Non-separable bidimensional filter banks associated with oversampled wavelet transforms

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Abstract:

We develop discrete wavelet transforms associated with oversampled affine systems. We give sufficient conditions for the construction of nonseparable bidimensional dyadic QMFs from separable bidimensional dyadic QMFs through oversampling.

Dyadic multiresolution analysis:

- Dyadic dilation: $Df(x) = \sqrt{2}f(2x)$
- Translation: $Tf(x) = f(x-1)$ (notation: $T_k := T^k$)
- $\{V_j\}_{j\in\mathbb{Z}}$ (closed subspaces of $L^2(\mathbb{R})$) form a *multiresolution analysis* if

  1. $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$;
  2. $f \in V_j \Leftrightarrow D^{-j}f \in V_0$;
  3. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
  4. $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
  5. $\exists\ \varphi \in V_0$ such that $\{T_k\varphi\}_{k\in\mathbb{Z}}$ is an ONB for V_0. (φ is the scaling function)

Dyadic multiresolution analysis:

- $D^{-1}\varphi \in V_{-1} \subset V_0$ leads to the low-pass filter m_0:

$$\hat{\varphi}(2\xi) = m_0(\xi)\,\hat{\varphi}(\xi)$$

- m_0 is 1-periodic and must satisfy the Smith-Barnwell equation:

$$\left|m_0(\xi)\right|^2 + \left|m_0(\xi + \frac{1}{2})\right|^2 = 1 \quad \text{a.e.}$$

- Defining ψ by $\hat{\psi}(2\xi) = e^{-2\pi i\xi}\overline{m_0(\xi + \frac{1}{2})}$ one obtains an orthonormal wavelet, i.e., $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ is an ONB for $L^2(\mathbb{R})$.

- A collection $\{h_j\}_{j\in J} \subset \mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $0 < A \le B < \infty$ such that for all $f \in \mathbb{H}$

$$A\|f\|_{\mathbb{H}}^2 \le \sum_{j\in J} |\langle f, h_j\rangle_{\mathbb{H}}|^2 \le B\|f\|_{\mathbb{H}}^2. \tag{1}$$

(*Tight frame* $\Leftrightarrow A = B$; *Parseval frame* $\Leftrightarrow A = B = 1$.)

Generalized low-pass filters:

- If N is a positive, odd integer, notice that $m(\xi) := m_0(N\xi)$ also satisfies the Smith-Barnwell equation.

$$\begin{aligned}\left|m(\xi)\right|^2 + \left|m(\xi + \frac{1}{2})\right|^2 &= \left|m_0(N\xi)\right|^2 + \left|m_0(N\xi + \frac{N}{2})\right|^2 \\ &= \left|m_0(N\xi)\right|^2 + \left|m_0(N\xi + \frac{1}{2})\right|^2 \\ &= 1.\end{aligned}$$

- Example: If $N = 3$ and m_0 is the Haar low-pass filter then one obtains

$$m(\xi) = \frac{1}{2}\Big(1 + e^{-2\pi i 3\xi}\Big).$$

  This example falls under the class of generalized low-pass filters studied by Paluszyński, Šikić, Weiss, and Xiao [7, 8], which can be used to produce Parseval frame wavelets.

Wavelets and translational oversampling:

- In 1994, Chui and Shi proved that if $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ is an ONB then $\{\frac{1}{\sqrt{N}} D^j T_{\frac{k}{N}} \psi\}_{j,k\in\mathbb{Z}}$ is a Parseval frame for any positive, odd integer N [4]. (2 and N must be relatively prime)

- In the case of an MRA wavelet with scaling function φ, we have the refinement equation,

$$D^j T_k \varphi = \sqrt{2} \sum_{m\in\mathbb{Z}} \alpha_{m-2k} D^{j+1} T_m \varphi,$$

  where $m_0(\xi) = \sum_{m\in\mathbb{Z}} \alpha_m e^{-2\pi i m \xi}$.

- For the $n\times$ oversampled system, this can be written as

$$D^j T_{\frac{k}{N}} \varphi = \sqrt{2} \sum_{(m-2k)\in N\mathbb{Z}} \alpha_{\frac{m-2k}{N}} D^{j+1} T_{\frac{m}{N}} \varphi,$$

Oversampled low-pass filter:

- Let $\{\tilde{\alpha}_m\}_{m\in\mathbb{Z}}$ be defined by

$$\tilde{\alpha}_m = \begin{cases} \alpha_{\frac{m}{N}}, & m \in N\mathbb{Z} \\ 0, & m \notin N\mathbb{Z}. \end{cases}$$

  This leads to another expression of the oversampled refinement,

$$D^j T_{\frac{k}{N}} \varphi = \sqrt{2} \sum_{m\in\mathbb{Z}} \tilde{\alpha}_{m-2k} D^{j+1} T_{\frac{m}{N}} \varphi,$$

  with an oversampled filter: $\tilde{m}_0(\xi) = \sum_{m\in\mathbb{Z}} \tilde{\alpha}_m e^{-2\pi i m\xi}$.

- Notice that $\tilde{m}_0(\xi) = m_0(N\xi)$.

- Hence $m(\xi) = \frac{1}{2}\left(1+e^{-2\pi i 3\xi}\right)$ is also the low-pass filter associated to the $3\times$ oversampled Haar wavelet.

Wavelet frames and matrix oversampling:

The oversampling result of Chui and Shi has been generalized to multi-generated affine frames in higher dimensions:

- Generators: $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$;
- Dilation: $Df(x) = \sqrt{|\det A|}f(Ax)$, A expansive, integer entries;
- Translations: $P^{-1}\mathbb{Z}^n$ where P has integer entries and $\det P \neq 0$;
- Oversampled affine system generated by Ψ relative to P:

$$X(\Psi, P) = \{|\det P|^{-\frac{1}{2}} D^j T_{P^{-1}k}\psi_\ell : 1 \leq \ell \leq L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\};$$

- $X(\Psi) := X(\Psi, I_n)$ is the usual affine system.

Second oversampling theorem:

- The relative primality condition of Chui and Shi's original result must be replaced by two *admissibility* conditions on P:
  - ⋆ PAP^{-1} must have integer entries; (automatic in scalar case)
  - ⋆ $P^{-1}\mathbb{Z}^n \bigcap A^{-1}\mathbb{Z}^n = \mathbb{Z}^n$. (relative primality condition)
- The following result was originally proven by Ron and Shen [9], but has been revisited by several others from various points of view: Chui et al. [3], Laugesen [6], Hernández et al. [5].

Theorem 1 (Second Oversampling Theorem). *If P satisfies the above admissibility conditions and $X(\Psi, I_n)$ is a frame then $X(\Psi, P)$ is a frame with the same upper and lower frame bounds.*

Multiresolution analysis:

- Dual scaling functions: $\varphi, \tilde{\varphi}$
- Refinable dual generating families: $\Psi := \{\psi_\ell\}_\ell, \tilde{\Psi} := \{\tilde{\psi}_\ell\}_\ell$
- Filters: $m_\ell, \tilde{m}_\ell$, $0 \leq \ell \leq L$, satisfying the generalized Smith-Barnwell equations,

$$\sum_{\ell=0}^{L} \overline{m_\ell(\xi)} \tilde{m}_\ell(\xi + (A^T)^{-1}\vartheta_s) = \delta_{0,s}, \quad 0 \leq s \leq m-1, \tag{2}$$

such that

$$\hat{\psi}_\ell(A^T\xi) = m_\ell(\xi)\hat{\varphi}(\xi) \text{ and } \hat{\tilde{\psi}}_\ell(A^T\xi) = \tilde{m}_\ell(\xi)\hat{\tilde{\varphi}}(\xi) \tag{3}$$

for $0 \leq \ell \leq L$ and a.e. $\xi \in \mathbb{R}^n$. ($\psi_0 := \varphi$, $\tilde{\psi}_0 := \tilde{\varphi}$)

- $\{\vartheta_s\}_{p=0}^{a-1}$ is a set of coset representatives of $\mathbb{Z}^n/A^T\mathbb{Z}^n$ and $a := |\det A|$. ($\vartheta_0 := 0$)

Oversampled filters:

- Original perfect reconstruction filters, $0 \le \ell \le L$,

$$m_\ell(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k e^{-2\pi i \langle \xi, k \rangle}.$$

- Following the one-dimensional case, define oversampled filters by

$$m_\ell^P(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k^P e^{-2\pi i \langle \xi, k \rangle},$$

where

$$\alpha_{\ell;r}^P := \begin{cases} \alpha_{\ell;s} & r = Ps, s \in \mathbb{Z}^n \\ 0 & \text{otherwise} \end{cases}.$$

Observe that $m_\ell^P(\xi) = m_\ell(P^T \xi)$.

Oversampled discrete wavelet transform:

Proposition 2 (Oversampled Wavelet Transform). *If* $X(\Psi, I_n)$ *and* $X(\tilde{\Psi})$ *are dual frames with the multiresolution structure described above and* P *is an admissible oversampling matrix, then the collections* $X(\Psi, P)$ *and* $X(\tilde{\Psi}, P)$ *are dual frames with the following analysis and synthesis relationships: (*$j \in \mathbb{Z},\ k \in \mathbb{Z}^n$*)*

$$\langle f, D^j T_{P^{-1}k} \psi_\ell \rangle = \sqrt{|\det A|} \sum_{r \in \mathbb{Z}^n} \overline{\alpha^P_{\ell;r}} \langle f, D^{j+1} T_{P^{-1}(r+\tilde{A}k)} \varphi \rangle, \qquad (4)$$

$0 \leq \ell \leq L$, *and*

$$\langle f, D^{j+1} T_{P^{-1}k} \varphi \rangle = \sqrt{|\det A|} \sum_{\ell=0}^{L} \sum_{r \in \mathbb{Z}^n} \tilde{\alpha}^P_{\ell;\tilde{A}r+k} \langle f, D^j T_{P^{-1}r} \psi_\ell \rangle, \qquad (5)$$

for each $f \in L^2(\mathbb{R}^n)$, *where* $\tilde{A} := PAP^{-1}$.

Oversampled discrete wavelet transform:

Proposition 3. *Suppose* $P, A \in GL_n(\mathbb{Z})$ *with* $p := |\det P|$. *Let* $\{\theta_r\}_{r=0}^{b-1}$ *be a complete set of distinct coset representatives of* $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ *with* $\theta_0 = 0$. *Suppose* $PAP^{-1} \in GL_n(\mathbb{Z})$. *Then* $\{A\theta_r\}_{r=0}^{b-1}$ *is a complete set of representatives of* $P^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ *if and only if* P *and* A *satisfy* $P^{-1}\mathbb{Z}^n \bigcap M^{-1}\mathbb{Z}^n = \mathbb{Z}^n$.

Remark 1. This proposition plays a role both in the proof of the Second Oversampling Theorem and in the proof of the Oversampled Wavelet Transform. In the latter case, the fact that dilation by A preserves coset representatives of $P^{-1}\mathbb{Z}^n$ leads to an equivalence between the perfect reconstruction equations of the original filters, $m_\ell, \tilde{m}_\ell$, and the oversampled filters, $m_\ell^P, \tilde{m}_\ell^P$.

The dyadic $L^2(\mathbb{R}^2)$ case:

- Dilation: $A = 2I_2$.
- Admissibility of P:
  - ⋆ $PAP^{-1} = A$ automatically has integer entries;
  - ⋆ A sufficient condition for $P^{-1}\mathbb{Z}^n \bigcap A^{-1}\mathbb{Z}^n = \mathbb{Z}^n$ is $|\det P|$ being odd.
- If $|\det P| = 2$ then $P^{-1}\mathbb{Z}^2 \subseteq \frac{1}{2}\mathbb{Z}^2 = A^{-1}\mathbb{Z}^2$, so P cannot be admissible.
- Hence if P is admissible $|\det P| \geq 3$ (measure of redundancy)

Nonseparable QMFs:

Let $m \in L^\infty(\mathbb{T}^n) \bigcap C(\mathbb{T}^n)$. Then m is an *n-dimensional QMF* if and only if $m(0) = 1$ and

$$\sum_{r=0}^{2^n-1} \left| m(\xi + \pi\vartheta_r) \right|^2 = 1 \tag{6}$$

for a.e. $\xi \in \mathbb{T}$, where $\{\vartheta_r\}_{r=0}^{2^n-1}$ is a complete set of coset representatives for $\mathbb{Z}^n/2\mathbb{Z}^n$.

Definition 1 (Ayache [1, 2]). Suppose that $m \in L^\infty(\mathbb{T}^2) \bigcap C(\mathbb{T}^2)$ is a bidimensional QMF, then m is *non-separable* if and only if there does not exist $Q \in \widetilde{SL}_2(\mathbb{Z})$ and univariate QMFs μ and λ such that

$$m(\xi) = \mu(Q\xi \cdot (1,0))\lambda(Q\xi \cdot (0,1)),$$

where $\xi := (\xi_1, \xi_2)$.

Nonseparable QMFs via oversampling:

Theorem 4. *Let m and ℓ be univariate QMFs and let $P \in GL_2(\mathbb{Z})$ with $|\det P|$ odd. The bidimensional filter $\mathcal{M}$ defined by*

$$\mathcal{M}(\xi_1, \xi_2) = m(p_{11}\xi_1 + p_{12}\xi_2)\ell(p_{21}\xi_1 + p_{22}\xi_2),$$

where $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, is a bidimensional QMF. Moreover, if PA is non-diagonal for all $A \in \widetilde{SL}_2(\mathbb{Z})$ then $\mathcal{M}$ is nonseparable in the sense of Definition 1.

Notation: $GL_n(\mathbb{Z})$ is the collection of $n \times n$ matrices with integer entries and nonzero determinant.

Nonseparable QMFs via oversampling:

Lemma 5. $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL_2(\mathbb{Z})$ *can be column-reduced (using only column permutations and transvections) to a matrix of the form* $B = \begin{pmatrix} b_1 & 0 \\ b_2 & b_3 \end{pmatrix}$, *with* $b_1 = \gcd(a_1, a_2) > 0$ *and* $|b_2| < |b_3|$. *In particular, there exists* $U \in \widetilde{SL}_2(\mathbb{Z})$ *such that* $B = AU$.

Proposition 6. *Let* $P \in GL_2(\mathbb{Z})$. *If any column-reduced form of* P *has three non-zero entries, then* PA *is non-diagonal for all* $A \in \widetilde{SL}_2(\mathbb{Z})$.

Examples of nice oversampling matrices:

- $\det P = 3$:

$$P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -3 \end{pmatrix}$$

- another $\det P = 3$:

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$$

- $\det P = 7$:

$$P = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \simeq \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}$$

An example:

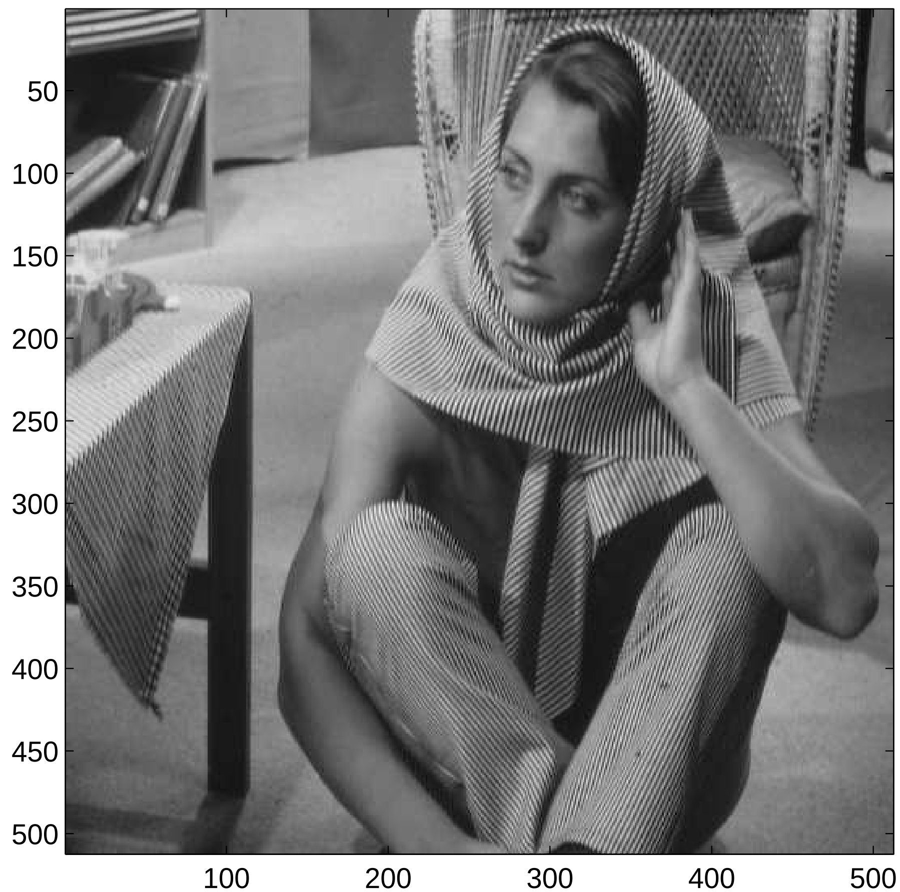
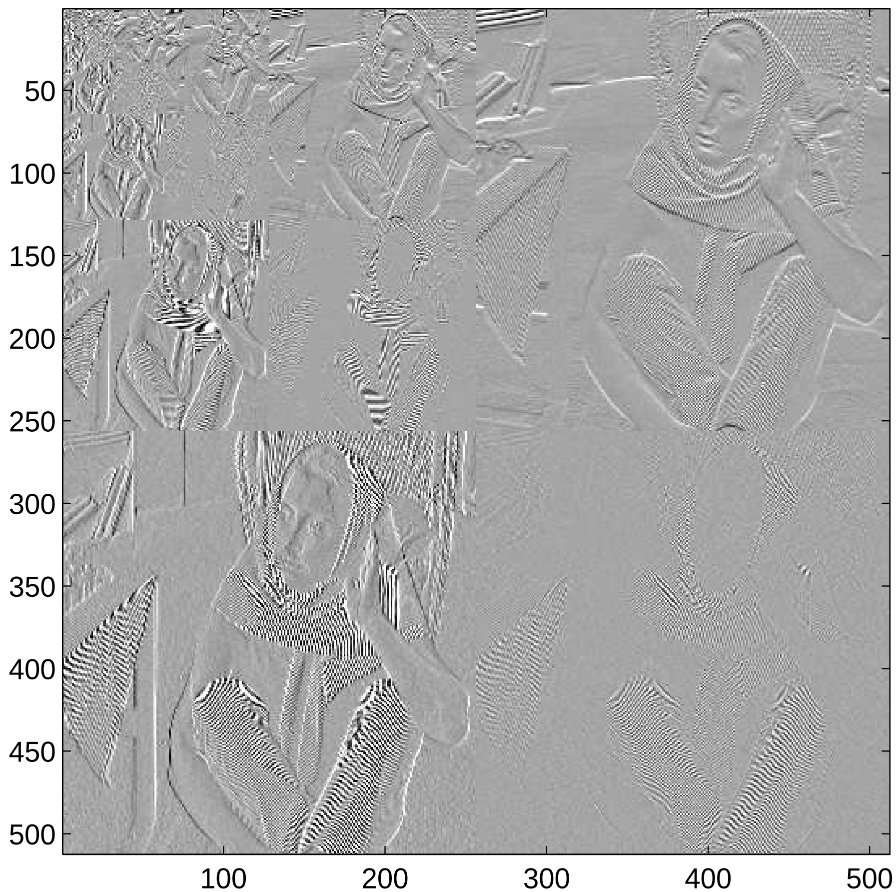
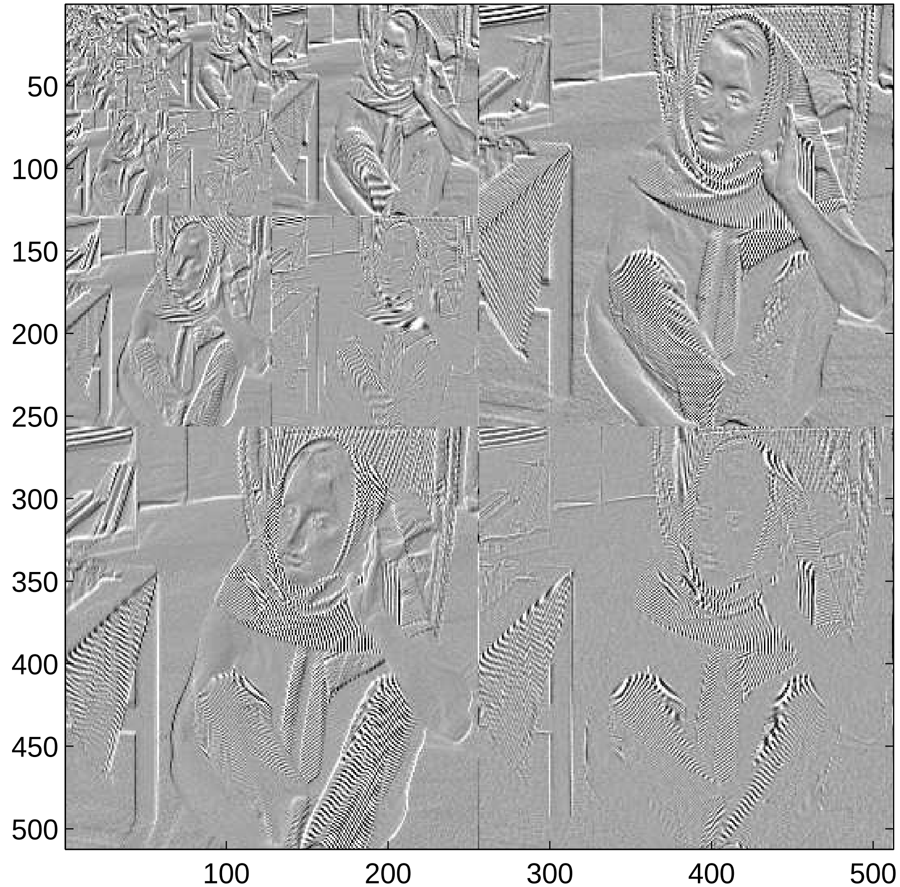


Figure 1: Original Barbara image.



$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Figure 2: Barbara image with Haar filters and no oversampling.



$$P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

Figure 3: Barbara image with Haar filters and oversampling by P.

Another example:

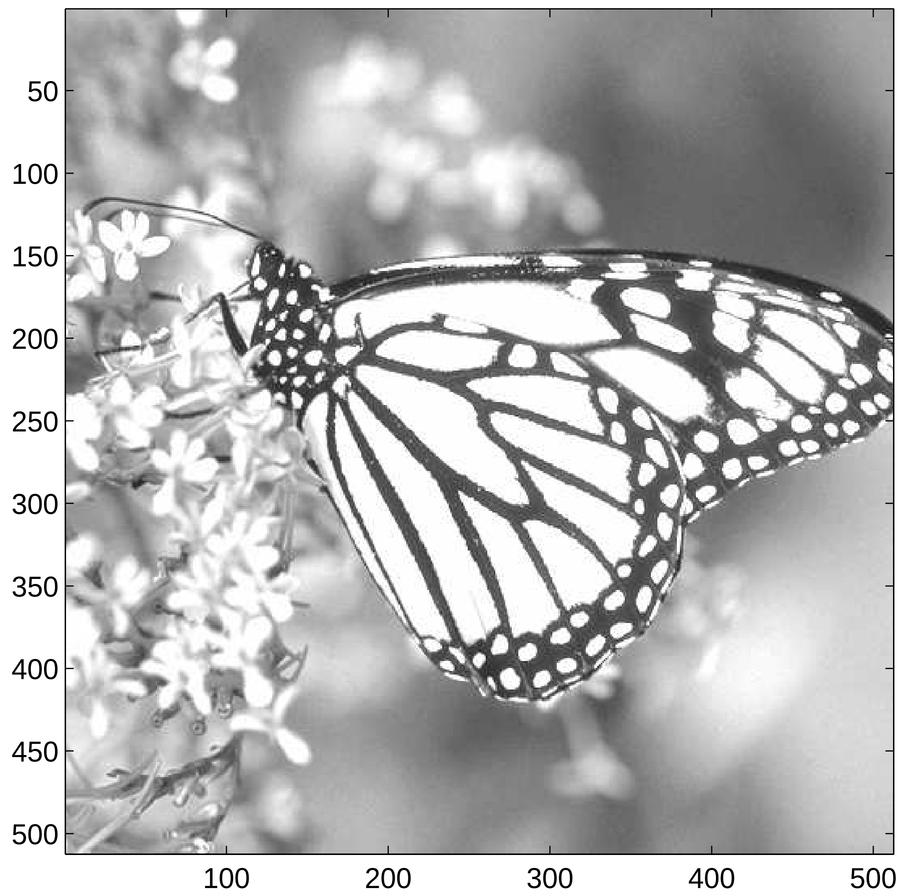
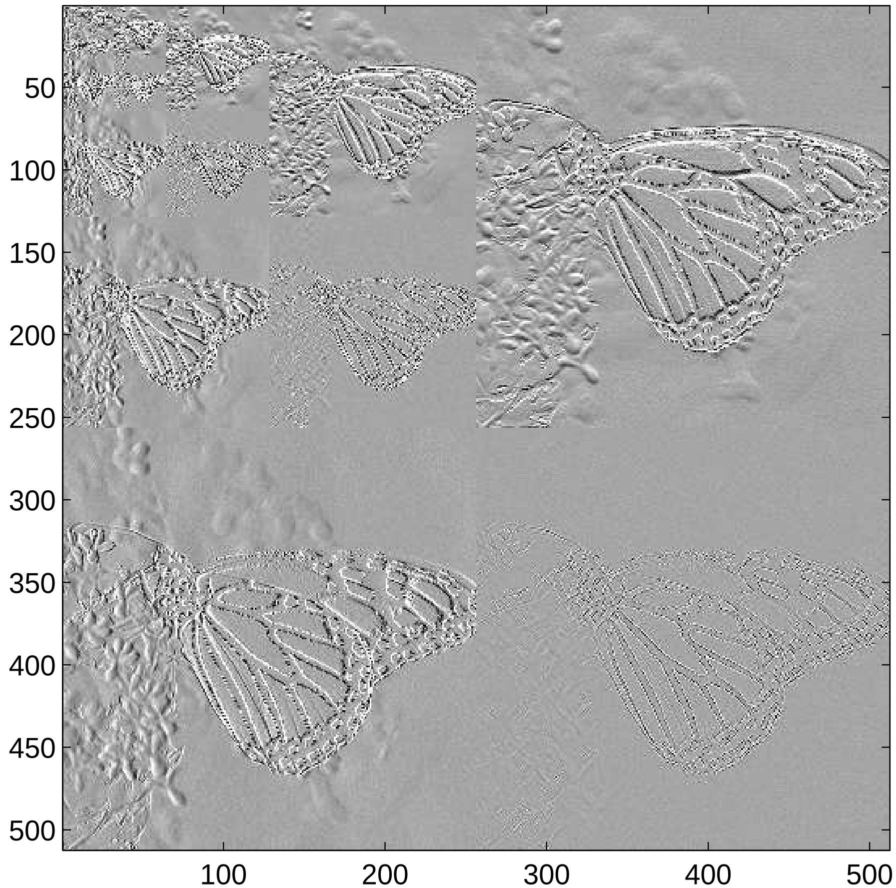
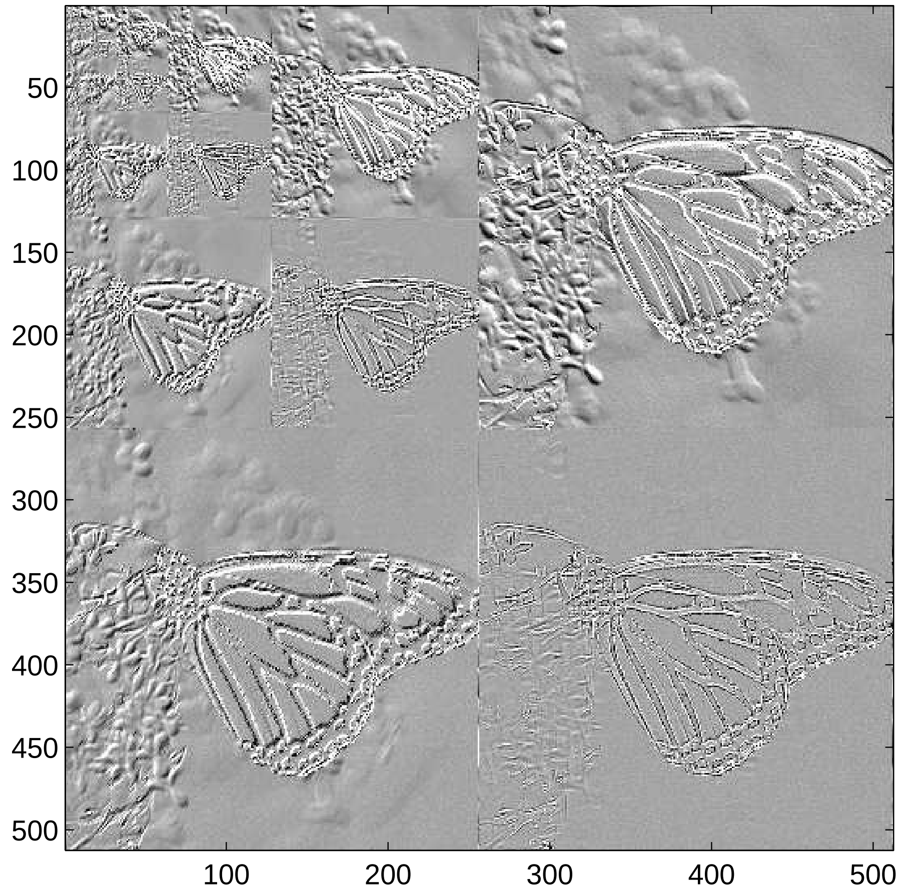


Figure 4: Original Monarch image.



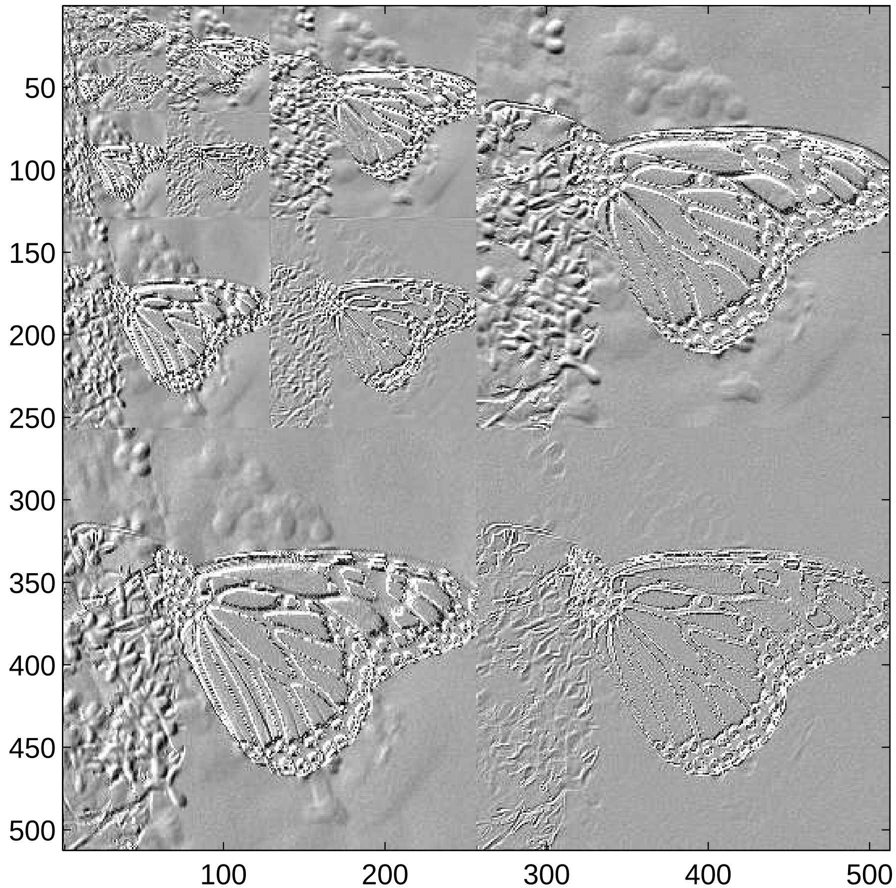
$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Figure 5: Monarch image with Haar filters and no oversampling.



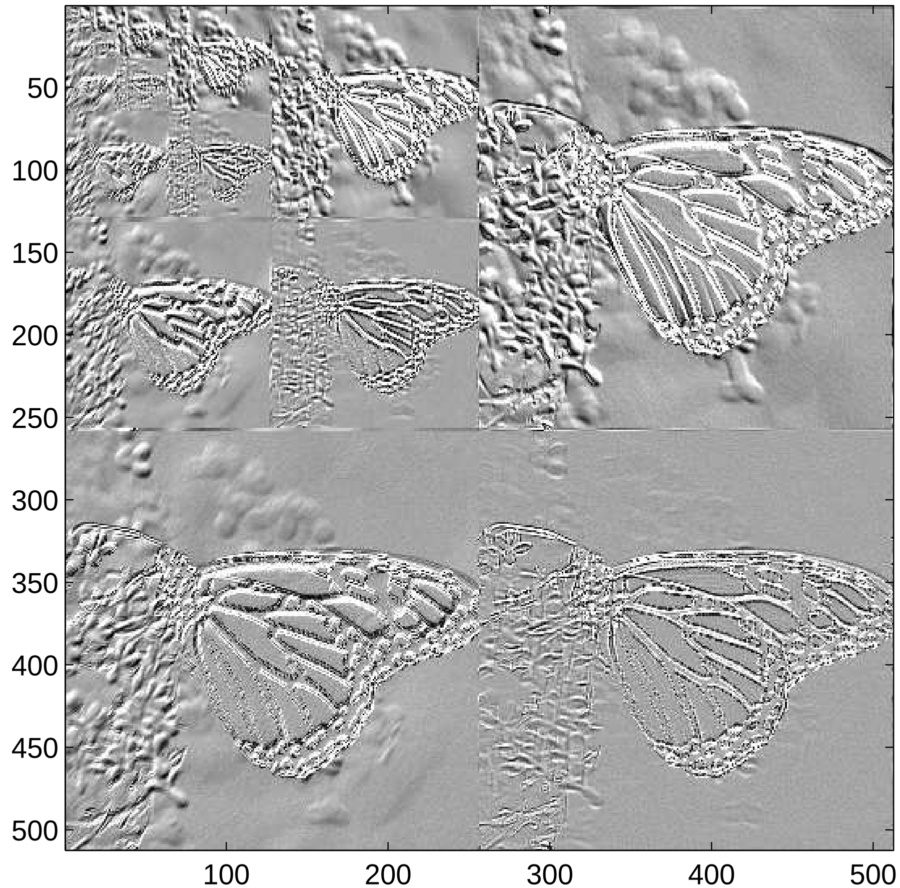
$$P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

Figure 6: Monarch image with Haar filters and oversampling by P.



$$P = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Figure 7: Monarch image with Haar filters and oversampling by P.



$$P = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$$

Figure 8: Monarch image with Haar filters and oversampling by P.

A final example:

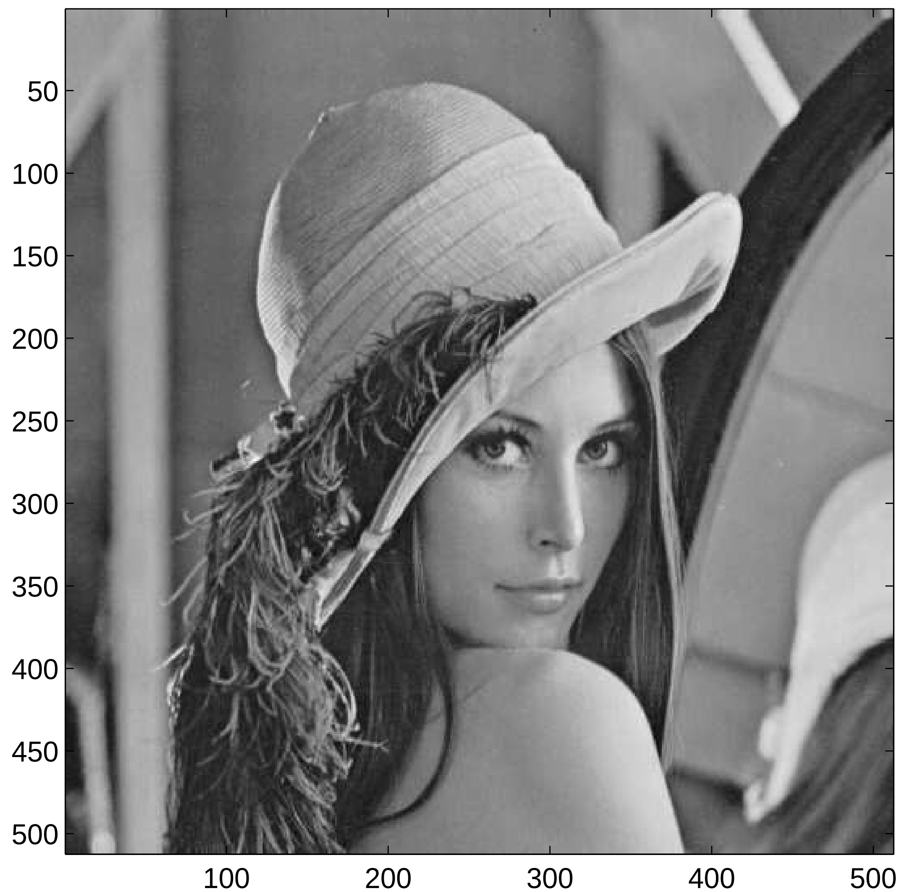
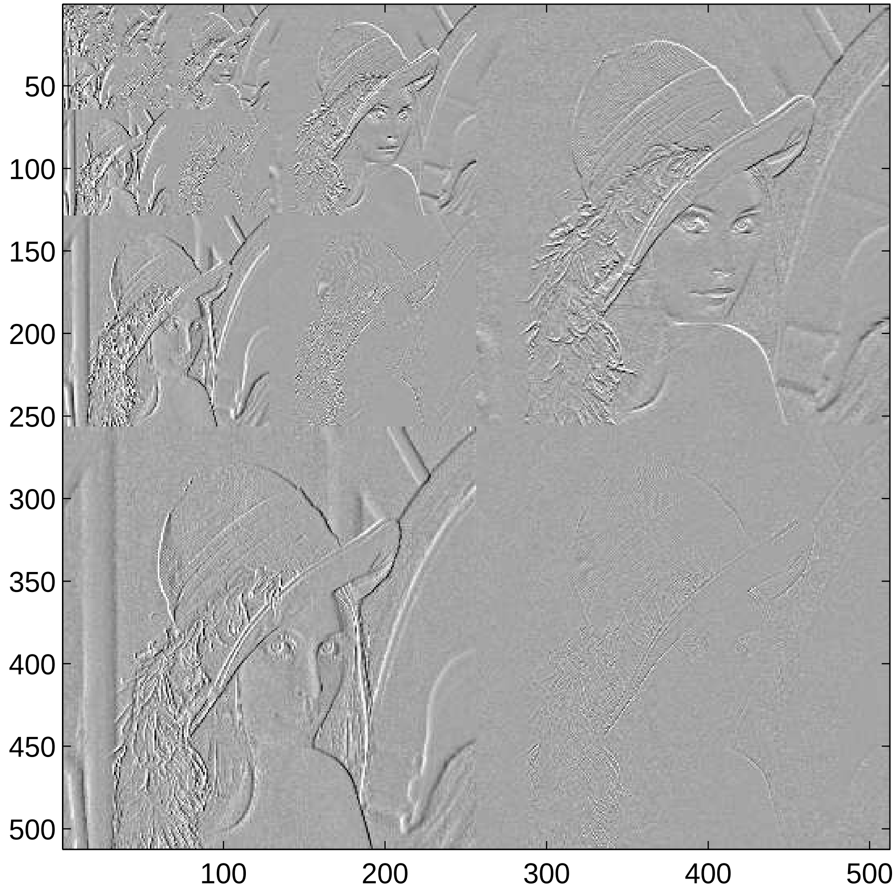
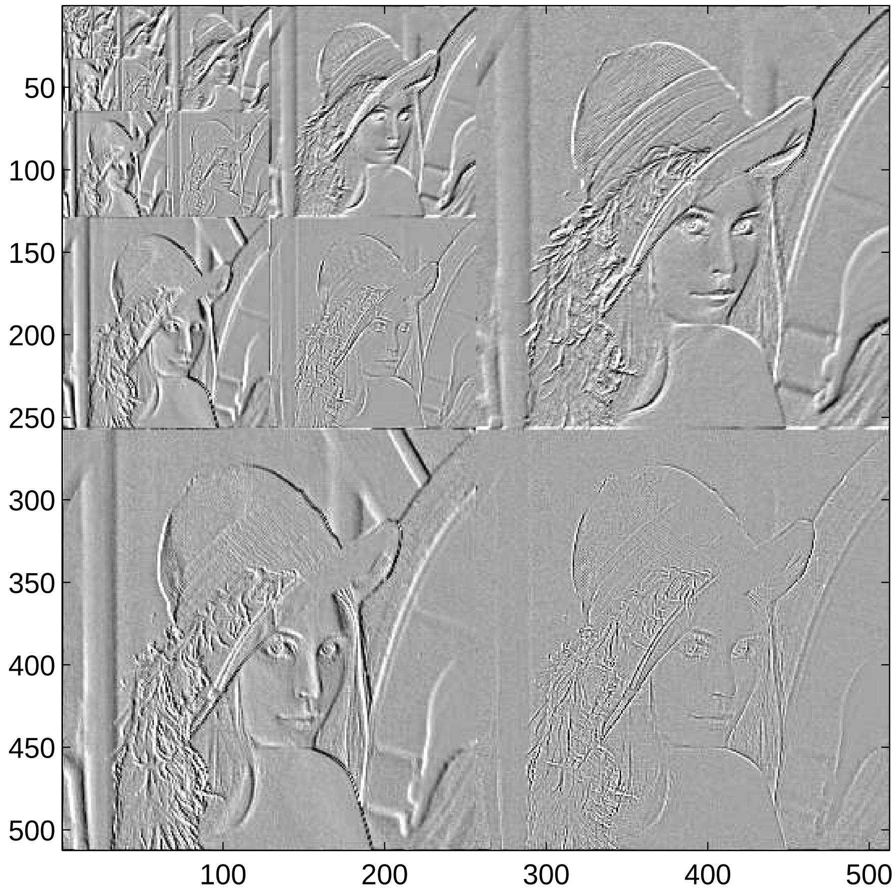


Figure 9: Original Lena Image.



$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Figure 10: Lena image with Haar filters and no oversampling.



$$P = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Figure 11: Lena image with Haar filters and oversampling by P.

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