ORDINAL RANKS ON WEAKLY COMPACT AND ROSENTHAL OPERATORS.

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ABSTRACT. Using the Schreier families $(\mathcal{S}_\xi)_{\xi<\omega_1}$, we define subclasses of weakly compact and Rosenthal operators between two Banach spaces. These subclasses give rise to ordinal ranks defined on each ideal. We prove several results concerning the analytic properties of this rank and give examples of spaces on which the ranks are bounded and unbounded.

1. INTRODUCTION

Given separable Banach spaces X and Y, we use the Schreier families $(\mathcal{S}_\xi)_{\xi<\omega_1}$ [3] to define subclasses of the weakly compact operators $\mathcal{W}(X,Y)$ and the Rosenthal operators $\mathcal{R}(X,Y)$. Recall that a bounded operator $T : X \to Y$ is Rosenthal if for each bounded sequence $(x_n)_{n=1}^\infty$ in X, $(Tx_n)_{n=1}^\infty$ has a weakly Cauchy subsequence [17]. For each ideal and ordinal $\xi < \omega_1$, we will use $\mathcal{S}_\xi$ to define an ordinal rank on $\mathcal{L}(X,Y)$, which we call the *local rank*. As $\mathcal{L}(X,Y)$ is a standard Borel space when endowed with the σ-algebra generated by the strong operator topology (SOT), we may apply techniques from Descriptive Set Theory to analyze these ranks.

Ordinal ranks, i.e. functions taking values in $\omega_1 \cup \{\omega_1\}$, are commonly used as tools to measure certain phenomena in Banach spaces, and can provide quantitative versions of particular properties [2, 6, 14, 18]. In [14], the authors study various ℓ_1-type indices on Banach spaces that have connections to the present work, and in [4, 8], a similar local rank, $\varrho_{\mathcal{SS}}$, is defined on the ideal $\mathcal{SS}(X,Y)$ of strictly singular operators from X to Y. In [9], P. Dodos and the first author study the rank $\varrho_{\mathcal{SS}}$ and prove the following:

- If $\mathcal{A}$ is an SOT-analytic subset of $\mathcal{SS}(X,Y)$, there is a countable ordinal ξ such that $\varrho_{\mathcal{SS}}(T) \le \xi$ for all $T \in \mathcal{A}$; that is, the rank $\varrho_{\mathcal{SS}}(T)$ satisfies *boundedness*.
- There are spaces X and Y such that $\mathcal{SS}(X,Y)$ is SOT-coanalytic non-Borel and $\varrho_{\mathcal{SS}}(T) \le 2$ for all $T \in \mathcal{SS}(X,Y)$; that is, the rank $\varrho_{\mathcal{SS}}(T)$ is not, in general, a coanalytic rank.

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One goal of the present paper is to prove the analogous results for the local ranks $\varrho_{\mathcal{W}}$ and $\varrho_{\mathcal{R}}$, defined on the spaces $\mathcal{W}(X,Y)$ and $\mathcal{R}(X,Y)$, respectively. We now briefly describe how the subclasses, which induce the local ranks, are defined. To do so, we need the following characterization of the sets $\mathcal{W}(X,Y)$ and $\mathcal{R}(X,Y)$.

Fact 1. *An operator $T : X \to Y$ is weakly compact if and only if for every normalized basic sequence (x_n) in X and $\varepsilon > 0$ there are scalars $(a_i)_{i=1}^{\ell}$ and a finite set $(n_i)_{i=1}^{\ell} \subset \mathbb{N}$ such that*

$$\|\sum_{i=1}^{\ell} a_i T x_{n_i}\| < \varepsilon \max_{1\leq k\leq \ell} |\sum_{i=k}^{\ell} a_i|.$$

Similarly, an operator $T : X \to Y$ is Rosenthal if and only if for every normalized basic sequence (x_n) in X and $\varepsilon > 0$ there are scalars $(a_i)_{i=1}^{\ell}$ and a finite set $(n_i)_{i=1}^{\ell} \subset \mathbb{N}$ such that

$$\|\sum_{i=1}^{\ell} a_i T x_{n_i}\| < \varepsilon \max_{1\leq k\leq \ell} \sum_{i=k}^{\ell} |a_i|.$$

In other words, an operator $T : X \to Y$ is weakly compact if and only if the image under T of every normalized basic sequence in X does not dominate the summing basis of c_0, and an operator $T : X \to Y$ is Rosenthal if and only if the image under T of every normalized basic sequence in X does not dominate the unit vector basis of ℓ_1. We prove this characterization in Proposition 8. Roughly speaking, Fact 1 states that showing an operator is weakly compact or Rosenthal reduces to finding, for each $\varepsilon > 0$ and normalized basic sequence (x_n), a finite subset $(n_i)_{i=1}^{\ell}$ satisfying certain properties. With this in mind, we may define subclasses of $\mathcal{W}(X,Y)$ and $\mathcal{R}(X,Y)$ by requiring that these finite subsets can always be chosen as elements of some prescribed collection of finite subsets of $\mathbb{N}$. In this way, membership by a weakly compact or Rosenthal operator in a particular subclass gives one additional information concerning the behavior of the operator.

Definition 2. *Let ξ be a countable ordinal, and let X and Y be separable Banach spaces. An operator $T \in \mathcal{W}(X,Y)$ is* $\mathcal{S}_\xi$-weakly compact *if for every normalized basic sequence (x_n) in X and $\varepsilon > 0$, there exists $(n_i)_{i=1}^{\ell} \in S_\xi$ and $(a_i)_{i=1}^{\ell} \in \mathbb{R}^{<\mathbb{N}}$ such that*

$$\|\sum_{i=1}^{\ell} a_i T x_{n_i}\| < \varepsilon \max_{1\leq k\leq \ell} |\sum_{i=k}^{\ell} a_i|.$$

An operator $T \in \mathcal{R}(X,Y)$ is $\mathcal{S}_\xi$-Rosenthal *if for every normalized basic sequence (x_n) in X and $\varepsilon > 0$, there exists $(n_i)_{i=1}^{\ell} \in S_\xi$ and $(a_i)_{i=1}^{\ell} \in \mathbb{R}^{<\mathbb{N}}$ such that*

$$\|\sum_{i=1}^{\ell} a_i T x_{n_i}\| < \varepsilon \max_{1\leq k\leq \ell} \sum_{i=k}^{\ell} |a_i|.$$

The ranks $\varrho_{\mathcal{W}}, \varrho_{\mathcal{R}} : \mathcal{W}(X,Y) \to \omega_1 \cup \{\omega_1\}$ are defined by:

$$\varrho_{\mathcal{W}}(T) = \inf\{\xi : T \in \mathcal{W}(X,Y) \text{ is } \mathcal{S}_\xi - \textit{weakly compact}\},$$

$$\varrho_{\mathcal{R}}(T) = \inf\{\xi : T \in \mathcal{R}(X,Y) \textit{ is } \mathcal{S}_\xi - \textit{Rosenthal}\},$$

where by convention $\inf \emptyset = \omega_1$.

We prove several results related to these subclasses including properties of the local ranks induced by the subclasses. The following is our first main result in this direction.

Theorem 3. *Let X and Y be separable Banach spaces and $\mathcal{G}$ stand for $\mathcal{W}$ or $\mathcal{R}$. There exists a coanalytic rank $r_{\mathcal{G}} : \mathcal{G}(X,Y) \to \omega_1$ such that*

$$\varrho_{\mathcal{G}}(T) \leq r_{\mathcal{G}}(T). \tag{1}$$

In particular, the rank $\varrho_{\mathcal{G}}$ satisfies boundedness; that is, if $\mathcal{A}$ is an analytic subset of $\mathcal{G}(X,Y)$, then $\sup\{\varrho_{\mathcal{G}}(T) : T \in \mathcal{A}\} < \omega_1$.

As singletons are Borel, we have the following immediate corollary.

Corollary 4. *Let X and Y be separable Banach spaces. If an operator $T : X \to Y$ is weakly compact, then there is a countable ordinal ξ such that T is $\mathcal{S}_\xi$-weakly compact. Likewise, if an operator $T : X \to Y$ is Rosenthal, then there is a countable ordinal ξ such that T is $\mathcal{S}_\xi$-Rosenthal.*

In section 2, we set notation and define several basic objects in Descriptive Set Theory. In section 3, we prove that the local ranks are bounded by a coanalytic rank. In section 4, we give several examples to illustrate that, depending on the spaces X and Y, the ranks $\varrho_{\mathcal{W}}$ and $\varrho_{\mathcal{R}}$ can be either bounded or unbounded on the spaces $\mathcal{W}(X,Y)$ and $\mathcal{R}(X,Y)$. The spaces $\mathcal{R}(X,Y)$ and $\mathcal{W}(X,Y)$ are SOT-coanalytic subsets of $\mathcal{L}(X,Y)$, however, we give examples of spaces X, Y and Z such that $\varrho_{\mathcal{W}}$ is not a coanalytic rank on $\mathcal{W}(X)$ and $\varrho_{\mathcal{R}}$ is a not a coanalytic rank on $\mathcal{R}(Y,Z)$.

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2. Notation and Preliminaries

Let $\mathbb{N}^{<\mathbb{N}}$ denote the set of all finite sequences of $\mathbb{N}$ and $[\mathbb{N}]^{<\mathbb{N}} \subset \mathbb{N}^{<\mathbb{N}}$ be the collection of strictly increasing sequences. Let $\prec$ be the natural ordering on $\mathbb{N}^{<\mathbb{N}}$ of strict end-extension. A tree on $\mathbb{N}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under initial segments. Notice that Tr is a closed subset of the compact metrizable space $2^{\mathbb{N}^{<\mathbb{N}}}$. Also notice that $[\mathbb{N}]^{<\mathbb{N}} \in \mathrm{Tr}$.

A subset of a tree is called a *chain* if it consists of pairwise comparable nodes. A maximal chain is called a *branch.* A tree T is called *ill-founded* if there is an infinite branch. A tree is called *well-founded* if it is not ill-founded. Let WF $\subset$ Tr denote the collection of all well-founded trees on $\mathbb{N}$.

For $T \in$ WF, the derivative T' is defined by $T' = \{s \in T : \exists t \in T \text{ with } s \prec t\}$. Using transfinite induction, we define, for each countable ordinal ξ, the iterated derivative T^ξ of T. If $\xi < \omega_1$ and T^ξ has been defined then we define $T^{\xi+1} = (T^\xi)'$. If $\xi < \omega_1$ is a limit ordinal and T^η has been defined for all $\eta < \xi$, then we define $T^\xi = \cap_{\eta<\xi} T^\eta$. The *order of* T, $o(T)$, is defined to be the least ordinal such that $T^\xi = \emptyset$. See [12] for further discussion of derivatives of trees.

Every subset of $\mathbb{N}$ is naturally identified with an element of $2^{\mathbb{N}}$. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.

- $\mathcal{F}$ is called *spreading* if for every $F = \{\ell_1 < \cdots < \ell_k\} \in \mathcal{F}$ and every $G = \{m_1 < \cdots < m_k\}$ with $\ell_i \le m_i$ for all $i \in \{1, \ldots, k\}$, we have $G \in \mathcal{F}$.
- $\mathcal{F}$ is called *hereditary* if $F \in \mathcal{F}$ and $G \subset F$ implies $G \in \mathcal{F}$.
- $\mathcal{F}$ is called *compact* if it is a compact subset of $2^{\mathbb{N}}$.
- $\mathcal{F}$ is called *regular* if it is spreading, hereditary and compact.

Every regular family $\mathcal{F}$ is a well-founded tree on $\mathbb{N}$, and so, its order $o(\mathcal{F})$ can be defined as above.

The Schreier families $(\mathcal{S}_\xi)_{\xi<\omega_1}$, are a particularly important collection of regular trees on $\mathbb{N}$ [3]. This collection has been defined as follows: Let $S_0 = \{\{n\} : n \in \mathbb{N}\}$. Let $\xi < \omega_1$ be some countable ordinal, and suppose $\mathcal{S}_\xi$ has be defined. Let

$$\mathcal{S}_{\xi+1} = \{\bigcup_{i=1}^{k} F_i : k \in \mathbb{N},\ n \le F_1 < \ldots < F_k \text{ and } F_i \in S_\xi,\ i \le k\}.$$

If ξ is a limit ordinal, let $(\xi_n)_{n\in\mathbb{N}}$ be a fixed increasing sequence of ordinals such that $\xi_n \to \xi$. Define

$$\mathcal{S}_\xi = \{F : n \le F \text{ and } F \in \mathcal{S}_{\xi_n} \text{ for some } n \in \mathbb{N}\}.$$

We will need the following facts concerning the families $(\mathcal{S}_\xi)_{\xi<\omega_1}$.

Fact 5. [6] *Let $\xi \le \zeta$ be countable ordinals.*

a) $o(S_\xi) = \omega^\xi$
b) *There is an $N \in \mathbb{N}$ such that $\mathcal{S}_\xi \cap \mathcal{P}([N, \infty)) \subset \mathcal{S}_\zeta$.*

2.1. **Polish spaces, standard Borel spaces, analytic and coanalytic sets.** A topological space P is called *Polish* if it is separable and homeomorphic to a complete metric space. Below we list a few important types of subsets of Polish spaces as well as properties of these subsets.

Fact 6. *In the following, P is a Polish space, τ is its topology and $Borel(\tau)$ is the σ-algebra generated by τ.*

a) *A subset A of P is* analytic *if there is a Polish space S and a Borel map $f : S \to P$ such that $f(S) = A$, where a map $f : P \to S$ is* Borel *if $f^{-1}(B)$ is Borel in S for every Borel set B of P.*

b) *The collection of all analytic sets P is closed under taking countable unions of elements.*
c) *A set is* coanalytic *if it is the complement of an analytic set.*
d) *If a subset of a Polish space is both analytic and coanalytic it is Borel.*
e) *A coanalytic set C of a Polish space P is called* coanalytic complete *if there is a Borel map $h : \mathrm{Tr} \to P$ such that $h^{-1}(C) = \mathrm{WF}$. Every coanalytic complete set is non-Borel, as $\mathrm{WF} \subset \mathrm{Tr}$ is not Borel.*
f) *Let $B \in Borel(\tau)$. There is a finer Polish topology τ' on P such that B is τ'-clopen, $Borel(\tau') = Borel(\tau)$ and B is a Polish space with the relative toplogy of τ'.*

The above statements, together with several useful characterizations of analytic, coanalytic and Borel sets can be found in [12]. It will be important for us that the space $2^{\mathbb{N}^{<\mathbb{N}}}$ under the product topology is a Polish space and $\mathrm{WF} \subset 2^{\mathbb{N}^{<\mathbb{N}}}$ is a coanalytic complete subset of $2^{\mathbb{N}^{<\mathbb{N}}}$. Let $\mathcal{NB}_X$ denote the collection of all normalized Schauder basic sequences in X. The set $\mathcal{NB}_X$ is a Borel subset of $X^{\mathbb{N}}$ [4]. By f), $\mathcal{NB}_X$ may be considered as a Polish space in its own right.

It is often the case that a particular topology on a set is not so important as the Borel sets given by that topology. For instance, a topological space may not be Polish, but the Borel sets for that topological space may be the same Borel sets for some Polish topology. A set X with a sigma algebra Σ is a *standard Borel space* if there is a Polish topology τ such that $Borel(\tau) = \Sigma$. Let X and Y be separable Banach spaces. The space $\mathcal{L}(X, Y)$ equipped with the σ-algebra generated by the open sets in the strong operator topology (SOT) is a standard Borel space [12, pg. 80]. A set $\mathcal{A} \subset \mathcal{L}(X, Y)$ is SOT-Borel if $\mathcal{A}$ is in the σ-algebra generated by the open sets in the strong operator topology; likewise, we define SOT-analytic and SOT-coanalytic.

Let C be a coanalytic subset of a standard Borel space P. A map $f : C \to \omega_1$ is a coanalytic rank on C if there are two binary relations $\leq_\Sigma$ and $\leq_\Pi$ on P which are analytic and coanalytic, respectively, such that for every $y \in C$ we have

$$f(x) \leq f(y) \iff (x \in C) \text{ and } f(x) \leq f(y) \iff x \leq_\Sigma y \iff x \leq_\Pi y.$$

It is a fundamental result that if f is a coanalytic rank and A is an analytic subset of C then

$$\sup\{f(x) : x \in A\} < \omega_1. \tag{2}$$

Given any rank f on C, which may or may not be coanalytic, we say that f satisfies *boundedness* if (2) holds for all analytic $A \subset C$.

3. Local Ranks on Operator Ideals

In the rest of the paper, (w_n) will always be a normalized 1-spreading basis of a Banach space W. That is, (w_n) is a normalized basis for W which is 1-equivalent to all of its subsequences.

Definition 7. *An operator $T \in \mathcal{L}(X,Y)$ will be called (w_n)-singular if for every normalized basic sequence $(x_n) \subset X$, the sequence (Tx_n) does not dominate (w_n). We denote the space of all (w_n)-singular operators from X to Y by $WS(X,Y)$.*

We are specifically interested in the weakly compact and Rosenthal operators, and the following proposition shows that these are specific examples of $WS(X,Y)$ operators given by choosing an appropriate basis (w_n). It may be of interest to consider different 1-spreading basic sequences, such as the unit vector basis of the ℓ_p spaces, for $1 < p < \infty$. Let (e_n) denote the unit vector basis of c_{00}. The summing basis of c_0 is $s_n = \sum_{i=1}^n e_i$ for all $n \in \mathbb{N}$. Note that $\| \sum_{i=1}^\infty a_i s_i\|_\infty = \sup_{k\in\mathbb{N}} | \sum_{i=k}^\infty a_i|$.

Proposition 8. *Let X and Y be Banach spaces.*

a) *If $W = c_0$ and $w_i = s_i$ for all $i \in \mathbb{N}$, then $WS(X,Y) = \mathcal{W}(X,Y)$; the weakly compact operators.*
b) *If $W = \ell_1$ and $w_i = e_i$ for all $i \in \mathbb{N}$, then $WS(X,Y) = \mathcal{R}(X,Y)$; the Rosenthal operators.*
c) *If $W = c_0$ and $w_i = e_i$ for all $i \in \mathbb{N}$, then $WS(X,Y) = \mathcal{K}(X,Y)$; the compact operators.*

Proof. Proof of a): First assume that $T : X \to Y$ is weakly compact. Fix a normalized basic sequence $(x_n)_{n\in\mathbb{N}}$ in X. We will show that $(Tx_n)_{n\in\mathbb{N}}$ does not dominate (s_n). Suppose, by passing to a subsequence and relabeling, there is a $y \in Y$ such that $Tx_n \to y$ weakly. By Mazur's theorem, there are finite subsets A and B of $\mathbb{N}$ with $\max A < \min B$ and scalars $(a_i)_{i\in A}, (b_i)_{i\in B} \subset [0,1]$ such that $\sum_{i\in A} a_i = \sum_{i\in B} b_i = 1$ and

$$\| \sum_{i\in A} a_i(Tx_i - y)\| < \frac{\varepsilon}{2} \text{ and } \| \sum_{i\in B} b_i(Tx_i - y)\| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} \| \sum_{i\in A} a_i Tx_i - \sum_{i\in B} b_i Tx_i\| &\leq \| \sum_{i\in A} a_i(Tx_i - y)\| + \| \sum_{i\in B} b_i(Tx_i - y)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \| \sum_{i\in A} a_i s_i - \sum_{i\in B} b_i s_i\|_\infty \end{aligned} \tag{3}$$

Thus (Tx_n) does not dominate (s_n).

Suppose T is not weakly compact. Let (x_n) be a normalized sequence such that (Tx_n) has no weakly convergent subsequence. We will show that (Tx_n) has a subsequence that dominates $(s_n)_{n\in\mathbb{N}}$.

Since the sequence (x_n) cannot have a weakly convergent subsequence, there are two possibilities: Either (x_n) has a subsequence equivalent to the unit vector basis of ℓ_1 or a subsequence that weak* converges to some $x^{**} \in X^{**} \setminus X$. In either case, by passing to a subsequence and relabeling, we can assume that (x_n) is basic. For (Tx_n) we have the same two cases. If (Tx_n) has a subsequence equivalent to the unit vector basis of ℓ_1, this subsequence dominates the summing basis for c_0 and we are done. Therefore, we assume, by passing to a subsequence and relabeling, that (Tx_n) converges weak* to some $y^{**} \in Y^{**} \setminus Y$ and that (Tx_n) is basic and C-equivalent to $(Tx_n - y^{**})$ for some $C > 0$ (see [1, pgs. 22-23]). Let $d = dist\{y^{**}, X\}$ and K be the basic constant of $(Tx_n - y^{**})$. Let $(a_i)_{i=1}^{\ell}$ be scalars. Then

$$\begin{aligned}
\|\sum_{i=1}^{\ell} a_i T x_i\| &\geq \frac{1}{C}\|\sum_{i=1}^{\ell} a_i (Tx_i - y^{**})\| \\
&\geq \frac{1}{C2K}\max_k \|\sum_{i=k}^{\ell} a_i (Tx_i - y^{**})\| \qquad (4)\\
&\geq \frac{d}{C2K}\max_k |\sum_{i=k}^{\ell} a_i|.
\end{aligned}$$

This finishes the proof of a).

Proof of b): First, note that if there is a normalized basic sequence (x_n) in X such that (Tx_n) dominates the unit vector basis of ℓ_1, then (x_n) is a bounded sequence such that (Tx_n) does not have a weakly Cauchy subsequence.

Now, assume (x_n) is a bounded sequence in X such that (Tx_n) has no weakly Cauchy subsequence. We may assume that (x_n) is normalized. Using Rosenthal's ℓ_1 theorem [17], passing to a subsequence and relabeling, we assume that (Tx_n) is equivalent to the unit vector basis of ℓ_1. It follows that (x_n) dominates the unit vector basis of ℓ_1 and is, therefore, normalized and basic, as desired.

Proof of c): If there is a normalized basic sequence (x_n) in X such that (Tx_n) dominates the unit vector basis of c_0, then T is not compact.

We now assume that T is not compact. There exists $\varepsilon > 0$ and a sequence (y_n) in B_X such that $\|Ty_n - Ty_m\| > \varepsilon$ for all $n \neq m$. After passing to a subsequence of (Ty_n), we may assume that either (Ty_n) is equivalent to the unit vector basis of ℓ_1 or that (Ty_n) is weakly Cauchy. If (Ty_n) is weakly Cauchy, then $(T(y_{2n} - y_{2n+1}))$ is seminormalized and weakly null, and hence $(T(y_{2n} - y_{2n+1}))$ has a seminormalized basic subsequence. Thus in either case, there exists a seminormalized sequence (z_n) in X such that (Tz_n) is a seminormalized basic sequence in Y. Again, after passing to a subsequence we may assume that $(z_{2n} - z_{2n+1})$ is seminormalized basic. Thus, setting $x_n = \frac{1}{\|z_{2n} - z_{2n+1}\|} z_{2n} - z_{2n+1}$, we get that (x_n) is a normalized basic sequence in X such that (Tx_n) is a seminormalized basic sequence in Y and hence dominates the unit vector basis for c_0.

□

The next proposition follows from standard techniques in Descriptive Set Theory. As the result is important for us, we have included the proof for the sake of completeness.

Proposition 9. *Let X and Y be separable. The set $WS(X,Y)$ is an SOT-coanalytic subset of $\mathcal{L}(X,Y)$.*

Proof. By definition we have

$$T \in \mathcal{L}(X,Y) \setminus WS(X,Y) \iff \exists (x_i)_i \in \mathcal{NB}_X,\ \exists m \in \mathbb{N} \text{ such that } \forall (a_i)_i \in \mathbb{Q}^{<\mathbb{N}},\ \|T\sum_i a_i x_i\| \geq \frac{1}{m}\|\sum_i a_i w_i\|.$$

The fact that there is a existential quantifier over the Borel set $\mathcal{NB}(X)$ and the remaining quantifiers are over countable sets, indicates that $\mathcal{L}(X,Y))\setminus WS(X,Y))$ is SOT-analytic. However, for the convenience of readers not familiar with descriptive set theory, we give a more detailed argument. For $m \in \mathbb{N}$, let

$$\mathcal{B}_m = \{(T,(x_n)) \in \mathcal{L}(X,Y) \times \mathcal{NB}_X : \|T\sum_i a_i x_i\| \geq \frac{1}{m}\|\sum_i a_i w_i\|, \forall (a_i)_i \in \mathbb{Q}^{<\mathbb{N}}\}.$$

Let π_1 be the projection of $\mathcal{L}(X,Y) \times \mathcal{NB}(X)$ onto $\mathcal{L}(X,Y)$. Then

$$\mathcal{L}(X,Y) \setminus WS(X,Y) = \bigcup_{m=1}^{\infty} \pi_1(\mathcal{B}_m).$$

Recall that the continuous image of a Borel set is analytic and that the countable union of analytic sets is analytic. Therefore, it suffices to show that $\mathcal{B}_m$ is Borel for each $m \in \mathbb{N}$. For each $\mathbf{a} = (a_i)_i \in \mathbb{Q}^{<\mathbb{N}}$ with $\mathbf{a} \neq 0$, the map $H_{\mathbf{a}} : \mathcal{L}(X,Y) \times \mathcal{NB}_X \to \mathbb{R}$ defined by

$$H_{\mathbf{a}}(T,(x_n)) = \frac{\|T\sum_i a_i x_i\|}{\|\sum_i a_i w_i\|}.$$

is Borel. Since $\mathcal{B}_m = \bigcap_{\mathbf{a}\in\mathbb{Q}^{<\mathbb{N}}} H_{\mathbf{a}}^{-1}[1/m,\infty)$, the claim follows. ☐

For each countable ξ, we use the Schreier family $\mathcal{S}_\xi$ to define what it means for a bounded operator to be $\mathcal{S}_\xi$-(w_i)-singular. This property is a quantified version of the property (w_i)-singular, and we will show later in Corollary 19 that if X and Y are separable Banach spaces then an operator $T \in \mathcal{L}(X,Y)$ is (w_i)-singular if and only if it is $\mathcal{S}_\xi$-(w_i)-singular for some countable ordinal ξ.

Definition 10. *Let ξ with $1 \leq \xi < \omega_1$. An operator $T \in \mathcal{L}(X,Y)$ is $\mathcal{S}_\xi$-(w_i)-singular if for every $\varepsilon > 0$ and normalized basic sequence $(x_n)_n \subset X$, there is a $(n_i)_{i=1}^{\ell} \in S_\xi$ and $(a_i)_{i=1}^{\ell} \in \mathbb{R}^{<\mathbb{N}}$ such that*

$$\|\sum_{i=1}^{\ell} a_i T x_{n_i}\| < \varepsilon \|\sum_{i=1}^{\ell} a_i w_i\|.$$

We denote the set of all $\mathcal{S}_\xi$-(w_n)-singular operators from X to Y by $WS_\xi(X,Y)$.

The next remark simply states that the subclasses are increasing with respect to the natural ordering and that every subclass contains the compact operators.

Remark 11. Let ξ, ζ be ordinals with $1 \leq \xi < \zeta < \omega_1$. Then

$$\mathcal{K}(X,Y) \subseteq WS_\xi(X,Y) \subseteq WS_\zeta(X,Y) \subseteq WS(X,Y).$$

As a consequence of the above remark, $\mathcal{K}_1(X,Y) = \mathcal{K}(X,Y)$ for all X and Y.

Proof. Let $T \in \mathcal{K}(X,Y)$ and $\varepsilon > 0$. There exists $c > 0$ such that $\|w_2 - w_1\| > c$. Let $(x_n) \in \mathcal{NB}_X$. Choose a subsequence (n_k) such that (Tx_{n_k}) is norm Cauchy. Let $i \in \mathbb{N}$ with $i \geq 2$ such that $\|Tx_{n_{i+1}} - Tx_{n_i}\| < \varepsilon c$. Then

$$\|Tx_{n_{i+1}} - Tx_{n_i}\| < \varepsilon \|w_2 - w_1\|.$$

Since $\{n_i, n_{i+1}\} \in \mathcal{S}_\xi$, we have $T \in WS_\xi(X,Y)$.

Let $\varepsilon > 0$, $(x_n) \in \mathcal{NB}_X$ and $T \in WS_\xi(X,Y)$. Using Fact 5 b) there is an $N \in \mathbb{N}$ such that $\mathcal{S}_\xi|_{[N,\infty)} \subset \mathcal{S}_\zeta$. Let $y_i = x_{N+i}$ for all $i \in \mathbb{N}$. Find $(n_i)_{i=1}^\ell \in S_\xi$ and $(a_i)_{i=1}^\ell$ such that

$$\|\sum_{i=1}^{\ell} a_i w_i\| = 1 \text{ and } \|\sum_{i=1}^{\ell} a_i T y_{n_i}\| = \|\sum_{i=1}^{\ell} a_i T x_{N+n_i}\| < \varepsilon.$$

Since $(N + n_i)_{i=1}^\ell \in \mathcal{S}_\xi|_{[N,\infty)} \subset \mathcal{S}_\zeta$, the claim is proved. □

Definition 12. *Let (w_n) be a 1-spreading basis of a Banach space W. Define the map $\varrho_{WS} : \mathcal{L}(X,Y) \to \omega_1 \cup \{\omega_1\}$ by*

$$\varrho_{WS}(T) = \inf\{\xi : T \in WS_\xi(X,Y)\}.$$

where by convention $\inf \emptyset = \omega_1$. In particular, if (w_i) is the summing basis of c_0, we denote ϱ_{WS} by $\varrho_{\mathcal{W}}$; similarly, if (w_i) is the unit vector basis of ℓ_1 we denote ϱ_{WS} by $\varrho_{\mathcal{R}}$. The ranks $\varrho_{\mathcal{W}}$ and $\varrho_{\mathcal{R}}$ are ordinal ranks defined on $\mathcal{W}(X,Y)$ and $\mathcal{R}(X,Y)$, respectively.

For $\mathcal{A} \subset \mathcal{L}(X,Y)$ let $\varrho_{WS}(\mathcal{A}) = \sup\{\varrho_{WS}(T) : T \in \mathcal{A}\}$.

Before proceeding, we make the following remarks comparing $\mathcal{W}(X,Y)$ and $\mathcal{R}(X,Y)$.

Remark 13. Let X and Y be separable Banach spaces.

a) For each ξ with $1 \leq \xi < \omega_1$, $\mathcal{W}_\xi(X,Y) \subset \mathcal{R}_\xi(X,Y)$.
b) In general, $\mathcal{W}(X,Y) \subset \mathcal{R}(X,Y)$. If Y is weakly sequentially complete then $\mathcal{W}(X,Y) = \mathcal{R}(X,Y)$.

The main theorem of this section is the following.

Theorem 14. *Let X and Y be separable Banach spaces. There exists a coanalytic rank* $\mathrm{r}_{WS} : WS(X,Y) \to \omega_1$ *such that*

$$\varrho_{WS}(T) \le \mathrm{r}_{WS}(T). \tag{5}$$

In particular, the rank ϱ_{WS} satisfies boundedness; that is, if $\mathcal{A}$ is an analytic subset of $WS(X,Y)$, then $\sup\{\varrho_{WS}(T) : T \in \mathcal{A}\} < \omega_1$.

There are two steps required to prove the above. The first is to produce the coanalytic rank; the second is to show that this rank bounds the local rank. We use the following fact to define the coanalytic rank.

Fact 15. [9, Fact 6] *Let X be a standard Borel space and $\mathcal{P}$ be an analytic subset of $X \times \mathrm{Tr}$. Then the set $\mathcal{P}^\sharp \subseteq X$ defined by*

$$x \in \mathcal{P}^\sharp \Leftrightarrow \forall S \in \mathrm{Tr}\ [(x,S) \in \mathcal{P} \Rightarrow S \in \mathrm{WF}]$$

is coanalytic. Moreover, there exists a coanalytic rank $r : \mathcal{P}^\sharp \to \omega_1$ such that for every $x \in \mathcal{P}^\sharp$ we have $\sup\{o(S) : S \in \mathrm{Tr}$ *and* $(x,S) \in \mathcal{P}\} \le r(x)$.

In order to use the above, we must assign to every $T \in \mathcal{L}(X,Y)$ a collection of trees on $\mathbb{N}$ such that every member of the collection is well-founded exactly when $T \in WS(X,Y)$. We do this in the next definition.

Definition 16. *For each $T \in \mathcal{L}(X,Y)$, $(x_n) \in \mathcal{NB}_X$ and $m \in \mathbb{N}$, let $R(T,(x_n),m)$ be the tree on $\mathbb{N}$ defined by the rule*

$$(6)\quad s \in R(T,(x_n),m) \quad\Leftrightarrow\quad \text{either } s = \varnothing \text{ or } s = (n_1 < \ldots < n_k) \in [\mathbb{N}]^{<\mathbb{N}} \text{ and}$$
$$\Big\| T\Big(\sum_{i=1}^k a_i x_{l_i}\Big)\Big\| \ge \frac{1}{m}\Big\|\sum_{i=1}^k a_i w_i\Big\|\ \forall a_1,\ldots,a_k \in \mathbb{Q} \text{ and}$$
$$(l_1 < \ldots < l_k) \in [\mathbb{N}]^{<\mathbb{N}} \text{ with } n_i \le l_i\ \forall 1 \le i \le k.$$

In the next remark, we collect a few important facts concerning the above trees. The proofs follow directly from standard arguments and so we omit them.

Remark 17. The following hold:

a) The map $\mathcal{L}(X,Y) \times \mathcal{NB}_X \times \mathbb{N} \ni (T,(x_n),m) \mapsto R(T,(x_n),m) \in \mathrm{Tr}$ is Borel.

b) Let $T \in \mathcal{L}(X,Y)$.

$$T \notin WS(X,Y) \iff \exists (x_n) \in \mathcal{NB}_X \text{ and } m \in \mathbb{N} \text{ with } R(T,(x_n),m) \notin \mathrm{WF}$$

c) Let $T \in \mathcal{L}(X,Y)$, $(x_n) \in \mathcal{NB}_X$ and $m \in \mathbb{N}$. The tree $R(T,(x_n),m)$ is a regular family (i.e. spreading, hereditary and compact).

The proof of Remark 17 b) uses the assumption that the basis (w_n) is 1-spreading. We apply Fact 15 to produce a coanalytic rank on $WS(X,Y)$. Let $\mathcal{P} \subset \mathcal{L}(X,Y) \times \mathrm{Tr}$ be defined by

$$(T,\mathfrak{R}) \in \mathcal{P} \Leftrightarrow \exists (x_n) \in \mathcal{NB}_X \text{ and } m \in \mathbb{N} \text{ such that } \mathfrak{R} = R(T,(x_n),m). \tag{7}$$

By Remark 17 a), the set $\mathcal{P}$ is analytic. By Remark 17 b), we have that

$$T \in WS(X,Y) \Leftrightarrow \forall \mathfrak{R} \in \mathrm{Tr}\ [(T,\mathfrak{R}) \in \mathcal{P} \Rightarrow \mathfrak{R} \in \mathrm{WF}].$$

Let $\widetilde{\mathrm{r}_{WS}} : WS(X,Y) \to \omega_1$ be the coanalytic rank given by Fact 15. Define

$$\mathrm{r}_{WS}(T) = \widetilde{\mathrm{r}_{WS}}(T) + 1.$$

Clearly, r_{WS} is a coanalytic rank on $WS(X,Y)$. Theorem 14 now follows as a corollary of the following claim.

Claim 18. *Let $T \in WS(X,Y)$.*

$$\begin{aligned} \varrho_{WS}(T) &\leq \sup\{\omega^\zeta : \zeta < \varrho_{WS}(T)\} \\ &\leq \sup\big\{o\big(R(T,(x_n),m)\big) : (x_n) \in \mathcal{NB}_X \text{ and } m \in \mathbb{N}\big\} + 1 \end{aligned} \tag{8}$$

Moreover, $\varrho_{WS}(T) \leq \mathrm{r}_{WS}(T)$.

Proof. The first inequality follows trivially. We will prove the second. For notational simplicity let $\xi = \varrho_{WS}(T)$. We may assume that $\xi > 1$. Fix a countable ordinal ζ such that $1 \leq \zeta < \xi$. Since $\varrho_{WS}(T) > \zeta$, we have $T \notin WS_\zeta(X,Y)$. Therefore, we may find $(x_n) \in \mathcal{NB}_X$ and $\varepsilon > 0$ such that for every non-empty set $(n_1, \dots, n_\ell) \in \mathcal{S}_\zeta$ and every $(a_i)_{i=1}^\ell \in c_{00}$ we have $\|T\sum_{i=1}^\ell a_i x_{n_i}\| \geq \varepsilon \|\sum_{i=1}^\ell a_i w_i\|$. Let $m \in \mathbb{N}$ such that $\varepsilon \geq m^{-1}$. The family $\mathcal{S}_\zeta$ is spreading and hereditary. Hence, $\mathcal{S}_\zeta \subset R(T,(x_n),m)$. Therefore by Fact 5 a)

$$\omega^\zeta = o(\mathcal{S}_\zeta) \leq o\big(R(T,(x_n),m)\big).$$

The second inequality follows. By the definition of r_{WS}, (see Fact 15) we have

$$\sup\big\{o\big(R(T,(x_n),m)\big) : (x_n) \in \mathcal{NB}_X \text{ and } m \in \mathbb{N}\big\} + 1 \leq \mathrm{r}_{WS}(T). \tag{9}$$

Thus we have that $\varrho_{WS}(T) \leq \mathrm{r}_{WS}(T)$. □

We finish this section with the following immediate corollary of Theorem 14.

Corollary 19. *The following hold:*

a) *If $T \in WS(X,Y)$ then $\varrho_{WS}(T) < \omega_1$. In other words, there is a countable ξ such that $T \in WS_\xi(X,Y)$.*
b) *If $WS(X,Y)$ is a SOT-Borel subset of $\mathcal{L}(X,Y)$ then $\varrho_{WS}(WS(X,Y)) < \omega_1$. In other words, there is a countable ξ such that $WS_\xi(X,Y) = WS(X,Y)$. In particular, this is the case whenever $WS(X,Y) = \mathcal{L}(X,Y)$.*

4. Applications and Examples

4.1. **Spaces not containing ℓ_1.** We begin with the following theorem.

Theorem 20. *Let X and Y be separable such that Y does not contain ℓ_1. Then*

$$\varrho_{\mathcal{W}}(\mathcal{W}(X,Y)) < \omega_1 \text{ and } \varrho_{\mathcal{R}}(\mathcal{R}(X,Y)) < \omega_1.$$

In other words, there is a countable ordinal ξ satisfying, $\mathcal{W}_\xi(X,Y) = \mathcal{W}(X,Y)$ and $\mathcal{R}_\xi(X,Y) = \mathcal{R}(X,Y)$.

To prove this theorem, we first recall the notion of ℓ_1^ξ spreading model for $1 \leq \xi < \omega_1$. A space X admits an ℓ_1^ξ-spreading model if there is a sequence (x_n) in X and a $\delta > 0$ such that for all $(a_i)_{i=1}^\ell \in c_{00}$ and $(n_i)_{i=1}^\ell \in S_\xi$ the following holds:

$$\|\sum_{i=1}^{\ell} a_i x_{n_i}\| \geq \delta \sum_{i=1}^{\ell} |a_i|. \tag{10}$$

Note that an ℓ_1^1 spreading model coincides with the usual notion of ℓ_1 spreading model. If a space admits an ℓ_1^ξ spreading model then the Bourgain ℓ_1 index [10] is greater than or equal to ω^ξ. If a space has unbounded Bourgain ℓ_1 index it must contain ℓ_1. This yields the following well-known fact.

Fact 21. *If X does not contain ℓ_1 then there is an minimum ξ with $0 \leq \xi < \omega_1$ such that for all $\zeta > \xi$, X does not admit an ℓ_1^ζ spreading model.*

In [7], the authors prove a dichotomy concerning ℓ_1^ξ spreading models (also see [6, Theorem III.3.11] for the exact statement). We need the following consequence of this dichotomy which is, in some sense, a quantized version of Mazur's theorem.

Theorem 22. [7][6, Theorem III.3.11] *Let ξ be a countable ordinal and (x_n) be weakly null sequence such that no subsequence admits an ℓ_1^ξ spreading model. Then for every $\varepsilon > 0$ there there is a $(n_i)_{i=1}^\ell \in S_\xi$ and convex scalars $(a_i)_{i=1}^\ell$ such that*

$$\max_{i \leq \ell} a_i < \varepsilon \text{ and } \|\sum_{i=1}^{\ell} a_i x_{n_i}\| < \varepsilon.$$

The proof of Theorem 20 immediately follows from combining Fact 21 with the following proposition.

Proposition 23. *Let ξ be a countable ordinal and suppose that Y admits no ℓ_1^ξ spreading model. Then $\varrho_{\mathcal{W}}(\mathcal{W}(X,Y)) \leq \xi$ and $\varrho_{\mathcal{R}}(\mathcal{R}(X,Y)) \leq \xi$; that is, $\mathcal{W}(X,Y) = \mathcal{W}_\xi(X,Y)$ and $\mathcal{R}(X,Y) = \mathcal{R}_\xi(X,Y)$.*

Proof. The case of $\mathcal{R}(X,Y)$ follows directly from the definition.

Let $T \in \mathcal{W}(X,Y)$ and (x_n) be a normalized basic sequence in X. By passing to a subsequence and relabeling we may assume that $Tx_n \to y$ weakly for some $y \in Y$. If $Tx_n \to y$ in norm, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|Tx_N - Tx_{N+1}\| < \varepsilon = \varepsilon\|s_N - s_{N+1}\|$. Note that $(N, N+1) \in S_\eta$ for all $1 \leq \eta < \omega_1$.

We now suppose that, after passing to a subsequence and relabeling, $(Tx_n - y)$ is seminormalized weakly null and C-basic for some constant $C \geq 1$. Let $\varepsilon > 0$. Theorem 22 states that there exists $(n_i)_{i=1}^\ell \in S_\xi$ and convex scalars $(a_i)_{i=1}^\ell$ such that

$$\max_{1 \leq i \leq \ell} a_i < \frac{\varepsilon}{12(\|y\|+1)} \text{ and } \|\sum_{i=1}^{\ell} a_i(Tx_{n_i} - y)\| < \frac{\varepsilon}{12C}.$$

Using the above, there is a $k \in \{1, \dots \ell\}$ such that

$$|\sum_{i=1}^{k} a_i - \sum_{i=k+1}^{\ell} a_i| \leq \frac{\varepsilon}{4(\|y\|+1)} \text{ and } \|\sum_{i=1}^{k} a_i s_i - \sum_{i=k+1}^{\ell} a_i s_i\|_\infty = |\sum_{i=k+1}^{\ell} a_i| \geq \frac{1}{2}.$$

Since $(Tx_i - y)$ is a basic sequence with basis constant at most C,

$$\|\sum_{i=1}^{k} a_i(Tx_{n_i} - y)\| < \frac{\varepsilon}{12} \text{ and } \|\sum_{i=k+1}^{\ell} a_i(Tx_{n_i} - y)\| < \frac{\varepsilon}{6}.$$

Therefore we have

$$\begin{aligned}\|\sum_{i=1}^{k} a_i Tx_{n_i} - \sum_{i=k+1}^{\ell} a_i Tx_{n_i}\| &\leq \|\sum_{i=1}^{k} a_i(Tx_{n_i} - y)\| + \|\sum_{i=k+1}^{\ell} a_i(Tx_{n_i} - y)\| \\ &\quad + \|(\sum_{i=1}^{k} a_i - \sum_{i=k+1}^{\ell} a_i)y\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4(\|y\|+1)}\|y\| \\ &< \frac{\varepsilon}{2} \leq \varepsilon|\sum_{i=k+1}^{\ell} a_i|\end{aligned}$$

Thus $T \in \mathcal{W}_\xi(X, Y)$. □

We recall that if ξ is a countable ordinal, then T_ξ denotes the Tsirelson space of order ξ.

Corollary 24. *Let ξ be a countable ordinal. Then $\xi < \varrho_{\mathcal{W}}(\mathcal{L}(T_\xi)) = \varrho_{\mathcal{R}}(\mathcal{L}(T_\xi)) < \omega_1$. In particular, there is a countable ordinal ζ such that*

$$\mathcal{W}_\xi(T_\xi) = \mathcal{R}_\xi(T_\xi) \subsetneq \mathcal{W}_\zeta(T_\xi) = \mathcal{R}_\zeta(T_\xi) = \mathcal{L}(T_\xi).$$

Proof. Since T_ξ is reflexive and has an unconditional basis, $\mathcal{W}_\xi(T_\xi) = \mathcal{R}_\xi(T_\xi)$ and $\mathcal{W}(T_\xi) = \mathcal{R}(T_\xi) = \mathcal{L}(T_\xi)$. Since T_ξ is asymptotic ℓ_1^ξ, the identity operator on T_ξ is in $\mathcal{L}(T_\xi) \setminus \mathcal{R}_\xi(T_\xi)$. Therefore,

$$\mathcal{W}_\xi(T_\xi) = \mathcal{R}_\xi(T_\xi) \subsetneq \mathcal{W}(T_\xi) = \mathcal{R}(T_\xi) = \mathcal{L}(T_\xi).$$

Corollary 19 b) yields that $\varrho_{\mathcal{W}}(\mathcal{L}(T_\xi)), \varrho_{\mathcal{R}}(\mathcal{L}(T_\xi)) < \omega_1$, as desired. □

Let (u_n) denote Pełczyński's universal basis [15] and U be the closed linear space of (u_n). By definition, every basic sequence is equivalent to a subsequence of (u_n) whose closed linear span is complemented in U. For this space we have the following proposition.

Proposition 25. *Let ξ be a countable ordinal.*

a) *There is a $T \in \mathcal{W}(U) \setminus \mathcal{W}_\xi(U)$.*
b) *There is a $T \in \mathcal{R}(U) \setminus \mathcal{R}_\xi(U)$.*

Moreover, both $\mathcal{W}(U)$ and $\mathcal{R}(U)$ are coanalytic non-Borel subsets of $\mathcal{L}(U)$.

Proof. Let ξ be a countable ordinal and U_ξ denote a complemented copy of T_ξ inside of U. Let P_ξ be the projection from U onto U_ξ. Recall that U_ξ is the closed linear span of some subsequence of (u_n). Since U_ξ is asymptotic ℓ_1^ξ and reflexive, $P_\xi \in \mathcal{W}(U) \setminus \mathcal{W}_\xi(U)$.

As $\xi < \omega_1$ was arbitrary, we have that $\varrho_{\mathcal{W}}(\mathcal{W}(U)) = \omega_1$. Corollary 19 b) yields that $\mathcal{W}(U)$ is coanalytic non-Borel. The case of $\mathcal{R}(U)$ is exactly the same. □

4.2. **Examples where $\mathcal{W}(X,Y)$ or $\mathcal{R}(X,Y)$ are SOT-Borel subsets.** If either X or Y is reflexive then $\mathcal{W}(X,Y) = \mathcal{L}(X,Y)$. Likewise, if either X or Y do not contain ℓ_1 then $\mathcal{R}(X,Y) = \mathcal{L}(X,Y)$. In each case the operator ideal is Borel and therefore $\varrho_{\mathcal{W}}(\mathcal{L}(X,Y)) \le \varrho_{\mathcal{R}}(\mathcal{L}(X,Y)) < \omega_1$. Below we provide two examples of non-reflexive spaces such that $\mathcal{W}(X)$ is a Borel subset of $\mathcal{L}(X)$.

In each example the space $\mathcal{W}(X)$ is a codimension-one subspace of $\mathcal{L}(X)$. We do not, however, have a general theorem that states that $\mathcal{W}(X)$ is a SOT-Borel subset of $\mathcal{L}(X)$ whenever $\mathcal{W}(X)$ is a hyperplane of $\mathcal{L}(X)$.

Example 26. Let J denote the quasi-reflexive space of R.C. James [11] (see [1, page 62], for a modern presentation). Let (e_n) be the unit vector basis of J and (e_n^*) be the associated biorthogonal functionals. The basis (e_n) is shrinking but not boundedly complete. In [13], the authors give the following characterization of the space of the weakly compact operators on J:

$$T \in \mathcal{W}(J) \iff \lim_{k\to\infty} \sum_{i=1}^{\infty} e_k^*(Te_i) = 0.$$

Note that for each $k \in \mathbb{N}$ the sum $\sum_{i=1}^{\infty} e_k^*(Te_i)$ converges since the sequence $(\sum_{i=1}^{n} e_i)_{n=1}^{\infty}$ is weakly Cauchy. It follows that $\mathcal{W}(J)$ is an SOT-Borel subset of $\mathcal{L}(J)$. Indeed, note that

$$T \in \mathcal{W}(J) \iff \forall m\ \exists K \text{ such that } \forall k \ge K, \forall n \in \mathbb{N}, |\sum_{i=1}^{n} e_k^*(Te_i)| \le \frac{1}{m}$$

Now, for each $m,k,n \in \mathbb{N}$, the set

$$A_{m,n,k} = \{T \in \mathcal{L}(J) : |\sum_{i=1}^{n} e_k^*(Te_i)| \le \frac{1}{m}\}$$

is SOT-closed in $\mathcal{L}(J)$. Thus $\mathcal{W}(J)$ is SOT-Borel.

Finally, since J since does not contain an ℓ_1 spreading model, Proposition 23 implies that $\mathcal{W}_1(J) = \mathcal{W}(J)$.

Example 27. In [8, Proposition 4.3], it is shown that for any Banach space X such that $\mathcal{L}(X) = \mathbb{R}I \oplus \mathcal{SS}(X)$, the set $\mathcal{SS}(X)$ is a SOT-Borel subset of $\mathcal{L}(X)$. Clearly if, in addition, $\mathcal{W}(X) = \mathcal{SS}(X)$ then $\mathcal{W}(X)$ is an SOT-Borel subset of $\mathcal{L}(X)$. In the papers [5, 16], the authors construct examples of non-reflexive HI spaces with the above property.

4.3. **Coanalytic complete ideals with bounded local rank.** In this section we show that the ranks $\varrho_{\mathcal{R}}$ and $\varrho_{\mathcal{W}}$ are not, in general, coanalytic ranks.

To get started, we recall the definiton of the James-tree space JT. Let JT be the completion of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ equipped with the norm

$$\|z\| = \sup\Big\{\Big(\sum_{i=1}^{d}\big(\sum_{t\in\mathfrak{s}_i} z(t)\big)^2\Big)^{1/2}\Big\}, \tag{11}$$

where the above supremum is taken over all families $(\mathfrak{s}_i)_{i=1}^d$ of pairwise incomparable non-empty segments of $\mathbb{N}^{<\mathbb{N}}$. A *segment* $\mathfrak{s}$ of $\mathbb{N}^{<\mathbb{N}}$ is a set consisting of pairwise comparable nodes of $\mathbb{N}^{<\mathbb{N}}$ satisfying

$$\forall s,t,s' \in \mathbb{N}^{<\mathbb{N}}\ (s \sqsubseteq t \sqsubseteq s' \text{ and } s,s' \in \mathfrak{s} \Rightarrow t \in \mathfrak{s}). \tag{12}$$

Let $(z_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ denote the standard Hamel basis of $c_{00}(\mathbb{N}^{<\mathbb{N}})$.

Let p and q with $1 \le p,q < \infty$ and define the following unconditional version of JT: Let $Z_{p,q}$ be the completion of $c_{00}(\mathbb{N}^{<\mathbb{N}})$ equipped with the norm

$$\|z\| = \sup\Big\{\Big(\sum_{i=1}^{d}\big(\sum_{t\in\mathfrak{s}_i} |z(t)|^p\big)^{q/p}\Big)^{1/q}\Big\}, \tag{13}$$

where the above supremum is taken over all families $(\mathfrak{s}_i)_{i=1}^d$ of pairwise incomparable non-empty segments of $\mathbb{N}^{<\mathbb{N}}$.

Theorem 28. [9, Theorem 4] *Let p such that $1 \le p < \infty$.*

a) *$\mathcal{SS}(\ell_p, Z_{p,2p})$ is a co-analytic complete (in particular, non-Borel) subset of $\mathcal{L}(\ell_p, Z_{p,2p})$.*
b) *$\mathcal{SS}(\ell_p, Z_{p,2p}) = \mathcal{SS}_2(\ell_p, Z_{p,2p})$.*

In particular, $\varrho_{\mathcal{SS}}$ is not a coanalytic rank on $\mathcal{SS}(X,Y)$.

Recall that a coanalytic set B of a Polish space P is coanalytic complete if there is a Borel map $H : \mathrm{Tr} \to P$ such that $H^{-1}(B) = \mathrm{WF}$. As WF is not Borel, coanalytic complete sets are not Borel. In the case of weakly compact operators, we have the following analogue of Theorem 28.

Theorem 29. *The following are satisfied:*

a) *$\mathcal{W}(JT)$ is a coanalytic complete subset of $\mathcal{L}(JT)$.*
b) *$\varrho_{\mathcal{W}}(\mathcal{W}(JT)) < \omega_1$; that is, there is countable ξ such that $\mathcal{W}(JT) = \mathcal{W}_\xi(JT)$.*

In particular, $\varrho_{\mathcal{W}}$ is not a coanalytic rank on $\mathcal{W}(JT)$.

Proof. It is well-known that JT does not contain ℓ_1. Therefore *b*) follows from Proposition 23.

Proposition 9 yields that $\mathcal{W}(JT)$ is a coanalytic subset of $\mathcal{L}(JT)$. Therefore, to prove item *a*), we need to define a Borel map $H : \mathrm{Tr} \to \mathcal{L}(X,Y)$ such that $H^{-1}(\mathcal{W}(X,Y)) = \mathrm{WF}$. To define H, we first set notation and collect facts concerning JT:

(i) For each $S \in \text{Tr}$, let $JT^S := \overline{span}\{z_t : t \in S\}$. If $S \in \text{WF}$, the space $JT^S = \overline{span}\{z_t : t \in S\}$ is reflexive.

(ii) Suppose $(\ell_i)_{i=1}^{\infty} \in [\mathbb{N}]^{\mathbb{N}}$. The sequence $(z_{(\ell_1,\dots,\ell_n)})_{n\in\mathbb{N}}$ is isometric to the summing basis of c_0.

(iii) For each $S \in \text{Tr}$, let P_S be the natural projection from JT to JT^S, note that $\|P_S\| = 1$.

Define $H : \text{Tr} \to \mathcal{L}(JT)$ by $H(S) = P_S$. It is easy to see that H is a Borel map. Let $S \in \text{Tr}$. We now that

$$S \in \text{WF} \iff P_S \in \mathcal{W}(JT). \tag{14}$$

Let $S \in \text{WF}$. By (i), JT^S is reflexive; so, $P_S \in \mathcal{W}(JT)$. Suppose that $S \notin \text{WF}$. Let $(\ell_i)_{i=1}^{\infty} \subset \mathbb{N}$ such that $(\ell_i)_{i=1}^{n} \in S$ for all $n \in \mathbb{N}$. By (ii), the sequence $\{z_{(\ell_1,\dots,\ell_n)} : n \in \mathbb{N}\}$ is isometric to the summing basis of c_0. Therefore P_S is not weakly compact. Thus (14) holds. We have that $H^{-1}(\mathcal{W}(JT)) = \text{WF}$, and hence $\mathcal{W}(JT)$ is a coanalytic complete subset of $\mathcal{L}(JT)$. $\square$

We have the following analogous result for $\mathcal{R}(X, Y)$.

Theorem 30. *The following are satisfied:*

a) *$\mathcal{R}(\ell_1, Z_{1,2})$ is a coanalytic complete subset of $\mathcal{L}(\ell_1, Z_{1,2})$.*

b) *$\mathcal{R}(\ell_1, Z_{1,2}) = \mathcal{R}_2(\ell_1, Z_{1,2})$.*

In particular, $\varrho_{\mathcal{R}}$ is not a coanalytic rank on $\mathcal{R}(\ell_1, Z_{1,2})$.

Proof. We will prove that for every countable ordinal ξ,

$$\mathcal{R}_\xi(\ell_1, Z_{1,2}) = \mathcal{SS}_\xi(\ell_1, Z_{1,2}). \tag{15}$$

This would imply that $\mathcal{R}(\ell_1, Z_{1,2}) = \mathcal{SS}(\ell_1, Z_{1,2})$, and our theorem would follow from Theorem 28. Thus we just need to prove (15).

Let ξ be a countable ordinal. Suppose $T \notin \mathcal{R}_\xi(\ell_1, Z_{1,2})$. There exists $\varepsilon > 0$ and $(x_i) \in \mathcal{NB}_{\ell_1}$ such that for each $F \in \mathcal{S}_\xi$ and scalars $(a_i)_{i\in F}$

$$\|\sum_{i\in F} a_i T x_{n_i}\| \geq \varepsilon \sum_{i\in F} |a_i|.$$

We thus have that,

$$\|\sum_{i\in F} a_i T x_{n_i}\| \geq \varepsilon \sum_{i\in F} |a_i| \geq \varepsilon \|\sum_{i\in F} a_i x_{n_i}\|.$$

It follows that $T \notin \mathcal{SS}_\xi(\ell_1, Z_{1,2})$.

Now suppose $T \notin \mathcal{SS}_\xi(\ell_1, Z_{1,2})$. Find $\varepsilon > 0$ and $(x_i) \in \mathcal{NB}_{\ell_1}$ such that for all $F \in \mathcal{S}_\xi$ and scalars $(a_i)_{i\in F}$,

$$\|\sum_{i\in F} a_i T x_i\| \geq \varepsilon \|\sum_{i\in F} a_i x_i\|.$$

Choose a subsequence (n_i) of $\mathbb{N}$ such that the following holds:

$$G \in S_\xi \implies \bigcup_{k\in G}\{n_{2k}, n_{2k+1}\} \in S_\xi. \tag{16}$$

By passing to a further subsequence of (n_k) and relabeling we can assume that the difference sequence $d_k = x_{n_{2k+1}} - x_{n_{2k}}$ is coordinate-wise convergent and seminormalized. By the Bessaga-Pełczyński selection principle [1, Theorem 1.3.10] we can assume that (d_n) is 2-equivalent to a block sequence of ℓ_1 and therefore 2-equivalent to the unit vector basis of ℓ_1. Let $G \in S_\xi$ and $(a_i)_{i\in G}$ be a scalar sequence. By (16), $\cup_{k\in G}\{n_{2k}, n_{2k+1}\} \in S_\xi$. Therefore

$$\|\sum_{i\in G} a_i T d_i\| = \|\sum_{i\in G} a_i T(x_{n_{2i+1}} - x_{n_{2i}})\| \geq \varepsilon\|\sum_{i\in G} a_i d_i\| \geq \frac{\varepsilon}{2}\sum_{i\in G}|a_i|.$$

It follows that $T \notin \mathcal{R}_\xi(\ell_1, Z_{1,2})$ □

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