Stable filtering schemes with rational dilations

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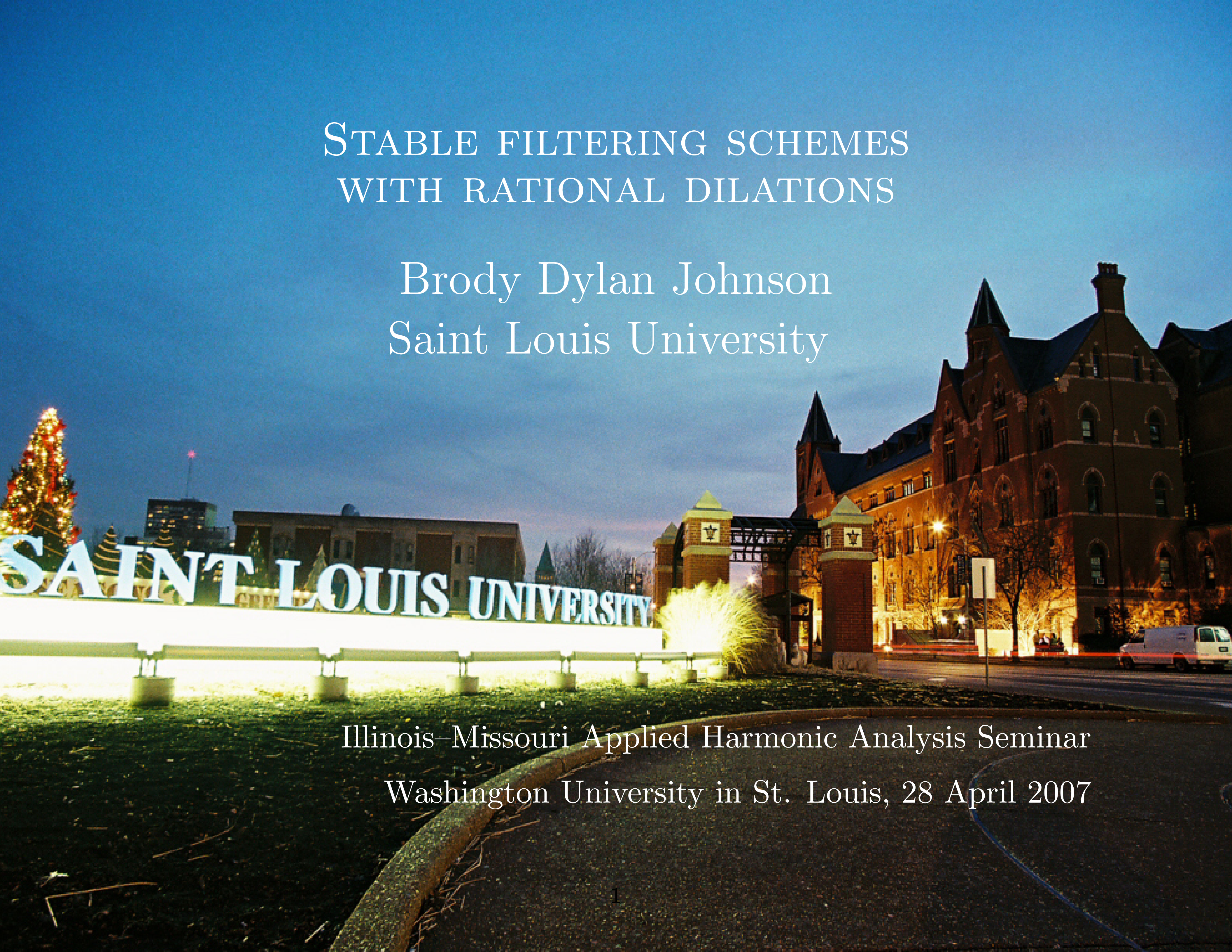
Illinois–Missouri Applied Harmonic Analysis Seminar
Washington University in St. Louis, 28 April 2007



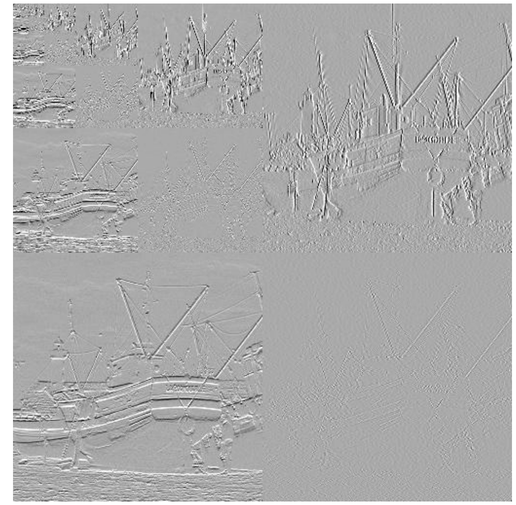
Outline:

- Motivation (4 slides)
- Multiresolution analysis (5 slides)
- Shift-invariant theory
  - Definitions (3 slides)
  - Machinery (2 slides)
- Projective frames for PSI space (5 slides)
- Haar example (2 slides)
- Filtering schemes
  - Correspondence with PSI decompositions (5 slides)
  - Example of a $\frac{3}{2}$-Discrete Wavelet Transform (3 slides)

Dyadic Discrete Wavelet Transform:

Original Boat Image

DWT at Scale 1



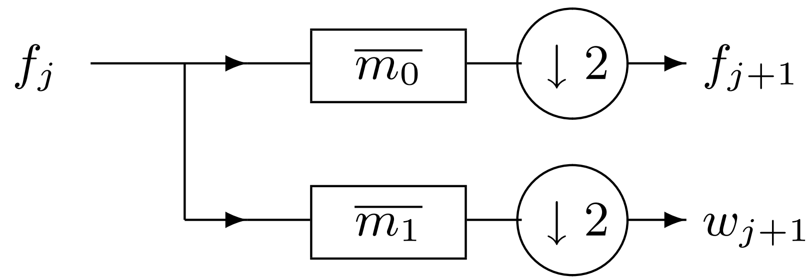
Full DWT

Remarks:

- In practice, the DWT is a filtering scheme implementing digital filters on a discrete, finite signal.
- Mathematically, the DWT may be thought of as a decomposition of a subspace of $L^2(\mathbb{R})$ into complementary components.

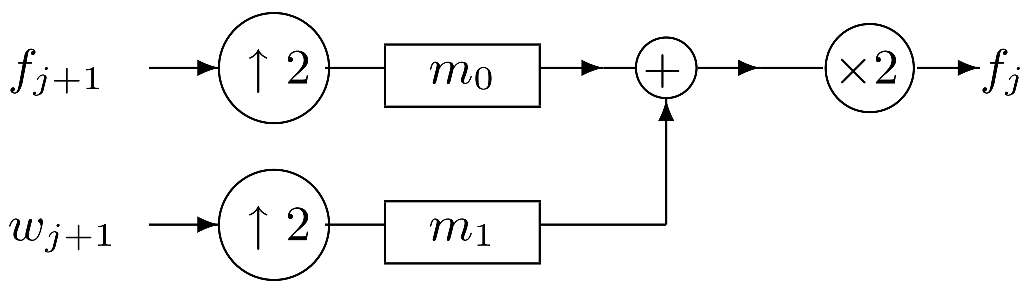
The filtering scheme:

- A sequence of samples f_0 is successively decomposed via low and high pass filters:



- The *wavelet coefficients* comprising the sequences $w_1, \ldots, w_N$ (where N is the number of scales) can then be compressed, analyzed, modified, etc. as desired.

- Ideally, m_0, m_1 will permit reconstruction of the original signal:



The subspace decomposition: (1 of 2)

- The sequence of samples $f_0 = \{f_{0,k}\}_{k\in\mathbb{Z}}$ may be associated to the function

$$f_0(x) = \sum_{k\in\mathbb{Z}} f_{0,k}(T^k\varphi)(x),$$

where φ is a *scaling function* with the property that $\{T^k\varphi\}_{k\in\mathbb{Z}}$ is a Riesz basis for its closed linear span,

$$V_0 = \overline{\text{span}}\,\{T^k\varphi : k \in \mathbb{Z}\}.$$

- The goal in this case is to decompose V_0 as $V_{-1} + W_{-1}$ with

$$\begin{aligned} V_{-1} &= \overline{\text{span}}\,\{D_2^{-1}T^k\varphi : k \in \mathbb{Z}\} \\ \& \quad W_{-1} &= \overline{\text{span}}\,\{D_2^{-1}T^k\psi : k \in \mathbb{Z}\} \end{aligned}$$

for some *wavelet generator* ψ associated with φ.

The subspace decomposition: (2 of 2)

- The function f_0 will be decomposed as

$$
\begin{aligned}
f_1(x) &= \sum_{k \in \mathbb{Z}} f_{1,k} (D_2^{-1} T^k \varphi)(x) \\
\& \quad w_1(x) &= \sum_{k \in \mathbb{Z}} w_{1,k} (D_2^{-1} T^k \varphi)(x).
\end{aligned}
$$

- It is desired that f_0 is recoverable from this decomposition in a stable manner, i.e., that the collections $\{D_2^{-1} T^k \varphi\}_{k \in \mathbb{Z}}$ and $\{D_2^{-1} T^k \psi\}_{k \in \mathbb{Z}}$ together form a frame for V_0.

- Ultimately, one hopes to obtain a decomposition of the form

$$L^2(\mathbb{R}) = \sum_{j \in \mathbb{Z}} W_j,$$

where $\{D_2^j T^k \psi\}_{j,k \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

Question:

What is the link between the filtering scheme and the subspace decomposition?

Answer: Multiresolution Analysis

Multiresolution analysis: (1 of 5)

Recall that a *multiresolution analysis (MRA)* consists of a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ and a scaling function $\varphi \in V_0$ satisfying

$$V_j \subseteq V_{j+1} \text{ for each } j \in \mathbb{Z}; \tag{1}$$

$$f \in V_j \text{ if and only if } D_a^{-j} f \in V_0 \text{ for each } j \in \mathbb{Z}; \tag{2}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}; \tag{3}$$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}); \tag{4}$$

$$\{T^k \varphi\}_{k\in\mathbb{Z}} \text{ is a Riesz basis for } V_0. \tag{5}$$

Here, $a > 1$, $D_a f(x) = \sqrt{a} f(ax)$, and $Tf(x) = f(x-1)$.

Multiresolution analysis: (2 of 5)

Again, fix $a = 2$.

- Since $V_{-1} \subseteq V_0$, it follows that

$$D_2^{-1}\varphi = \sum_{k\in\mathbb{Z}} c_k T^k \varphi,$$

  or $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$ where $m_0(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi}$ is the *low-pass filter.*

- Similarly, $W_{-1} \subseteq V_0$ leads to

$$D_2^{-1}\psi = \sum_{k\in\mathbb{Z}} d_k T^k \varphi,$$

  or $\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi)$ where $m_1(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi}$ is the *high-pass filter.*

Multiresolution analysis: (3 of 5)

Natural lines of inquiry:

- There has been extensive research on MRAs addressing necessary and sufficient conditions for the low-pass filter m_0 to give rise to a bonafide MRA. (generalizations: higher dimensions, integer dilations other than 2)
- The notion of an MRA itself has also been generalized, e.g., biorthogonal wavelets, TFWs, framelets, etc.
- There has been relatively little work on MRAs with rational dilation factors, e.g., Auscher [1], Daubechies [2].

Multiresolution analysis: (4 of 5)

Let $a = \frac{p}{q} > 1$ (in lowest terms) and suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA.

- Then $V_{-1} \subseteq V_0$ implies

$$D_a^{-1}\varphi = \sum_{k\in\mathbb{Z}} c_{0;k} T^k \varphi. \tag{6}$$

- Notice that $D_a^{-1}T^{q\ell} = T^{p\ell}D_a^{-1}$, so (6) yields

$$D_a^{-1}T^{q\ell}\varphi = \sum_{k\in\mathbb{Z}} c_k T^{k+p\ell}\varphi.$$

- Additional masks ($q-1$ in total) are required for other shifts:

$$D_a^{-1}T^m\varphi = \sum_{k\in\mathbb{Z}} c_{m;k} T^k \varphi, \quad 0 \le m \le q-1.$$

This greatly constrains the possible scaling functions, φ.

Multiresolution analysis: (5 of 5)

Theorem 1 (Auscher [1]). *If φ is a scaling function for an orthonormal MRA with dilation $a = \frac{p}{q}$ ($p, q > 1$ relatively prime integers), then φ has neither compact support nor exponential decay at ∞.*

Remark: Ideally the MRA structure should correspond to a rational filtering scheme that permits polynomial filters, such as those studied by Kovačević and Vetterli [3]. (more on this later)

Auscher's result shows that an alternative MRA structure is required if compactly supported scaling functions and, hence, polynomial filters, are to be compatible with rational dilations.

Shift-invariant space (1 of 5):

- Given $\Phi = \{\phi_1, \ldots, \phi_n\} \in L^2(\mathbb{R})$ let

$$X(\Phi; p) = \{T^{pk}\phi_\ell : 1 \leq \ell \leq n, k \in \mathbb{Z}\}.$$

- The $p\mathbb{Z}$ *shift-invariant space* generated by Φ is

$$V(\Phi; p) = \overline{\text{span}}\, X(\Phi; p).$$

- The functions $\phi_1, \ldots, \phi_n$ will be referred to as the *generators* of $V(\Phi; p)$.

- If Φ consists of a single generating function, then $V(\Phi; p)$ is referred to as a principal shift-invariant (PSI) space.

Shift-invariant space (2 of 5):

Definition 1. Fix $p \in \mathbb{N}$. Define the *p-bracket product* of $f, g \in L^2(\mathbb{R})$ by

$$[\hat{f}, \hat{g}]_p(\xi) = \sum_{k \in \mathbb{Z}} \hat{f}(\xi + k/p)\overline{\hat{g}(\xi + k/p)}. \tag{7}$$

When $p = 1$ the subscript p will often be omitted.

Properties:

- $f, g \in L^2(\mathbb{R})$ implies $[\hat{f}, \hat{g}]_p \in L^1(\mathbb{T}_p)$, where $\mathbb{T}_p \simeq [0, \frac{1}{p})$;
- $\left\langle [\hat{f}, \hat{g}]_p, \sqrt{p} e^{2\pi i p k \xi} \right\rangle = \sqrt{p} \langle f, T^{-pk} g \rangle$;
- $f, g \in L^2(\mathbb{R})$ and $[\hat{g}, \hat{g}]_p \in L^\infty(\mathbb{T}_p)$ implies $[\hat{f}, \hat{g}]_p \in L^2(\mathbb{T}_p)$ with

$$\|[\hat{f}, \hat{g}]_p\|_2 \leq \|[\hat{g}, \hat{g}]_p\|_\infty \|f\|_2.$$

- For $\xi \in \mathbb{T}$, $[\widehat{D_p f}, \widehat{D_p g}](\xi) = \frac{1}{p}[\hat{f}, \hat{g}]_p(\xi/p)$.

Shift-invariant space (3 of 5):

More terminology:

- The *p-spectrum of* ϕ_ℓ, $1 \le \ell \le n$ is

$$\sigma_{\phi_\ell;p} = \left\{ \xi \in [0, 1/p] : [\hat{\phi}_\ell, \hat{\phi}_\ell]_p(\xi) \neq 0 \right\},$$

  while the *p-spectrum of* Φ is

$$\sigma_{\Phi;p} = \bigcup_{n=1}^{N} \sigma_{\phi_n;p}.$$

- The p-Gramian matrix of Φ is defined by

$$G_{\Phi;p}(\xi) = \frac{1}{p} \begin{pmatrix} [\hat{\phi}_1, \hat{\phi}_1]_p(\xi) & \cdots & [\hat{\phi}_N, \hat{\phi}_1]_p(\xi) \\ \vdots & \ddots & \vdots \\ [\hat{\phi}_1, \hat{\phi}_N]_p(\xi) & \cdots & [\hat{\phi}_N, \hat{\phi}_N]_p(\xi) \end{pmatrix}.$$

Shift-invariant space (4 of 5):

Theorem 2 (Ron & Shen [4]). *Fix $p \in \mathbb{N}$ and let $\Phi = \{\phi_1, \ldots, \phi_n\} \in L^2(\mathbb{R})$. Then $X(\Phi; p)$ is a frame for $V(\Phi; p)$ if and only if $1/\lambda^+$ and Λ are essentially bounded on $\sigma_{\Phi;p}$. If either condition holds, then*

$$A = \operatorname*{ess\,inf}_{\xi \in \sigma_{\Phi;p}} \lambda^+(\xi) \quad \textit{and} \quad B = \operatorname*{ess\,sup}_{\xi \in \sigma_{\Phi;p}} \Lambda(\xi),$$

respectively, are the lower and upper frame bounds of $X(\Phi; p)$.

Remark:

Here, $\lambda(\xi)$, $\Lambda(\xi)$, and $\lambda^+(\xi)$ are the smallest, largest, and smallest *nonzero* eigenvalues of $G_{\Phi;p}(\xi)$, respectively.

Shift-invariant space (5 of 5):

Question: When can a PSI space $V(\varphi; q)$ be recovered as an FSI space $V(\Phi; p)$?

Lemma 3. *Let $\varphi \in L^2(\mathbb{R})$, fix $n, p, q \in \mathbb{N}$ such that $p = nq$, and let $\Phi = \{\phi_1, \dots \phi_n\} \subseteq V(\varphi; q)$. Suppose that $\sigma_{\varphi;q} = \mathbb{T}_q$, then $V(\Phi; p) = V(\varphi; q)$ if and only if the determinant of*

$$\mathcal{M}(\xi) = \begin{pmatrix} m_1(\xi) & \cdots & m_n(\xi) \\ \vdots & \ddots & \vdots \\ m_1(\xi + \frac{n-1}{p}) & \cdots & m_n(\xi + \frac{n-1}{p}) \end{pmatrix}$$

is nonzero for almost every $\xi \in \mathbb{T}_p$. Here, m_k is the $1/q$-periodic function such that $\hat{\phi}_k(\xi) = m_k(\xi)\hat{\varphi}(\xi)$, $1 \leq k \leq n$.

Frame decompositions of PSI space (1 of 5):

Ingredients:

- Dilation $a = p/q > 1$, where $p, q \in \mathbb{N}$ (lowest terms);
- $\varphi \in L^2(\mathbb{R})$ such that

$$0 < A \leq [\hat{\varphi}, \hat{\varphi}](\xi) \leq B < \infty, \quad a.e.\ \xi \in \mathbb{T}, \tag{8}$$

  i.e., $X(\varphi)$ is a Riesz basis for $V(\varphi)$;

- Denote by S the frame operator of $X(\varphi)$, given by

$$Sf = \sum_{k \in \mathbb{Z}} \langle f, T^k \varphi \rangle T^k \varphi;$$

- Under the Fourier transform,

$$\widehat{Sf} = [\hat{f}, \hat{\varphi}]\, \hat{\varphi}.$$

Frame decompositions of PSI space (2 of 5):

Projective frame:

- In the MRA case, one would decompose $V(\varphi)$ in terms of

  $$\{D_a^{-1}T^k\varphi : k \in \mathbb{Z}\}$$

  and a corresponding "wavelet" component.

- Without the nestedness property of an MRA, there is no guarantee that any of the functions in this collection even belong to $V(\varphi)$! (φ is not even assumed to be refinable.)

- This obstacle can be overcome by mapping each function into $V(\varphi)$ via the frame operator S. (S preserves compact support, while the orthogonal projection may not.)

Frame decompositions of PSI space (3 of 5):

Low-pass component:

- Define $\Phi_0 = \{\phi_0, \ldots, \phi_{q-1}\}$ where

$$\phi_\ell = D_a^{-1} T^\ell \varphi, \quad 0 \le \ell \le q-1,$$

  and notice

$$\bigcup_{\ell=0}^{q-1} X(D_a^{-1} T^\ell \varphi; p) = \{D_a^{-1} T^k \varphi : k \in \mathbb{Z}\};$$

- Under the action of S (S commutes with T) this generates a natural subspace of $V(\varphi)$:

$$V(S\Phi_0; p) \subseteq V(\varphi).$$

- $V(\varphi)$ is the $p\mathbb{Z}$ shift-invariant space generated by $\{T^r \varphi : 0 \le r \le p-1\}$, i.e., $V(S\Phi_0; p)$ is $p - q$ generators short.

Frame decompositions of PSI space (4 of 5):

High-pass component:

- Let $\Phi_1 = \{D_a^{-1}\psi_1, \ldots, D_a^{-1}\psi_{p-q}\}$ be a given collection of potential wavelet generators under the convention that

$$\phi_\ell = D_a^{-1}\psi_{\ell-q+1}, \quad q \leq \ell \leq p-1. \tag{9}$$

- The goal is to characterize when these generators fill out Φ_0 to provide a set of spanning generators for $V(\varphi)$.

- Let $\Phi = \Phi_0 \cup \Phi_1$. The next theorem describes conditions under which $X(S\Phi; p)$ is a frame for $V(\varphi)$.

Frame decompositions of PSI space (5 of 5):

Theorem 4. *Let* $\varphi \in L^2(\mathbb{R})$ *such that* (8) *holds and let* Φ *as above. Moreover, for* $0 \le \ell \le p-1$, *let* m_ℓ *be the* 1*-periodic function* $[\hat{\phi}_\ell, \hat{\varphi}]$ *so that* $\widehat{S\phi_\ell} = m_\ell \hat{\varphi}$. *Define* $\mathcal{M}(\xi)$ *by*

$$\mathcal{M}(\xi) = \frac{1}{\sqrt{p}} \begin{pmatrix} m_0(\xi) & \cdots & m_{p-1}(\xi) \\ \vdots & \ddots & \vdots \\ m_0(\xi + \frac{p-1}{p}) & \cdots & m_{p-1}(\xi + \frac{p-1}{p}) \end{pmatrix} \tag{10}$$

and let $\lambda_{\mathcal{M}}$ *and* $\Lambda_{\mathcal{M}}$ *be the smallest and largest eigenvalue functions of* $\mathcal{M}^*\mathcal{M}$ *over* $\mathbb{T}_p$. *Then* $X(S\Phi; p)$ *is a frame for* $V(\varphi)$ *if and only if* $1/\lambda_{\mathcal{M}}$ *and* $\Lambda_{\mathcal{M}}$ *are essentially bounded on* $\mathbb{T}_p$. *If either condition holds, let*

$$\lambda_A = \operatorname*{ess\,inf}_{\xi \in \sigma_{\Phi;p}} \lambda_{\mathcal{M}}(\xi) \quad \textit{and} \quad \lambda_B = \operatorname*{ess\,sup}_{\xi \in \sigma_{\Phi;p}} \Lambda_{\mathcal{M}}(\xi),$$

then $X(S\Phi; p)$ *is a frame for* $V(\varphi)$ *with lower and upper bounds* $\lambda_A A$ *and* $\lambda_B B$, *respectively.*

Haar example (1 of 2):

- Fix $a = \frac{3}{2}$ and let $\varphi = \chi_{[0,1)}$;
- Observe that

$$D_a^{-1}\varphi = \sqrt{\frac{2}{3}}\chi_{[0,\frac{3}{2})} \quad \text{and} \quad D_a^{-1}T\varphi = \sqrt{\frac{2}{3}}\chi_{[\frac{3}{2},3)}$$

and let $\phi_0 = D_a^{-1}\varphi$ and $\phi_1 = D_a^{-1}T\varphi$.

- This leads to the filters

$$m_0(\xi) = [\hat{\phi}_0, \hat{\varphi}](\xi) = \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{6}}e^{-2\pi i\xi},$$

$$m_1(\xi) = [\hat{\phi}_1, \hat{\varphi}](\xi) = \sqrt{\frac{1}{6}}e^{-2\pi i\xi} + \sqrt{\frac{2}{3}}e^{-2\pi i2\xi}.$$

Haar example (2 of 2):

- Choose $\hat{\phi}_2(\xi) = \widehat{D^{-1}\psi}(\xi) = m_2(\xi)\hat{\varphi}(\xi)$, where

$$m_2(\xi) \quad = \quad -\sqrt{1/6} + \sqrt{2/3}e^{-2\pi i\xi} - \sqrt{1/6}e^{-2\pi i 2\xi}.$$

  It is easy to see that $m_2(\xi) = [\hat{\phi}_2, \hat{\varphi}]$.

- Let $\Phi = \{\phi_0, \phi_1, \phi_2\}$ then it can be verified directly that $X(S\Phi; 3)$ is a frame for $V(\varphi)$. If $f = \sum_{k\in\mathbb{Z}} f_k T^k \varphi$ then

$$\sum_{\ell=0}^{2} \sum_{k\in\mathbb{Z}} |\langle f, T^{3k} S\phi_\ell \rangle|^2 = \sum_{k\in 3\mathbb{Z}} \frac{5}{6} f_k^2 + f_{k+1}^2 + \frac{5}{6} f_{k+2}^2 + \frac{1}{3} f_k \overline{f_{k+2}}.$$

  It follows that $X(S\Phi; 3)$ is a frame for $V(\varphi)$ with lower bound $\frac{2}{3}$ and upper bound 1 since

$$\frac{2}{3}\|f\|^2 \leq \sum_{\ell=0}^{2} \sum_{k\in\mathbb{Z}} |\langle f, T^{3k} S\phi_\ell \rangle|^2 \leq \|f\|^2.$$

QUESTION:

Given a frame decomposition as in Theorem 4 is there a corresponding filtering scheme?

ANSWER: Yes

Rational filtering schemes (1 of 5):

1. Let $f \in V(\varphi)$, i.e.,
$$f = \sum_{k\in\mathbb{Z}} f_k T^k \varphi.$$

2. Apply the frame operator of $X(\Phi_0; p)$ to f,
$$g_0 := S_{X(\Phi_0;p)} f = \sum_{\ell=0}^{q-1} \sum_{k\in\mathbb{Z}} \left[\sum_{m\in\mathbb{Z}} f_m \langle \varphi, T^{pk-m} \phi_\ell \rangle \right] T^{pk} \phi_\ell.$$

3. In $\ell^2(\mathbb{Z})$ this is equivalent to (recall: $\phi_\ell = D^{-1} T^\ell \varphi$)
$$\{g_{0,k}\}_{k\in\mathbb{Z}} = \sum_{\ell=0}^{q-1} T^\ell \uparrow_q \downarrow_p (\{f_m\} * \{\langle \varphi, T^m \phi_\ell \rangle\}),$$
where $g_0 = \sum_{k\in\mathbb{Z}} g_{0,k} D^{-1} T^k \varphi$.

Rational filtering schemes (2 of 5):

4. Apply the frame operator of $X(D^{-1}\psi_\ell; p)$ to f, $1 \le \ell \le p-q$,

$$g_\ell := S_{\psi_\ell;p} f = \sum_{k\in\mathbb{Z}} \left[\sum_{m\in\mathbb{Z}} f_m \langle \varphi, T^{pk-m} D^{-1}\psi_\ell\rangle \right] D^{-1} T^{qk} \psi_\ell$$

5. In $\ell^2(\mathbb{Z})$ this yields

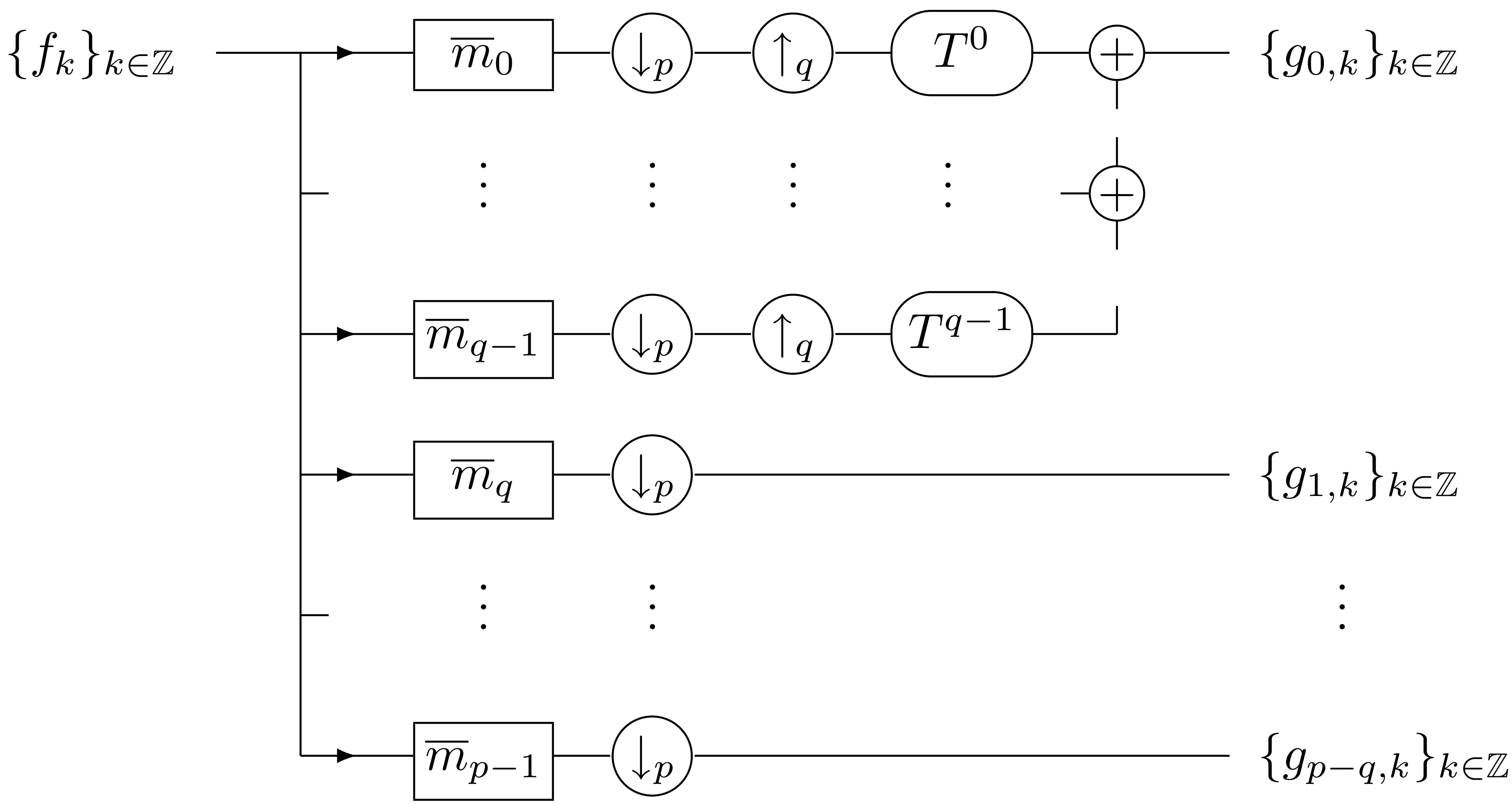
$$\{g_{\ell,k}\}_{k\in\mathbb{Z}} = \downarrow_p (\{f_m\} * \{\langle \varphi, T^m \phi_{\ell+q}\rangle\}), \quad 1 \le \ell \le p-q,$$

   where $g_\ell = \sum_{k\in\mathbb{Z}} g_{\ell,k} T^{qk} D^{-1}\psi$.

6. The action of the frame operators for $X(\Phi_0; p)$ and $X(D^{-1}\psi_\ell; p)$, $1 \le \ell \le p-q$, on $f \in V(\varphi)$ is described by a subband filtering scheme using the filters $m_0, \ldots, m_{p-1}$ of Theorem 4.

Rational filtering schemes (3 of 5):

Analysis Stage:



Rational filtering schemes (4 of 5):

- Define the *filterbank analysis operator* by

$$F : \ell^2(\mathbb{Z}) \to \bigoplus_{\ell=0}^{p-q} \ell^2(\mathbb{Z})$$
$$\{f_k\}_k \mapsto \oplus_{\ell=0}^{p-q} \{g_{\ell,k}\}_{k\in\mathbb{Z}},$$

  which follows the above notation.

- The filtering scheme will be referred to as *stable* if there exist constants $0 < A \leq B < \infty$ such that for any $\{f_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$,

$$A \left\|\{f_k\}_k\right\|^2 \leq \left\|F\{f_k\}_k\right\|^2 \leq B \left\|\{f_k\}_k\right\|^2 . \tag{11}$$

- Stability can be reformulated as a frame identity for a specific system of translates in $\ell^2(\mathbb{Z})$ and thus guarantees reconstruction after the analysis stage.

Rational filtering schemes (5 of 5):

Theorem 4 guarantees stability of the associated filtering scheme.

Theorem 5. *Let Φ, $\mathcal{M}$, $\lambda_{\mathcal{M}}$, and $\Lambda_{\mathcal{M}}$ be as in Theorem 4. Then the induced filtering scheme is stable if and only if $1/\lambda_{\mathcal{M}}$ and $\Lambda_{\mathcal{M}}$ are essentially bounded on $\mathbb{T}_p$. If either condition holds, let*

$$\lambda_A = \operatorname*{ess\,inf}_{\xi \in \sigma_{\Phi;p}} \lambda_{\mathcal{M}}(\xi) \quad \textit{and} \quad \lambda_B = \operatorname*{ess\,sup}_{\xi \in \sigma_{\Phi;p}} \Lambda_{\mathcal{M}}(\xi),$$

then the filterbank analysis operator satisfies

$$\lambda_A \left\|\{f_k\}_k\right\|^2 \leq \left\|F\{f_k\}_k\right\|^2 \leq \lambda_B \left\|\{f_k\}_k\right\|^2, \quad \forall \, \{f_k\}_k \in \ell^2(\mathbb{Z}).$$

Discrete wavelet transform (1 of 3):

Haar $\frac{3}{2}$-filters.

Discrete wavelet transform (2 of 3):

Haar $\frac{3}{2}$-filters.

Discrete wavelet transform (3 of 3):

Haar $\frac{3}{2}$-filters.

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