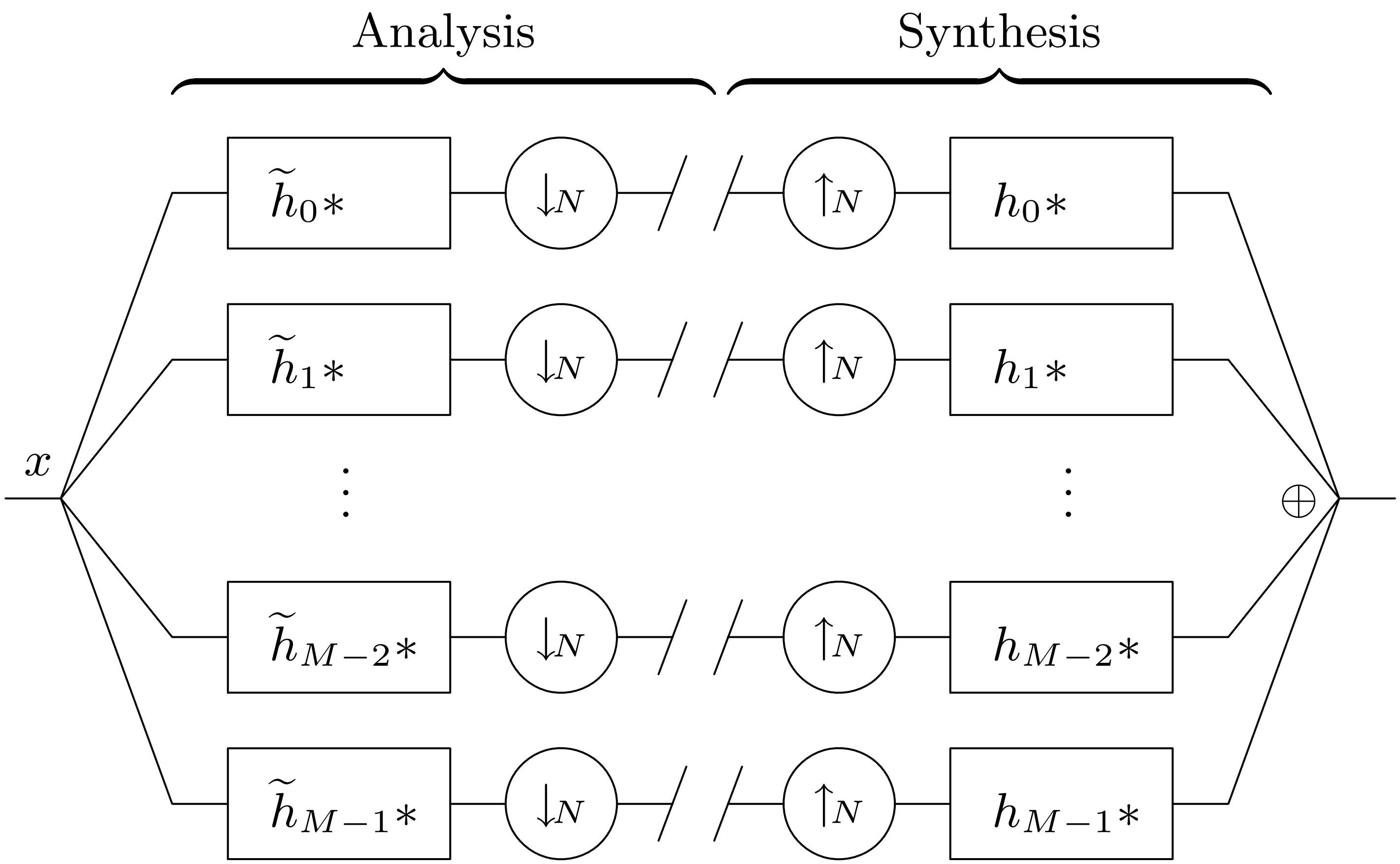
$$X_C(\{h_m\}_{m=0}^{M-1}, N) = \{T_k h_m : k \in N\mathbb{Z}_d, 0 \leq m \leq M-1\}.$$

Let S be the frame operator of $X_C(\{h_m\}_{m=0}^{M-1}, N)$, then

$$Sx = \sum_{m=0}^{M-1} \sum_{k \in N\mathbb{Z}_d} \langle x, T_k h_m \rangle T_k h_m = \sum_{m=0}^{M-1} \Big(\uparrow\downarrow_N (x * \tilde{h}_m) \Big) * h_m.$$

The latter form reveals the convolutional nature of shift-invariant systems. In this sense S may be thought of as a *filter bank frame operator*, where $H := \{h_m\}_m$ is the collection of filters.

Filterbank frame operator:



Basic ideas:

- $S_{X_C}x = \sum_{k \in N\mathbb{Z}_d} T_k \sum_{m=0}^{M-1} \langle T_{-k}x, h_m \rangle h_m = \sum_{k \in N\mathbb{Z}_d} T_k S_H T_{-k} x,$ where, again, $H = \{h_m\}_{m=0}^{M-1}$.

- One may construct familiar examples using the above identity. Let $d = 2^p$, $N = 2$, $M = 2$:

$$L_H^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \Longrightarrow S_H = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}_{d \times d} \Longrightarrow S_{X_C} = I_d.$$

This is the discrete Haar basis.

Mercedes-Benz example:

The so called Mercedes-Benz frame is the $\frac{3}{2}$-tight frame for $\mathbb{R}^2$ associated with the third-roots of unity.

Let $d = 2^p$, $N = 2$, $M = 3$:

$$L_H^* = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow S_H = \begin{pmatrix} \frac{3}{2} I_2 & 0 \\ 0 & 0 \end{pmatrix}_{d \times d} \Longrightarrow S_{X_C} = \frac{3}{2} I_d.$$

Characterizing shift-invariant tight frames:

- Given $a_0 \geq \cdots \geq a_{M-1} > 0$, constrain filter lengths:

$$\|h_m\| = a_m, \quad 0 \leq m \leq M-1.$$

- $X_C(H)$ will satisfy FFI if

$$Na_0^2 \leq \sum_{m=0}^{M-1} a_m^2. \quad \text{(easier since } N \leq d\text{)}$$

- One can still prove that if a set of filters H is a minimizer then each element $T_{Nk}h_m$ of X_C is an eigenvector of S_{X_C}, but the pertubation arguments used in the previous results do not seem to work.

Modulated filter representation:

- For $0 \le k \le d-1$, let

$$H^*_{\text{mod}}(k) = \sqrt{\frac{d}{N}} \begin{bmatrix} \widehat{h}_0(k + \frac{0d}{N}) & \dots & \widehat{h}_{M-1}(k + \frac{0d}{N}) \\ \vdots & & \vdots \\ \widehat{h}_0(k + \frac{(N-1)d}{N}) & \dots & \widehat{h}_{M-1}(k + \frac{(N-1)d}{N}) \end{bmatrix}_{N \times M}$$

- Let H^*_{mod} be the $d \times M \frac{d}{N}$ matrix given by

$$H^*_{\text{mod}} = \begin{bmatrix} H^*_{\text{mod}}(0) & & 0 \\ & \ddots & \\ 0 & & H^*_{\text{mod}}(\frac{d}{N} - 1) \end{bmatrix}$$

The modulated filter representation here is adapted from that of the $\ell^2(\mathbb{Z})$ setting, e.g. in the work of Vetterli [4].

Modulated filter representation:

Proposition 4. The synthesis operator $L^*_{X_C}$ of $X_C(\{h_m\}_m, N)$ may be written as

$$L^*_{X_C} = U_1 H^*_{\text{mod}} U_2,$$

where U_1, U_2 are unitary. In particular, the frame operator S_{X_C} is unitarily equivalent to $H^*_{\text{mod}} H_{\text{mod}}$.

- The unitary operators involve the Fourier transform as well as perfect shuffle operators (Strohmer [3]).
- H^*_{mod} can be interpreted as the tensor sum of $\frac{d}{N}$ synthesis operators for collections of M vectors in $\mathbb{H}^N$.

Modulated filter representation:

- Transforming the problem:

$$X_C(\{h_m\}_m, N) \subset \mathbb{H}^d \quad \Longleftrightarrow \quad X_j = \{x_{m,j}\}_m \subset \mathbb{H}^N,$$

$0 \le j \le \frac{d}{N} - 1$, where $x_{m,j}$ is the mth column of $H^*_{\text{mod}}(j)$.

- Constraints for $0 \le m \le M - 1$:

$$\|h_m\|^2 = a_m^2 \quad \Longleftrightarrow \quad \sum_{j=0}^{\frac{d}{N}-1} \|x_{m,j}\|^2 = \frac{d}{N} a_m^2.$$

- The convolutional frame potential problem is converted into a "shared-constraint" version of the Casazza et al. minimization problem.

Relating the problems:

Proposition 5. Let $\{h_m\}_m$ and X_j as above.

(a) $\mathrm{FP}\big(X_C(\{h_m\}_m, N)\big) = \sum_{j=0}^{\frac{d}{N}-1} \mathrm{FP}\big(X_j\big)$;

(b) The frame bounds of $X_C(\{h_m\}_m, N)$ are the minimum/maximum of the frame bounds of the collections X_j.

Solution of the shared-constraint problem:

Let $\mathcal{A} = \left\{ \{x_{m,j}\}_{m,j} \subset \mathbb{H}^N : \sum_{j=0}^{\frac{d}{N}-1} \|x_{m,j}\|^2 = \frac{d}{N} a_m^2, 0 \leq m \leq M-1 \right\}$.

Theorem 6. Suppose $M \geq N$ where $N \mid d$ and let $a_0 \geq a_1 \geq \cdots \geq a_{M-1} > 0$. If the collections X_j form a minimizer of the combined frame potential

$$\sum_{j=0}^{\frac{d}{N}-1} \mathrm{FP}(X_j)$$

over $\mathcal{A}$ and FFI is satisfied, then each collection X_j is a tight frame with a common frame bound.

Final characterization:

Let $\mathcal{A} = \Big\{ \{h_m\}_m \subset \mathbb{H}^d : \|h_m\| = a_m, 0 \leq m \leq M-1 \Big\}$.

Corollary 7. Suppose $M \geq N$ where $N \mid d$ and $\{a_m\}_m$ satisfies FFI. If $\{h_m\}_m$ is a minimizer of $\mathrm{FP}\big(X_C(\{h_m\}_m, N)\big)$ over $\mathcal{A}$ then $X_C(\{h_m\}_m)$ is a tight frame for $\mathbb{H}^d$.

- one may thus search for convolutional tight frames using the frame potential
- underdetermined case and situations where FFI does not hold are analogous

References

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