2. If \( n = 2 \) then clearly \( A_2 \) is the only subgroup of index 2 in \( S_2 \). Assume \( n > 2 \). Then \( S_n \) contains a 3-cycle \( \sigma \). Let \( H \) be a subgroup of \( S_n \) of index 2. Suppose \( \sigma \notin H \). Clearly then \( \sigma^{-1} \notin H \). Now \( \sigma H = \sigma^{-1} H = \sigma^2 H = (\sigma H)^2 = H \), a contradiction. Thus \( H \) contains each 3-cycle \( \sigma \in S_n \). Therefore, \( H \) generates \( A_n \). But as both \( H \) and \( A_n \) have index 2 in \( S_n \), \( |A_n| = |H| \). This yields \( H = A_n \).

4. There are many ways to show that the group \( A_4 \) has no subgroup of order 6. (Note that this example shows that the converse of Lagrange’s Theorem is not true, in general).

Proof 1: First observe that if \( G \) is a finite group and \( N \) is a subgroup of \( G \) such that \( [G : N] = 2 \), then each element of odd order in \( G \) must lie in \( N \). If \( A_4 \) has a subgroup with index 2 then all elements of \( A_4 \) with odd order will be in that subgroup. But \( A_4 \) contains 8 elements of order 3 (there are 8 different 3-cycles), and so not all elements of odd order can lie in the subgroup of order 6. Therefore, \( A_4 \) has no subgroup of order 6.

Proof 2. Let \( H \) be a subgroup of \( A_4 \) of order 6. Then \( H \) is isomorphic to \( \mathbb{Z}_6 \) or \( S_3 \). Since \( A_4 \) has no element of order 6, \( H \) can’t be isomorphic to \( \mathbb{Z}_6 \). In \( S_3 \) there are three elements of order 2. The group \( A_4 \) has three elements of order 2, \((1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)\). So these must lie in \( H \). But then these three elements of order 2 together with identity element will form a subgroup of \( H \) of order 4. But a group of order 6 can’t have a subgroup of order 4. Thus such a subgroup \( H \) doesn’t exist.