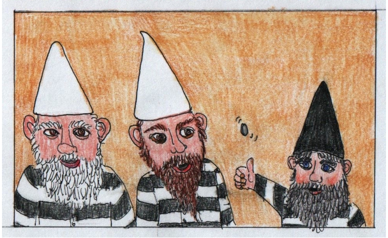
The Hat Problem

Brody Dylan Johnson
Saint Louis University



The Hat Problem[1]

A group of prisoners is allowed to play a game for their freedom. The prisoners are donned with either a black hat or a white hat and, while they cannot see their own hat, they can see the remaining hats. The two colors are equally likely. The prisoners play as a team and win when at least one prisoner guesses the color of his or her own hat without any incorrect guesses being made. The prisoners may strategize before the game, but cannot communicate in any fashion once the game begins.

[1] Todd Ebert, University of California, Santa Barbara (1998).

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- The prisoners WIN when at least one correct guess is made and no incorrect guesses are made.
- A prisoner sees all of the hats, EXCEPT his or her own.
- No communication is allowed between prisoners, including whether or not others have elected to GUESS or PASS.

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  - A single prisoner guessing leads to a $\frac{1}{2}$ probability of victory.

Is it clear that this is the optimal strategy for $n = 2$?

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- Each possible outcome can be expressed as a 3-digit binary number:

$$\underbrace{\begin{matrix} 010 & 100 & 101 \\ 001 & 110 & 011 \end{matrix}}_{\text{WIN}} \quad \underbrace{\begin{matrix} 111 \\ 000 \end{matrix}}_{\text{LOSE}}$$

Thus, the probability of winning with this strategy is $\frac{3}{4}$.

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3. Each WIN results from a single correct guess, while each LOSS stems from *three* incorrect guesses.
4. Winning one more game requires one more correct guess (a total of 7), but now we have only 1 possible game to lose and 7 incorrect guesses to make. There are only 3 prisoners so this is impossible.

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- It is possible to have X wins among the 2^N possible games provided that the losses allow room for X incorrect guesses, i.e.,

$$2^N - X \geq \frac{X}{N} \quad \text{or} \quad X \leq \frac{N \cdot 2^N}{N+1}.$$

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$$2^N - X \geq \frac{X}{N} \quad \text{or} \quad X \leq \frac{N \cdot 2^N}{N+1}.$$

- This argument gives an upper bound on the probability of victory:

$$P(\text{Win}) \leq \frac{N}{N+1}.$$

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2. If a strategy leads to 13 wins in these 16 outcomes, then at least 13 incorrect guesses must be made in the 3 losses.

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2. If a strategy leads to 13 wins in these 16 outcomes, then at least 13 incorrect guesses must be made in the 3 losses.
3. Four prisoners can make only 12 guesses in 3 games, leading to a contradiction.

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Five Prisoners: A Simple Approach

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- How do we know when the team wins?

Five Prisoners: Too Simple?

- Assume that the first three hats are all white. Each of the other two prisoners will guess that their hat is black, so the only time they win is when both of those hats ARE black. In other words, they win when exactly three hats have the same color.

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- There are $\binom{5}{3} = 10$ outcomes where three hats are white and an equal number where three hats are black. There are $2^5 = 32$ possible outcomes so the probability of winning with this strategy is $20/32 = \frac{5}{8}$.
- This is worse than the $n = 3$ strategy $\frac{3}{4}$ and way worse than the upper bound of $\frac{5}{6}$. This is because the right/wrong guesses are not distributed effectively.

What Do We Know So Far?

- The following table summarizes what we know for $1 \leq N \leq 7$:

N	1	2	3	4	5	6	7
$P(\text{WIN})$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	?	?	?
$N/(N+1)$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$

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- The theoretical maximum is achieved for $N = 1, 3$, but for $N = 2, 4$ there is no way to achieve the theoretical maximum.
- It is natural to wonder, then, for which N can one achieve the theoretical maximum? If not, what is the best one can do?

Vector Spaces over a Finite Field

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- It turns out that one can obtain strategies for the N-prisoner Hat Problem by examining subspaces of N-dimensional vector spaces over $\mathbb{F}_2$.
- Recall that the term *subspace* refers to a collection of vectors which is closed under addition and scalar multiplication.

Vector Spaces over $\mathbb{F}_2$

- Consider the field $\mathbb{F}_2 = \{0, 1\}$ with the addition/multiplication tables:

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- $\mathbb{F}_2^n$ is an n-dimensional vector space over $\mathbb{F}_2$. We may write vectors in $\mathbb{F}_2^n$ as n-digit binary numbers, but must be careful to add digit-wise. Example:

$$101 + 001 = 100 \qquad 101 + 001 \neq 110$$

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  - If the errors are small, there will only be one code vector which is *close* to the given vector, allowing for recovery of the data.

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Proof.

Each bit in which $\vec{c}_3$ differs from $\vec{c}_1$ falls into one of two groups: (a) $\vec{c}_3(k) = \vec{c}_2(k)$ which implies $\vec{c}_2(k) \neq \vec{c}_1(k)$ and (b) $\vec{c}_3(k) \neq \vec{c}_2(k)$. □

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  - If the two bits visible do not make a codeword, the prisoner does not guess.

(7,4) Hamming code:

- Let $n = 7$ and consider the code $\mathcal{C}$ corresponding to the subspace generated by $\{1000110, 0100101, 0010011, 0001111\}$. (Every codeword is a linear combination of these four vectors.)

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- Our message will be the four coefficients used to construct a codeword as a linear combination. Let $\vec{a} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$. (4-digit binary message) We will transmit the codeword

$$\vec{c}_{\vec{a}} = \vec{a}G = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

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- There is another subspace of $\mathbb{F}_2^7$, $\mathcal{C}^\perp$, so that the combined span of $\mathcal{C}$ and $\mathcal{C}^\perp$ is all of $\mathbb{F}_2^7$ and, moreover, every vector in $\mathcal{C}$ is orthogonal to every vector in $\mathcal{C}^\perp$. ($\mathcal{C}^\perp$ is 3-dimensional)

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- In our example $\mathcal{C}^\perp$ is generated by $\{1101100, 1011010, 0111001\}$. The orthogonality condition tells us that if $\vec{c} \in \mathcal{C}$, then

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \vec{c}H^T = \vec{c}\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}^T.$$

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- Notice that columns 3, 6, and 7 of H sum to $\vec{0}$. Also observe that no *two* columns of H sum to $\vec{0}$.

How close can two codewords be?

Proposition

There is a nonzero codeword $\vec{c} \in \mathcal{C}$ such that $w(\vec{c}) \leq d$ if and only if there exists a set of d columns of H that are linearly dependent.

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($\Longrightarrow$): Suppose there is a vector $\vec{c}$ with $w(\vec{c}) \leq d$. The linear combination only involves $w(\vec{c}) \leq d$ columns and $\vec{c}H^T = \vec{0}$ is merely expressing the linear dependence of these $w(\vec{c})$ columns. □

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($\Longleftarrow$): Suppose there exist d columns $\vec{H}_{k_1}, \ldots, \vec{H}_{k_d}$ of H and scalars $a_{k_1}, \ldots, a_{k_d}$ (at least one nonzero) so that

$$\sum_{n=1}^{d} a_{k_n} \vec{H}_{k_n} = \vec{0}.$$

Choose $\vec{c}$ so that $\vec{c}(k_n) = a_{k_n}$ with $\vec{c}(k) = 0$ otherwise. Then $\vec{c}$ belongs to the code because $\vec{c}H^T = \vec{0}$ and $w(\vec{c}) \leq d$ by construction. □

$(7, 4)$ Hamming code:

- In our $(7, 4)$ Hamming code no two columns are linearly dependent, but there are triplets of columns which are dependent. Proposition 5.1 implies that no nonzero codeword $\vec{c}$ has $w(\vec{c}) \leq 2$.

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- Recall that $d(\vec{c}_1, \vec{c}_2) = w(\vec{c}_1 - \vec{c}_2)$, so we know that $d(\vec{c}_1, \vec{c}_2) > 2$ for all the codewords in our code. In other words, we must change at least *three* bits to change one codeword into another.

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- **Conclusion:** *If a word is one flip away from a codeword, then that codeword is the unique codeword closest to the given word.*

Dissecting the $(7, 4)$ Hamming code:

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Conclusion: *Every word is either a codeword or one flip away from a uniquely defined codeword.*

Seven Prisoner Strategy:

- Assign each guest a bit number and give each guest a list of codewords and the associated one-flip-away words. If a prisoner sees six correct digits then (s)he guesses that his or her bit will result in a non-codeword. If a prisoner sees that the word cannot be a codeword, (s)he simply elects not to guess.

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- LOSING: In 16 cases (the codewords) everyone sees 6 correct digits and guesses incorrectly that the full word is not a codeword. (All seven prisoners guess wrong.)
- WINNING: In the remaining 112 cases only one digit actually is "wrong" so only the prisoner wearing this hat makes a guess. In this case the single guess is correct and the team wins.

Optimality and Extensions

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- As N increases $P(\textsc{Win})$ approaches 1.

Conclusion

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- If the hat colors are not equally likely, how will the optimal strategy be affected?