Solution 5

A. Let $G$ be a group of order $p^n$ where $p$ is a prime and $n \geq 2$. Then by Sylow’s First Theorem, $G$ has a normal subgroup of order $p^{n-1}$. Hence $G$ is not simple.

B. Let $G$ be a group of order $pq$ where $p, q$ are primes and $p > q$. Then $G$ has a Sylow $p$-subgroup. The number of Sylow $p$-subgroups, $n_p = 1 + kp$ and $n_p$ divides $pq$. This gives, $k = 0$ and hence $G$ has a unique Sylow $p$-subgroup $P$. Therefore, $P$ must be normal in $G$. Hence $G$ is not simple.

C. Let $G$ be a group of order $p^2q$ where $p, q$ are primes. We claim that $G$ has either a unique Sylow $p$-subgroup or a unique Sylow $q$-subgroup. Let $n_p$ and $n_q$ be the number of Sylow $p$-subgroups and Sylow $q$-subgroups, respectively. Suppose, if possible, that $n_p > 1$ and $n_q > 1$. We have $n_p$ divides $q$ and hence $n_p = q$. As $n_p = 1 + kp$, $q > p$. Also, as $n_q$ divides $p^2$, we have $n_q = p$ or $p^2$. Let $Q_1$ and $Q_2$ be two distinct Sylow $q$-subgroups. Then $Q_1 \cap Q_2 = \langle 1 \rangle$ by Lagrange’s Theorem. Hence, there are $n_q(q - 1)$ distinct elements of order $q$ in $G$. If $n_q = p^2$, then $p^2 = p^2(q - 1) = p^2$ elements are left to form a unique Sylow $p$-subgroup, a contradiction to the assumption that $n_p > 1$. Hence, $n_q = p$. This gives $p > q$, again a contradiction. Hence, $G$ has either a unique Sylow $p$-subgroup or a unique Sylow $q$-subgroup.

D. Let $G$ be a group of order $pqr$ where $p, q, r$ are primes and $p > q > r$. Let $n_p$, $n_q$ and $n_r$ be the number of Sylow $p$-subgroups, Sylow $q$-subgroups, and Sylow $r$-subgroups, respectively. We claim that $G$ has either a unique Sylow $p$-subgroup or a unique Sylow $q$-subgroup or a unique Sylow $r$-subgroup. Suppose, if possible, that $n_p > 1$, $n_q > 1$ and $n_r > 1$. Since any two distinct Sylow $p$-subgroups of $G$ intersect trivially by Lagrange’s Theorem, $G$ contains $n_p(p - 1)$ distinct elements of order $p$. Similarly, $G$ contains $n_q(q - 1)$ distinct elements of order $q$ and $n_r(r - 1)$ distinct elements of order $r$. Therefore, $pqr \geq 1 + n_p(p - 1) + n_q(q - 1) + n_r(r - 1)$. Clearly, as $n_p$ divides $qr$ and $n_p > 1$, we have $n_p = qr$. Similarly, note that $n_q \geq p$ and $n_r \geq q$. So, we have $pqr \geq 1 + qr(p - 1) + p(q - 1) + q(r - 1)$. This gives, $(p - 1)(q - 1) \leq 0$, a contradiction.

From (A), (C), and (D) above it follows that if $G$ is a group of order $pqr$ where $p, q, r$ are primes then $G$ is not simple.

E. If $G$ is a simple group of order 60, then $G \cong A_5$.

Proof. The number of Sylow $p$-subgroups $n_2 = 3, 5$ or 15. Let $P$ be a Sylow 2-subgroup of order 4 and let $N = N_G(P)$. First, observe that $G$ has no proper subgroup $H$ of index less than 5. If $H$ were a subgroup of $G$ of index 4, 3, or 2 then $G$ would have a normal subgroup $K$ contained in $H$ with $G/K$ isomorphic to a subgroup of $S_4$, $S_3$ or $S_2$. Since $K \neq G$ and $G$ is simple, $K = \langle e \rangle$. But this is impossible as $|G| = 60$. This argument shows that $n_2 \neq 3$. If $n_2 = 5$, then $N$ has index 5 in $G$ and so the action of $G$ on the set of left cosets of $N$ in $G$ induces a homomorphism from $G$ to $S_5$. Since $G$ is simple, the kernel is trivial and hence $G$ is isomorphic to a subgroup of $S_5$. Identify $G$ with this isomorphic copy so that we may assume $G \leq S_5$. If $G$ is not contained in $A_5$, then $S_5 = GA_5$ and by Second
Isomorphism Theorem, $A_5 \cap G$ is of index 2 in $G$, a contradiction as $G$ is simple. Hence, $G \leq A_5$. As $|G| = |A_5|$, the isomorphic copy of $G$ in $S_5$ coincides with $A_5$. Finally, assume $n_2 = 15$. If for every pair of distinct Sylow 2-subgroups $P$ and $Q$ of $G$, $P \cap Q = 1$, then the number of nonidentity elements in Sylow 2-subgroups of $G$ would be 45. But as $n_5 = 6$, the number of elements of order 5 in $G$ is 24. Since $45 + 24 = 69 > |G|$, we have a contradiction. This proves that there exist distinct Sylow 2-subgroups $P$ and $Q$ with $|P \cap Q| = 2$. Let $M = N_G(P \cap Q)$. Since $P$ and $Q$ are abelian (being of order 4), $P$ and $Q$ are subgroups of $M$ and since $G$ is simple, $M \neq G$. Thus 4 divides $|M|$ and $|M| > 4$ (otherwise $P = Q = M$). The only possibility is $|M| = 12$, i.e. $M$ has index 5 in $G$ (recall that $M$ cannot have index 3 or 1). The above argument applied now to $M$ in place of $N$ gives $G \cong A_5$. 