

# UNIFORMLY FACTORING WEAKLY COMPACT OPERATORS

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ABSTRACT. Let  $X$  and  $Y$  be separable Banach spaces. Suppose  $Y$  either has a shrinking basis or  $Y$  is isomorphic to  $C(2^{\mathbb{N}})$  and  $\mathcal{A}$  is a subset of weakly compact operators from  $X$  to  $Y$  which is analytic in the strong operator topology. We prove that there is a reflexive space with a basis  $Z$  such that every  $T \in \mathcal{A}$  factors through  $Z$ . Likewise, we prove that if  $\mathcal{A} \subset \mathcal{L}(X, C(2^{\mathbb{N}}))$  is a set of operators whose adjoints have separable range and is analytic in the strong operator topology then there is a Banach space  $Z$  with separable dual such that every  $T \in \mathcal{A}$  factors through  $Z$ . Finally we prove a uniform version of this result in which we allow the domain and range spaces to vary.

## 1. INTRODUCTION

Recall that if  $X$  and  $Y$  are Banach spaces then a bounded operator  $T : X \rightarrow Y$  is called *weakly compact* if  $\overline{T(B_X)}$  is weakly compact, where  $B_X$  is the unit ball of  $X$ . If there exists a reflexive Banach space  $Z$  and bounded operators  $T_1 : X \rightarrow Z$  and  $T_2 : Z \rightarrow Y$  with  $T = T_2 \circ T_1$  then  $T_1$  and  $T_2$  are both weakly compact by Alaoglu's theorem and hence  $T : X \rightarrow Y$  is weakly compact as well. Thus it is immediate that any bounded operator which factors through a reflexive Banach space is weakly compact. In their seminal 1974 paper [11], Davis, Figiel, Johnson and Pełczyński proved that the converse is true as well. That is, every weakly compact operator factors through a reflexive Banach space. Likewise, every bounded operator whose adjoint has separable range factors through a Banach space with separable dual. Using the DFJP interpolation technique, in 1988 Zippin proved that every separable reflexive Banach space embeds into a reflexive Banach space with a basis and that every Banach space with separable dual embeds into a Banach space with a shrinking basis [30].

For each separable reflexive Banach space  $X$  we may choose a reflexive Banach  $Z$  with a basis such that  $X$  embeds into  $Z$ . It is natural to consider when the choice of  $Z$  can be done uniformly. That is, given a set of separable reflexive Banach spaces  $\mathcal{A}$ , when does there exist a reflexive Banach space  $Z$  with a basis such that  $X$  embeds into  $Z$  for every  $X \in \mathcal{A}$ ? Szlenk proved that there does not exist a Banach space  $Z$  with separable dual such that every separable reflexive Banach

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space embeds into  $Z$  [29]. Bourgain proved further that if  $Z$  is a separable Banach space such that every separable reflexive Banach space embeds into  $Z$  then every separable Banach space embeds into  $Z$  [9]. Thus, any uniform embedding theorem must consider strict subsets of the set of separable reflexive Banach spaces. In his Phd thesis, Bossard developed a framework for studying sets of Banach spaces using descriptive set theory [8, 7]. In this context, it was shown in [14] and [28] that if  $\mathcal{A}$  is an analytic set of separable reflexive Banach spaces then there exists a separable reflexive Banach space  $Z$  such that  $X$  embeds into  $Z$  for all  $X \in \mathcal{A}$ , and in [14] and [16] it was shown that if  $\mathcal{A}$  is an analytic set of Banach spaces with separable dual then there exists a Banach space  $Z$  with separable dual such that  $X$  embeds into  $Z$  for all  $X \in \mathcal{A}$ . In particular, solving an open problem posed by Bourgain [9], there exists a separable reflexive Banach space  $Z$  such that every separable uniformly convex Banach space embeds into  $Z$  [27]. As the set of all Banach spaces which embed into a fixed Banach space is analytic in the Bossard framework, these uniform embedding theorems are optimal.

The goal for this paper is to return to the original operator factorization problem with the same uniform perspective that was applied to the embedding problems. That is, given separable Banach spaces  $X$  and  $Y$  and a set of weakly compact operators  $\mathcal{A} \subset \mathcal{L}(X, Y)$ , we want to know when does there exist a reflexive Banach space  $Z$  such that  $T$  factors through  $Z$  for all  $T \in \mathcal{A}$ . We are able to answer this question in the following cases.

**Theorem 1.** *Let  $X$  and  $Y$  be separable Banach spaces and let  $\mathcal{A}$  be a set of weakly compact operators from  $X$  to  $Y$  which is analytic in the strong operator topology. Suppose either  $Y$  has a shrinking basis or  $Y$  is isomorphic to  $C(2^{\mathbb{N}})$ . Then there is a reflexive Banach space  $Z$  with a basis such that every  $T \in \mathcal{A}$  factors through  $Z$ .*

**Theorem 2.** *Let  $X$  be a separable Banach space and let  $\mathcal{A} \subset \mathcal{L}(X, C(2^{\mathbb{N}}))$  be a set of bounded operators whose adjoints have separable range which is analytic in the strong operator topology. Then there is a Banach space  $Z$  with a shrinking basis such that every  $T \in \mathcal{A}$  factors through  $Z$ .*

The idea of factoring all operators in a set through a single Banach space has been considered previously for compact operators and compact sets of weakly compact operators [3, 18, 26]. In particular, Johnson constructed a reflexive Banach space  $Z_K$  such that if  $X$  and  $Y$  are Banach spaces and either  $X^*$  or  $Y$  has the approximation property then every compact operator  $T : X \rightarrow Y$  factors through  $Z_K$  [20]. Later, Figiel showed that if  $X$  and  $Y$  are Banach spaces and  $T : X \rightarrow Y$  is a compact operator, then  $T$  factors through a subspace of  $Z_K$  [15]. It is particularly interesting that the space  $Z_K$  is independent of the Banach spaces  $X$  and  $Y$ . In [10], Brooker proves that for every countable ordinal  $\alpha$ , if  $X$  and  $Y$  are separable Banach spaces and  $T : X \rightarrow Y$  is a bounded operator with Szlenk index at most  $\omega^\alpha$  then  $T$  factors through a Banach space with separable dual and Szlenk index

at most  $\omega^{\alpha+1}$ . This result, combined with the embedding result in [16] gives that for every countable ordinal  $\alpha$ , there exists a Banach space  $Z$  with a shrinking basis such that every bounded operator with Szlenk index at most  $\omega^\alpha$  factors through a subspace of  $Z$ . In section 4 we present generalizations of Theorems 1 and 2 where the Banach space  $X$  is allowed to vary.

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## 2. PRELIMINARIES

A topological space  $P$  is called a *Polish* space if it is separable and completely metrizable. A set  $X$ , together with a  $\sigma$ -algebra  $\Sigma$ , is called a *standard Borel space* if the measurable space  $(X, \Sigma)$  is Borel isomorphic to a Polish space. A subset  $A \subset X$  is said to be *analytic* if there exists a Polish space  $P$  and a Borel map  $f : P \rightarrow X$  with  $f(P) = A$ . A subset of  $X$  is said to be *coanalytic* if its complement is analytic.

Given some Polish space  $X$ , we will be studying sets of closed subspaces of  $X$ . Thus, from a descriptive set theory point of view, it is natural to assign a  $\sigma$ -algebra to the set of closed subsets of  $X$  which then forms a standard Borel space. Let  $F(X)$  denote the set of closed subspaces of  $X$ . The *Effros-Borel  $\sigma$ -algebra*,  $E(X)$ , is defined as the collection of sets with the following generator

$$\{\{F \in F(X) : F \cap U \neq \emptyset\} : U \subset X \text{ is open}\}.$$

The measurable space  $(F(X), E(X))$  is a standard Borel space. If  $X$  is a Banach space, then  $Subs(X)$  denotes the standard Borel space consisting of the closed subspaces of  $X$  endowed with the relative Effros-Borel  $\sigma$ -algebra. As every separable Banach space is isometric to a subspace of  $C(2^{\mathbb{N}})$ , the standard Borel space  $SB = Subs(C(2^{\mathbb{N}}))$  is of particular importance when studying sets of separable Banach spaces.

If  $X$  and  $Y$  are separable Banach spaces, then the space  $\mathcal{L}(X, Y)$  of all bounded linear operators from  $X$  to  $Y$  carries a natural structure as a standard Borel space whose Borel sets coincide with the Borel sets generated by the strong operator topology (i.e. the topology of pointwise convergence on nets). In this paper when we refer to a Borel subset of  $\mathcal{L}(X, Y)$  it is understood that this is with respect to the Borel  $\sigma$ -algebra generated by the strong operator topology. There are several papers in which  $\mathcal{L}(X, Y)$  is considered with this structure [4, 5, 6].

Both the set of all separable reflexive Banach spaces and the set of all Banach spaces with separable dual are coanalytic subsets of  $SB$ . This fact is essential in the proofs of the universal embedding theorems for analytic sets of separable reflexive Banach spaces and analytic sets of Banach spaces with separable dual [14],[16],[28].

Thus, we will naturally need the following theorem to prove our universal factorization results for analytic sets of weakly compact operators and analytic sets of operators whose adjoints have separable range.

**Proposition 3.** *For  $X, Y \in SB$  the following are coanalytic subsets of  $\mathcal{L}(X, Y)$ .*

- (a) *The set of weakly compact operators.*
- (b) *The set of operators whose adjoints have separable range (these operators are called Asplund operators).*

Before proving Proposition 3, we will need to introduce some more results from descriptive set theory. Given a Polish space  $E$ , let  $K(E)$  be the space of all compact subsets of  $E$ . The space  $K(E)$  is Polish when equipped with the Vietoris topology, which is the topology on  $K(E)$  generated by the sets

$$\{\{K \in K(E) : K \cap U \neq \emptyset\} : U \subset E \text{ is open}\} \text{ and } \{\{K \in K(E) : K \subseteq U\} : U \subset E \text{ is open}\}.$$

When studying sequences in the unit ball of a separable Banach space  $X$ , we note that the space  $B_X^{\mathbb{N}}$  is a Polish space when endowed with the product topology. We will always consider  $B_{\ell_\infty}$ , the ball of  $\ell_\infty$ , to be equipped with the weak\* topology. In [7] (also see [12, Theorem 2.11]) Bossard proves the following theorem. This result is used to prove that the Szlenk index is a  $\Pi_1^1$  rank of the collection of Banach spaces with separable duals.

**Theorem 4** ([7]). *The set*

$$\Sigma = \{K \in K(B_{\ell_\infty}) : K \text{ is norm-separable}\}$$

*is coanalytic in the Vietoris topology of  $K(B_{\ell_\infty})$ .*

We will show that for all  $X, Y \in SB$  the set of operators whose adjoints have separable range is coanalytic in  $\mathcal{L}(X, Y)$  by showing that the set is Borel reducible to  $\Sigma \subset K(B_{\ell_\infty})$ . To do this, we will define a map  $\Phi : \mathcal{L}(X, Y) \rightarrow K(B_{\ell_\infty})$ , and use the following theorem to show that it is Borel.

**Theorem 5.** [22, Theorem 28.8] *Let  $X$  and  $Y$  be Polish spaces and  $A \subset Y \times X$  be such that for each  $y \in Y$  the set  $A_y = \{x \in X : (y, x) \in A\}$  is compact. Consider the map  $\Phi_A : Y \rightarrow K(X)$  defined by  $\Phi_A(y) = A_y$ . Then  $A$  is Borel if and only if  $\Phi_A$  is a Borel map.*

By the Kuratowski and Ryll-Nardzewski selection theorem [23] we can find a sequence of Borel maps  $(s_n)_{n \in \mathbb{N}}$  such that  $s_n : F(C(2^{\mathbb{N}})) \rightarrow C(2^{\mathbb{N}})$  for each  $n \in \mathbb{N}$  and  $(s_n(E))_{n=1}^\infty$  is dense in  $E$ . In addition, for all  $n \in \mathbb{N}$  let  $d_n : SB \rightarrow C(2^{\mathbb{N}})$  be a Borel map such that  $(d_n(X))_{n \in \mathbb{N}}$  is dense in  $B_X$  for all  $X \in SB$  and for  $p, q \in \mathbb{Q}$  and  $m, k \in \mathbb{N}$  if  $qd_m(X) + pd_k(X) \in B_X$  then there is an  $\ell \in \mathbb{N}$  with  $d_\ell(X) = qd_m(X) + pd_k(X)$ . We will also assume that  $d_n(X) \neq 0$  for all  $X \in SB$  and  $n \in \mathbb{N}$ . Working with the sequences  $(s_n)$  and  $(d_n)$  will be easier for us than dealing with the Effros-Borel  $\sigma$ -algebra or Vietoris topology directly.

*Proof of Proposition 3.* Item (a) is proved in [6, Proposition 9] and follows from the fact that weakly compact operators are exactly those operators that take bounded sequences in  $X$  to sequences that do not dominate the summing basis of  $c_0$ .

The proof of (b) requires a bit more effort, but it follows the same outline as the proof that the collection of all spaces with separable dual (SD) is coanalytic [8]. Let  $\mathcal{A}(X, Y)$  denote the collection of operators in  $\mathcal{L}(X, Y)$  whose adjoints have separable range. Let  $B_{\mathcal{L}(X, Y)}$  denote the unit ball of  $\mathcal{L}(X, Y)$ . It is enough to prove that the collection of  $T \in \mathcal{A}(X, Y)$  with  $\|T\| \leq 1$  is coanalytic as a subset of  $B_{\mathcal{L}(X, Y)}$ . For  $T \in B_{\mathcal{L}(X, Y)}$  and  $y^* \in B_{Y^*}$  let

$$f_{T^*y^*} = \left( \frac{T^*y^*(d_i(X))}{\|d_i(X)\|} \right)_{i=1}^{\infty} \in B_{\ell_{\infty}}.$$

For  $T \in B_{\mathcal{L}(X, Y)}$  let  $K_T = \{f_{T^*y^*} : y^* \in B_{Y^*}\}$ . Notice that  $K_T$  can be identified with  $T^*(B_{Y^*})$  via the homeomorphism  $T^*(y^*) \mapsto f_{T^*y^*}$ . Here  $T^*(B_{Y^*})$  is endowed with the weak\* topology. So,  $K_T$  is compact in  $B_{\ell_{\infty}}$  with the weak\* topology. Define  $D_{\mathcal{L}} \subset B_{\mathcal{L}(X, Y)} \times B_{\ell_{\infty}}$  as follows

$$(T, f) \in D_{\mathcal{L}} \iff f \in K_T.$$

Using the following characterization, the set  $D_{\mathcal{L}}$  is Borel.

$$(T, f) \in D_{\mathcal{L}} \iff \forall n, m, k \in \mathbb{N} \forall q, p \in \mathbb{Q} \text{ we have}$$

$$(pTd_n(X) + qTd_m(X) = Td_k(X) \implies$$

$$p\|d_n(X)\|f(n) + q\|d_m(X)\|f(m) = \|d_k(X)\|f(k)).$$

Notice that for each  $T \in B_{\mathcal{L}(X, Y)}$  the set  $D_T = \{f : (T, f) \in D_{\mathcal{L}}\}$  is equal to  $K_T$  and is therefore compact. Applying Theorem 5,  $\Phi : \mathcal{L}(X, Y) \rightarrow K(B_{\ell_{\infty}})$  defined by  $\Phi(T) = K_T$  is a Borel map. Finally, note that

$$T \in \mathcal{A}(X, Y) \iff \Phi(T) = K_T \in \Sigma = \{K \in K(B_{\ell_{\infty}}) : K \text{ is norm-separable}\}.$$

Using Proposition 4 we have that  $\mathcal{A}(X, Y)$  with  $\|T\| \leq 1$  is Borel reducible to a coanalytic set and is hence itself coanalytic.  $\square$

Concerning reflexive Banach spaces with bases as well as Banach spaces with bases and separable dual, Argyros and Dodos [2, Theorems 83, 84 and 91] proved the following deep theorem (see also [12, Theorems 7.4 and 7.8]).

**Theorem 6** ([2]). *Let  $\mathcal{A} \subset SB$  be an analytic collection of reflexive Banach spaces (resp. Banach spaces with shrinking bases) such that each  $X \in \mathcal{A}$  has a basis. Then there is a reflexive Banach space  $Z_{\mathcal{A}}$  (resp. Banach space with shrinking bases) with a basis that contains every  $X \in \mathcal{A}$  as a complemented subspace.*

Although it is possible for us to apply Theorem 6 as a black box, we give some brief description here about how the space  $Z_{\mathcal{A}}$  is constructed. Let  $\mathcal{A} \subset SB$  be an analytic collection of reflexive spaces. Since the map from  $S_{C(2^{\mathbb{N}})}^{\mathbb{N}}$  to  $SB$  given by  $(x_n)_{n \in \mathbb{N}} \mapsto [x_n]_{n \in \mathbb{N}}$  is Borel and the set of basic sequences in a Banach space is

Borel, we obtain an analytic set  $\mathcal{B}$  of basic sequences in  $S_{C(2^{\mathbb{N}})}$  such that for every reflexive Banach space  $X \in \mathcal{A}$  there exists  $(x_n) \in \mathcal{B}$  such that  $(x_n)$  is a basis for  $X$  and for every  $(x_n) \in \mathcal{B}$  we have that  $[x_n] \in \mathcal{A}$ . Instead of working with an analytic collection of Banach spaces  $\mathcal{A}$ , we can now work with an analytic collection of basic sequences  $\mathcal{B}$ . Therefore the following theorems imply the previous one.

**Theorem 7** ([2]). *Let  $\mathcal{A} \subset C(2^{\mathbb{N}})^{\mathbb{N}}$  be an analytic collection of shrinking and boundedly complete basic sequences. There is a reflexive Banach space  $Z$  with a basis  $(z_n)$  such that if  $(x_n) \in \mathcal{A}$  then there exists a subsequence  $(k_n)$  of  $\mathbb{N}$  such that  $(x_n)$  is equivalent to  $(z_{k_n})$  and  $[z_{k_n}]$  is complemented in  $Z$ .*

**Theorem 8** ([2]). *Let  $\mathcal{A} \subset C(2^{\mathbb{N}})^{\mathbb{N}}$  be an analytic collection of shrinking basic sequences. There is a Banach space  $Z$  with a shrinking basis  $(z_n)$  such that if  $(x_n) \in \mathcal{A}$  then there exists a subsequence  $(k_n)$  of  $\mathbb{N}$  such that  $(x_n)$  is equivalent to  $(z_{k_n})$  and  $[z_{k_n}]$  is complemented in  $Z$ .*

To prove these theorems, Argyros and Dodos, give a procedure to amalgamate an analytic collection of basic sequences  $\mathcal{B}$  into a tree basis  $(x_\alpha)_{\alpha \in Tr}$ , where  $Tr$  is a finitely branching tree. That is, they construct  $(x_\alpha)_{\alpha \in Tr} \subset S_{C(2^{\mathbb{N}})}$  such that  $(x_\alpha)_{\alpha \in Tr} \subset S_{C(2^{\mathbb{N}})}$  is a basic sequence under any ordering which preserves the tree order,  $[x_\alpha]$  is reflexive, and every  $(x_n) \in \mathcal{B}$  is equivalent to a branch of  $(x_\alpha)_{\alpha \in Tr}$ . Moreover, every branch of the tree is bounded complete and shrinking basis from our analytic collection. Furthermore, if  $(\alpha_n)_{n \in \mathbb{N}}$  is a branch of  $Tr$  then the restriction operator  $P : [x_\alpha]_{\alpha \in Tr} \rightarrow [x_{\alpha_n}]_{n \in \mathbb{N}}$  given by  $P(\sum a_\alpha x_\alpha) = \sum a_{\alpha_n} x_{\alpha_n}$  is a bounded projection.

Given an analytic collection  $\mathcal{A}$  of weakly compact operators, our goal is to obtain an analytic collection  $\mathcal{B}$  of normalized shrinking and boundedly complete basic sequences such that for every  $T \in \mathcal{A}$ , there exists  $(x_n) \in \mathcal{B}$  such that  $T$  factors through  $[x_n]$ . We then are able to apply Theorem 7 and obtain a separable reflexive Banach space  $Z$  such that every  $T \in \mathcal{A}$  factors through a complemented subspace of  $Z$ . Hence, every  $T \in \mathcal{A}$  factors through  $Z$  itself. This idea of creating a complementably universal Banach space  $Z$  in order to lift operators defined on an analytic collection of Banach spaces with bases was used by Dodos in [13], where he characterizes the sets of separable Banach spaces  $\mathcal{C}$  for which there exist a separable Banach space  $Z$  such that  $\ell_1$  does not embed into  $Z$  and every  $X \in \mathcal{C}$  is a quotient of  $Z$ .

### 3. PARAMETRIZED FACTORIZATION

**Notation 1.** In the rest of the paper we set the following notation.

- (a)  $X$  denotes a separable Banach space,  $Y$  denotes a Banach space with a Schauder basis and  $y_0 \in Y$ . The vector  $y_0$  will be prescribed depending on the space  $Y$  which we are considering.

- (b) Let  $T \in \mathcal{L}(X, Y)$ . Denote by  $(y_n^T)_{n \in \mathbb{N}}$  a normalized basis of  $Y$  that depends on  $T$  and for  $k \in \mathbb{N}$ , let  $P_k^T : Y \rightarrow [y_n^T : n \leq k]$  be the natural projection.
- (c) Define the following set depending on  $T$  and  $y_0$ .

$$E_T := \overline{\text{co}(T(B_X) \cup \{\pm y_0\})}.$$

- (d) The following set depends on the basis  $(y_n^T)$  and  $E_T$

$$W_T = \bigcup_{k \in \mathbb{N}} P_k^T(E_T).$$

Note that  $W_T$  is closed, bounded, convex and symmetric. Also,  $P_k^T(W_T) \subset W_T$  for each  $k \in \mathbb{N}$ .

- (e) Let  $W \subset Y$  be closed, convex, bounded and symmetric and for each  $m \in \mathbb{N}$  define

$$W^m := 2^m W + 2^{-m} B_Y.$$

- (f) Let  $\|\cdot\|_{W^m}$  denote the Minkowski gauge norm of the set  $W^m$ . That is,

$$\|y\|_{W^m} = \inf\{\lambda > 0 : \frac{y}{\lambda} \in W^m\}.$$

- (g) Let

$$Z_T = \{z \in Y : \sum_{m=1}^{\infty} \|z\|_{W_T^m}^2 < \infty\} \text{ and } \|z\|_T = \left( \sum_{m=1}^{\infty} \|z\|_{W_T^m}^2 \right)^{\frac{1}{2}}.$$

The following items are proved in [11]. The reader may also consult [12, Appendix B] for a nice treatment of this material.

**Theorem 9** ([11]). *The following hold.*

- (a) *There exist  $T_1 : X \rightarrow Z_T$  and  $T_2 : Z_T \rightarrow Y$  such that  $T = T_2 T_1$ ; in other words,  $T$  factors through  $Z_T$ . Furthermore,  $T_2$  is constructed to be one-to-one.*
- (b) *If  $y_0 = \sum_k a_k y_k^T$  and  $a_k \neq 0$  for each  $k \in \mathbb{N}$  then  $y_n^T \in \text{span } W_T$  for each  $n \in \mathbb{N}$ . Let  $z_n^T = T_2^{-1}(y_n^T)$  for each  $n \in \mathbb{N}$  (this is well defined as  $T_2$  is one-to-one). The sequence  $(z_n^T)_{n \in \mathbb{N}}$  is a (not normalized) basis for  $Z_T$ .*
- (c) *The space  $Z_T$  is reflexive if and only if  $W_T$  is weakly compact.*
- (d) *If  $T$  is weakly compact and  $(y_n^T)$  shrinking then  $W_T$  weakly compact.*

We sketch of the first part of (b). Note that  $y_0 \in W_T$  and  $P_1^T(y_0) = a_1 y_1^T$ . Hence  $y_1^T \in \text{span } W_T$  because  $P_1^T(W_T) \subset W_T$ . Also, for  $n > 1$ ,  $(P_n^T - P_{n-1}^T)y_0 = a_n y_n^T \in W_T - W_T$ . Thus  $y_n^T \in \text{span } W_T$  for each  $n \in \mathbb{N}$ .

Item (b) above essentially states that including a vector  $y_0$  in  $W_T$  that is supported on all coordinates of the basis of  $Y$  ensures that the interpolation space  $Z_T$  has a basis, which we denote  $(z_n^T)$ . Therefore whenever we refer to  $Z_T$  with basis  $(z_n^T)$  it is assumed that the vector  $y_0$  contained in  $E_T$  satisfies the assumptions of item (b) with respect to the basis  $(y_n^T)$  of  $Y$ .

In the next lemma, however,  $y_0$  can be any vector in  $Y$ .

**Lemma 10.** *Let  $\mathcal{B} \subset \mathcal{L}(X, Y)$  be Borel. Then the following hold:*

- (a) *The map  $\mathcal{B} \ni T \mapsto E_T \in F(Y)$  is Borel.*
- (b) *The map  $\mathcal{B} \ni T \mapsto W_T \in F(Y)$  is Borel. Moreover, for each  $m \in \mathbb{N}$  the map  $\mathcal{B} \ni T \mapsto W_T^m \in F(Y)$  is Borel.*
- (c) *The map  $\mathcal{B} \times Y \ni (T, y) \mapsto \|y\|_{W_T^m}$  is Borel.*

The proof of Lemma 10 will rely on the following elementary observation from Descriptive Set Theory [12, page 83].

**Fact 11.** *Suppose  $E$  is a standard Borel space,  $P$  is a Polish space and for each  $n \in \mathbb{N}$ ,  $f_n : E \rightarrow P$  is a Borel map. Then the map  $\Phi : E \rightarrow F(P)$  defined by  $\Phi(x) = \overline{\{f_n(x) : n \in \mathbb{N}\}}$  for all  $x \in E$  is Borel.*

*Proof of Lemma 10(a).* Let  $(x_n)_{n \in \mathbb{N}}$  be dense in  $B_X$  and let  $U$  be a non-empty open subset of  $Y$ . Notice that

$$E_T \cap U \neq \emptyset \iff \exists n \in \mathbb{N}, q_1, q_2 \in \mathbb{Q} \cap [0, 1], \delta \in \{\pm 1\} \text{ with} \\ q_1 + q_2 = 1, q_1(Tx_n) + \delta q_2 y_0 \in U.$$

For  $n \in \mathbb{N}$  and  $q_1, q_2 \in \mathbb{Q} \cap [0, 1]$  we define  $\tau_{n, q_1, q_2} : \mathcal{B} \rightarrow Y^2$  and  $p : Y^2 \rightarrow Y$  by

$$\tau_{n, q_1, q_2}(T) = (q_1 Tx_n, \delta q_2 y_0) \text{ and } p((z_1, z_2)) = z_1 + z_2.$$

Using the definition of strong operator topology, the map  $\tau_{n, q_1, q_2}$  is Borel. The map  $p$  is continuous, and hence  $p \circ \tau_{n, q_1, q_2}$  is Borel. The set  $\{p \circ \tau_{n, q_1, q_2}(T) : n \in \mathbb{N} \text{ and } q \in \mathbb{Q} \cap [0, 1]\}$  is dense in  $E_T$ . Hence, the map  $\mathcal{B} \ni T \mapsto E_T \in F(Y)$  is Borel by Fact 11.  $\square$

*Proof of Lemma 10(a).* Let  $(x_n)_{n \in \mathbb{N}}$  be dense in  $B_X$  and let  $U$  be a non-empty open subset of  $Y$ . For  $n, k \in \mathbb{N}$  and  $q_1, q_2 \in \mathbb{Q} \cap [0, 1]$  we define the map  $f_{n, k, q} : \mathcal{B} \rightarrow Y$  by

$$f_{n, k, q}(T) = P_k(q_1 Tx_n + q_2 y_0).$$

Using the same argument used in the proof of Lemma 10(b), we have that  $f_{n, k, q}$  is a Borel map. The set  $\{f_{n, k, q}(T) : n, k \in \mathbb{N} \text{ and } q_1, q_2 \in \mathbb{Q} \cap [0, 1]\}$  is dense in  $W_T$ . Hence, the map  $\mathcal{B} \ni T \mapsto W_T \in F(Y)$  is Borel by Fact 11. The same argument gives that the map  $\mathcal{B} \ni T \mapsto W_T^m \in F(Y)$  is Borel for each  $m \in \mathbb{N}$ .  $\square$

*Proof of Lemma 10(c).* Let  $r \in \mathbb{R}$  with  $r > 0$  and notice that for  $(W, y) \in F(Y) \times Y$

$$\|y\|_W < r \iff \exists q \in \mathbb{Q} \text{ with } 0 < q < r \text{ and } y \in qW.$$

Thus, the map  $F(Y) \times Y \ni (W, y) \mapsto \|y\|_W$  is Borel as  $qW$  is closed. The map  $(T, y) \mapsto (W_T^m, y)$  is Borel by part (c). Hence, the map  $(T, y) \mapsto \|y\|_{W_T^m}$  is Borel.  $\square$

The next remark follows directly from the definition of the basis (see Theorem 9(b)).



**Remark 12.** Suppose  $(y_n^T)$  is a basis for  $Y$  and  $y_0 = \sum_{k \in \mathbb{N}} a_k y_k^T$  where  $a_k \neq 0$  for all  $k \in \mathbb{N}$ . Let  $(z_n^T)$  be the basis give by Theorem 9(b). Then a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $C(2^{\mathbb{N}})$  is 1-equivalent to  $(z_n^T)_{n \in \mathbb{N}}$  if and only if for each  $(a_n)_n \in c_{00}$

$$\sum_{m=1}^{\infty} \left\| \sum_{n=1}^{\infty} a_n y_n^T \right\|_{W_T^m}^2 = \left\| \sum_{n=1}^{\infty} a_n x_n \right\|^2$$

**Lemma 13.** Let  $\mathcal{B} \subset \mathcal{L}(X, Y)$  be a Borel set, suppose that the map  $\mathcal{B} \ni T \mapsto (y_n^T)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  is Borel and the vector  $y_0 \in Y$  has the property that for every  $T \in \mathcal{B}$ ,  $y_0 = \sum_{k \in \mathbb{N}} a_k y_k^T$  with  $a_k \neq 0$ . Then the following set

$$\mathcal{F} = \{(T, (x_n)) \in \mathcal{B} \times C(2^{\mathbb{N}})^{\mathbb{N}} : (z_n^T) \text{ is 1-equivalent to } (x_n)\}$$

is Borel in  $\mathcal{L}(X, Y) \times C(2^{\mathbb{N}})^{\mathbb{N}}$ .

*Proof.* For  $k, p, N \in \mathbb{N}$  and  $a = (a_1, \dots, a_k) \in \mathbb{Q}^k$ , we let

$$A_{k,N,a} = \left\{ (T, (x_n)) \in \mathcal{B} \times C(2^{\mathbb{N}})^{\mathbb{N}} : \sum_{1 \leq m \leq N} \left\| \sum_{n=1}^k a_n y_n^T \right\|_{W_T^m}^2 \leq \left\| \sum_{n=1}^k a_n x_n \right\|^2 \right\} \text{ and}$$

$$B_{k,p,N,a} = \left\{ (T, (x_n)) \in \mathcal{B} \times C(2^{\mathbb{N}})^{\mathbb{N}} : \left\| \sum_{n=1}^k a_n x_n \right\|^2 - \frac{1}{p} \leq \sum_{1 \leq m \leq N} \left\| \sum_{n=1}^k a_n y_n^T \right\|_{W_T^m}^2 \right\}.$$

The sets  $A_{k,N,a}, B_{k,p,N,a} \subset \mathcal{B} \times C(2^{\mathbb{N}})^{\mathbb{N}}$  are Borel as the maps  $T \mapsto (y_n^T)_{n \in \mathbb{N}}$  and  $(T, y) \mapsto \|y\|_{W_T^m}$  are Borel. By Remark 12, we have that  $\mathcal{F} = \bigcap_{k,N \in \mathbb{N}; a \in \mathbb{Q}} A_{k,N,a} \cap \bigcap_{k,p \in \mathbb{N}; a \in \mathbb{Q}} \bigcup_{N \in \mathbb{N}} B_{k,p,N,a}$ , and hence  $\mathcal{F}$  is Borel.  $\square$

The next proposition is our main tool for proving Theorems 1 and 2.

**Proposition 14.** Suppose that  $\mathcal{B} \subset \mathcal{L}(X, Y)$  is a Borel collection of weakly compact operators (resp. operators whose adjoints have separable range) such that

- (a) The map  $\mathcal{B} \ni T \mapsto (y_n^T)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  is Borel.
- (b) The vector  $y_0 \in Y$  has the property that for every  $T \in \mathcal{B}$ ,  $y_0 = \sum_{k \in \mathbb{N}} a_k y_k^T$  with  $a_k \neq 0$ .
- (c) The basis  $(z_n^T)$  (see Theorem 9(b)) is boundedly complete and shrinking (resp. shrinking).

Then there is a reflexive space (resp. space with separable dual) with a basis  $Z_{\mathcal{B}}$  such that each  $T \in \mathcal{B}$  factors through  $Z_{\mathcal{B}}$ .

*Proof.* We prove the weakly compact case. The case of operators whose adjoints have separable range is analogous. By Lemma 13, the set

$$\{(T, (x_n)) \in \mathcal{B} \times C(2^{\mathbb{N}})^{\mathbb{N}} : (z_n^T) \text{ is 1-equivalent to } (x_n)\}$$

is Borel in  $\mathcal{L}(X, Y) \times C(2^{\mathbb{N}})^{\mathbb{N}}$ . Hence, the set

$$\mathcal{Z}_{\mathcal{B}} = \{(x_n) \in C(2^{\mathbb{N}})^{\mathbb{N}} : \exists T \in \mathcal{B} \text{ such that } (z_n^T) \text{ is 1-equivalent to } (x_n)\}$$

is analytic in  $C(2^{\mathbb{N}})^{\mathbb{N}}$ . By Theorem 7 there is a reflexive space  $Z_{\mathcal{B}}$  such that if  $(z_n^T) \in Z_{\mathcal{B}}$  the space  $Z_T$  is isomorphic to a complemented subspace of  $Z_{\mathcal{B}}$ . That is, there exists an embedding  $I_T : Z_T \rightarrow Z_{\mathcal{B}}$  and a bounded projection  $P_T : Z_{\mathcal{B}} \rightarrow I_T(Z_T)$ . Given the factorization  $T_1 : X \rightarrow Z_T$  and  $T_2 : Z_T \rightarrow Y$  with  $T = T_2 T_1$ , we now have the factorization  $I_T T_1 : X \rightarrow Z_{\mathcal{B}}$  and  $T_2 I_T^{-1} P_T : Z_{\mathcal{B}} \rightarrow Y$  with  $T = T_2 I_T^{-1} P_T I_T T_1$ . Thus, each  $T \in \mathcal{B}$  factors through  $Z_{\mathcal{B}}$ .  $\square$

**Theorem 15.** *Suppose  $Y$  has a shrinking basis and  $\mathcal{A} \subset \mathcal{L}(X, Y)$  is an analytic collection of weakly compact operators. Then there is a reflexive space with a basis  $Z_{\mathcal{A}}$  such that each  $T \in \mathcal{A}$  factors through  $Z_{\mathcal{A}}$ .*

*Proof.* By Proposition 3 the collection of all weakly compact operators from  $X$  to  $Y$  (for separable  $X$  and  $Y$ ) is coanalytic. Using Lusin's separation theorem [22, Theorem 28.1] there is a Borel set  $\mathcal{B}$  of weakly compact operators such that  $\mathcal{A} \subset \mathcal{B}$ .

It suffices to show that the hypotheses of Proposition 14 are satisfied for  $\mathcal{B}$ . Let  $(y_n)$  be a shrinking basis for  $Y$ . Let  $y_0 = \sum_{k \in \mathbb{N}} \frac{1}{2^k} y_k$ . For each  $T \in \mathcal{B}$ , set  $y_n^T = y_n$  for each  $n \in \mathbb{N}$ . Clearly,  $T \mapsto (y_n^T)_{n \in \mathbb{N}}$  is Borel, as it is constant. Using Theorem 9, for each  $T \in \mathcal{B}$  the space  $Z_T$  is reflexive and has a basis  $(z_n^T)_{n \in \mathbb{N}}$ . Therefore the hypotheses of Proposition 14 are satisfied. This finishes the proof.  $\square$

Next we prove Theorem 1 and Theorem 2 for  $Y = C(2^{\mathbb{N}})$ . We will use the method of slicing and selection developed by Ghoussoub, Maurey and Schachermayer [17]. This method was used to give alternate proofs of Zippin's theorems that every reflexive separable Banach space embeds into a reflexive Banach space with a basis and every Banach space with separable dual embeds into a Banach space with a shrinking basis. Dodos and Ferenczi [14] showed that it is possible to parametrize this slicing and selection procedure. We will use their parametrized selection in our proof. They proved that given an analytic collection  $\mathcal{A}$  of separable reflexive Banach space (respectively Banach spaces with separable dual), there exists an analytic collection  $\mathcal{A}'$  of separable reflexive Banach spaces with bases (respectively Banach spaces with shrinking bases) such that for all  $X \in \mathcal{A}$  there exists  $Z \in \mathcal{A}'$  such that  $X$  embeds into  $Z$ . Before proceeding to the proof, we must introduce several notions involved in the slicing and selection procedure.

Let  $E$  be a compact metric space. A metric  $\Delta : E \times E \rightarrow \mathbb{R}$  is a *fragmentation* if for every non-empty closed subset  $K$  of  $E$  and  $\varepsilon > 0$  there exists an open subset  $V$  of  $E$  with  $K \cap V \neq \emptyset$  and such that  $\sup\{\Delta(x, y) : x, y \in K \cap V\} \leq \varepsilon$ . Recall that  $K(E)$  is the space of all compact subsets of  $E$ . In [17] they prove the following.

**Theorem 16** ([17]). *Let  $E$  be a compact metric space and  $\Delta$  be a fragmentation on  $E$ . Then there is a function  $s_{\Delta} : K(E) \rightarrow E$  called the dessert selection satisfying the following:*

- (i) *For every non-empty  $K \in K(E)$ , we have  $s_{\Delta}(K) \in K$ .*
- (ii) *If  $K \subset C$  are in  $K(E)$  and  $s_{\Delta}(C) \in K$ , then  $s_{\Delta}(K) = s_{\Delta}(C)$ .*

(iii) If  $(K_m)$  are descending in  $K(E)$  and  $K = \bigcap_m K_m$ , then

$$\lim_m \Delta(s_\Delta(K_m), s_\Delta(K)) = 0.$$

**Definition 17.** Let  $Z$  be a standard Borel space. A parametrized Borel fragmentation on  $E$  is a map  $\mathcal{D} : Z \times E \times E \rightarrow \mathbb{R}$  such that for each  $z \in Z$ , setting  $\mathcal{D}_z(\cdot, \cdot) := \mathcal{D}(z, \cdot, \cdot)$  the following are satisfied.

- (1) For  $z \in Z$ , the map  $\mathcal{D}_z : E \times E \rightarrow \mathbb{R}$  is a fragmentation on  $E$ .
- (2) The map  $\mathcal{D}$  is Borel.

Let  $\mathcal{D}$  be a parametrized Borel fragmentation on a compact metric space  $E$  with respect to some standard Borel space  $Z$ . Define  $s_{\mathcal{D}} : Z \times K(E) \rightarrow E$  by  $s_{\mathcal{D}}(z, K) = s_{\mathcal{D}_z}(K)$  where  $s_{\mathcal{D}_z}$  is the dessert selection associated to the fragmentation  $\mathcal{D}_z$  and given by Theorem 16. We need the following important theorem of Dodos.

**Theorem 18.** [12, Theorem 5.8] Let  $E$  be a compact metrizable space and  $Z$  be a standard Borel space. Let  $\mathcal{D} : Z \times E \times E \rightarrow \mathbb{R}$  be a parametrized Borel fragmentation. Then the parametrized dessert selection  $s_{\mathcal{D}} : Z \times K(E) \rightarrow E$  associated to  $\mathcal{D}$  is Borel.

For convenience, we restate Theorem 2.

**Theorem 19.** Let  $X$  be a separable Banach space and let  $\mathcal{A} \subset \mathcal{L}(X, C(2^{\mathbb{N}}))$  be a set of bounded operators whose adjoints have separable range which is analytic in the strong operator topology. Then there is a Banach space  $Z$  with a shrinking basis such that every  $T \in \mathcal{A}$  factors through  $Z$ .

*Proof.* Let  $\mathcal{A} \subset \mathcal{L}(X, C(2^{\mathbb{N}}))$  be an analytic collection of operators whose adjoints have separable range. Using Proposition 3 the space of all operators whose adjoints have separable range is coanalytic. Therefore we may apply Lusin's theorem [22, Lemma 28.1] to find a Borel set  $\mathcal{B}$  of operators whose adjoints have separable range such that  $\mathcal{A} \subset \mathcal{B}$ .

Our tool is Proposition 14. The main step in the proof is to define a parametrized Borel fragmentation and use the associated parametrized dessert selection to pick a basis  $(y_n^T)_{n \in \mathbb{N}}$  of  $C(2^{\mathbb{N}})$  such that  $T \mapsto (y_n^T)_{n \in \mathbb{N}}$  is Borel. To begin, however, we fix  $y_0 \in C(2^{\mathbb{N}})$  to be the sum of a function that separates points in  $2^{\mathbb{N}}$  and the constant function **1**. Recall that the closed set  $E_T$  (used below) contains  $y_0$ . Once we define the basis  $(y_n^T)$  for each  $T$  we must show that the assumptions of Proposition 14(b) are satisfied for  $y_0$ .

Define the map  $\mathcal{D} : \mathcal{B} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}} \rightarrow \mathbb{R}$  by

$$\mathcal{D}(T, \sigma, \tau) = \sup\{|s_n(E_T)(\sigma) - s_n(E_T)(\tau)| : n \in \mathbb{N}\}$$

We claim that for each  $T \in \mathcal{B}$ ,  $\mathcal{D}_T = \mathcal{D}(T, \cdot, \cdot)$  is a fragmentation.

To see this, we will follow the argument in [17]. It will be convenient to define a new operator  $T_0 : X \oplus_1 \ell_1^2 \rightarrow C(2^{\mathbb{N}})$  by  $T_0(x, a, b) = T(x) + ay_0 + bId$  for

all  $(x, a, b) \in X \oplus_1 \ell_1^2$ . Here  $Id$  is the identity function on  $2^{\mathbb{N}}$ . Note that  $T_0^*$  has separable range because  $T^*$  has separable range. As  $Id \in T_0(B_{X \oplus_1 \ell_1^2})$ , the following defines a metric on  $C(2^{\mathbb{N}})$ ,

$$\Delta(\sigma, \tau) = \sup\{|f(\sigma) - f(\tau)| : f \in T_0(B_{X \oplus_1 \ell_1^2})\} \quad \text{for all } \sigma, \tau \in C(2^{\mathbb{N}}).$$

As  $\overline{co(T(B_X) \cup \{\pm y_0\})} = E_T \subseteq \overline{T_0(B_{X \oplus_1 \ell_1^2})}$ , we have that if  $\Delta$  is a fragmentation then  $\mathcal{D}_T = \mathcal{D}(T, \cdot, \cdot)$  is a fragmentation. We denote the point evaluation at  $\sigma \in 2^{\mathbb{N}}$  by  $\delta_\sigma$ . Thus,  $\Delta(\sigma, \tau) = \|T_0^*(\delta_\sigma) - T_0^*(\delta_\tau)\|$ . As  $T_0^*$  has separable range, the metric  $\Delta$  will be separable on  $2^{\mathbb{N}}$ . Given  $\varepsilon > 0$  and  $\sigma \in 2^{\mathbb{N}}$ , we have that the closed  $\varepsilon$ -ball about  $\sigma$  in the  $\Delta$  metric is given by

$$B_\Delta(\sigma, \varepsilon) := \{\tau \in 2^{\mathbb{N}} : \Delta(\sigma, \tau) \leq \varepsilon\} = \bigcap_{f \in T_0(B_{X \oplus_1 \ell_1^2})} \{\tau \in 2^{\mathbb{N}} : |f(\sigma) - f(\tau)| \leq \varepsilon\}.$$

Thus,  $B_\Delta(\sigma, \varepsilon)$  is closed in the usual topology on  $2^{\mathbb{N}}$ . Let  $\varepsilon > 0$  and  $K \subseteq 2^{\mathbb{N}}$  be closed and non-empty. We let  $A \subset K$  be a countable subset which is dense in the  $\Delta$  metric. Thus,  $K \subseteq \bigcup_{\sigma \in A} B_\Delta(\sigma, \varepsilon/2)$ . By the Baire Category Theorem, there exists  $\sigma \in A$  such that  $B_\Delta(\sigma, \varepsilon/2) \cap K$  is not relatively nowhere dense. Thus, there exists a non-empty open set  $V \subseteq B_\Delta(\sigma, \varepsilon/2) \cap K$ , as  $B_\Delta(\sigma, \varepsilon/2) \cap K$  is closed. We have that  $K \cap V \neq \emptyset$  and  $\sup\{\Delta(x, y) : x, y \in K \cap V\} \leq \varepsilon$ . Thus,  $\Delta$  is a fragmentation.

Invoking the Borelness of the maps  $(s_n)_{n \in \mathbb{N}}$  and the map  $T \mapsto E_T$ , we have that  $\mathcal{D}$  is a parametrized Borel fragmentation according to Definition 17. By Theorem 18 there is a Borel map  $s : \mathcal{B} \times K(2^{\mathbb{N}}) \rightarrow 2^{\mathbb{N}}$  such that  $s_T : K(2^{\mathbb{N}}) \rightarrow 2^{\mathbb{N}}$  defined by  $s_T(K) = s(T, K)$  is a dessert selection associated to the fragmentation  $\mathcal{D}_T$ . We will use  $s_T$  to select a basis for  $C(2^{\mathbb{N}})$ .

Define a sequence  $(t_n^T)_{n=1}^\infty$  in  $2^{<\mathbb{N}}$  as follows: Let  $t_1^T = \emptyset$ . Let  $\phi : 2^{<\mathbb{N}} \rightarrow \mathbb{N} \cup \{0\}$  denote the unique bijection satisfying  $\phi(s) < \phi(t)$  if either  $|s| < |t|$ , or  $|s| = |t|$  and  $s <_{\text{lex}} t$ . Fix  $n \in \mathbb{N}$  with  $n \geq 2$  and  $t = \phi^{-1}(n-1)$ . By Theorem 16 there is a unique  $i_t \in \{0, 1\}$  such that  $t \hat{\prec} i_t \prec s_T(V_t)$ , where  $V_s := \{\sigma \in 2^{\mathbb{N}} : s \prec \sigma\}$  for  $s \in 2^{<\mathbb{N}}$ . Set

$$(1) \quad t_n^T = t \hat{\prec} j \text{ where } j = i_t + 1 \pmod{2} \text{ and } y_n^T = \chi_{V_{t_n^T}}.$$

Note that  $y_1^T \equiv \mathbf{1}$ , the constant function 1. As in (see [17] and [12, Claim 5.13 pg. 79])  $(y_n^T)_{n=1}^\infty$  is a normalized monotone basis of  $C(2^{\mathbb{N}})$ . Also note that for  $T \in \mathcal{B}$ , by definition,  $y_0 = \sum_{k \in \mathbb{N}} a_k y_n^T$  and  $a_k \neq 0$  for each  $k \in \mathbb{N}$ . Therefore we may apply Theorem 9(b) to see that the corresponding sequence  $(z_n^T)_{n \in \mathbb{N}}$  is a basis for  $Z_T$ . In [17, Theorem III.1, page 503] or [12, page 80] they prove  $(z_n^T)_{n \in \mathbb{N}}$  is a shrinking basis for  $Z_T$ . It remains to prove the next claim.

**Claim 20.** *The map  $\mathcal{B} \ni T \mapsto (y_n^T)_{n=1}^\infty \in C(2^{\mathbb{N}})^{\mathbb{N}}$  is Borel.*

*Proof.* It is enough to show that for each  $n \in \mathbb{N}$  the map  $T \mapsto y_n^T$  is (call it  $\psi$ ) Borel. Fix  $n \in \mathbb{N}$ . If  $n = 1$  let  $t = \emptyset$ ; otherwise, let  $t = \phi^{-1}(n-1)$ . Let

$$B_0 = \{T \in \mathcal{B} : t \hat{\prec} 1 \prec s(T, V_t)\} \text{ and } B_1 = \mathcal{B} \setminus B_0.$$

Let

$$f_t : \mathcal{B} \rightarrow \mathcal{B} \times K(2^{\mathbb{N}}) \text{ be defined by } f_t(T) = (T, V_t).$$

Then  $B_0$  and  $B_1$  are Borel since  $B_0 = f_t^{-1}(s^{-1}(V_{t \smallfrown 1}))$  and  $B_1 = f_t^{-1}(s^{-1}(V_{t \smallfrown 0}))$ . By definition

$$\psi(T) = y_n^T = \begin{cases} \chi_{V_{t \smallfrown 0}} & T \in B_0 \\ \chi_{V_{t \smallfrown 1}} & T \in B_1. \end{cases}$$

Since  $\psi^{-1}(\chi_{V_{t \smallfrown 0}}) = B_0$  and  $\psi^{-1}(\chi_{V_{t \smallfrown 1}}) = B_1$ , our claim is proved.  $\square$

Finally, invoking Proposition 14, the proof is complete.  $\square$

*Proof of Theorem 1.* Now assume that  $\mathcal{A} \subset \mathcal{L}(X, C(2^{\mathbb{N}}))$  is an analytic collection of weakly compact operators. This proof follows the same outline as the proof of Theorem 2. Indeed, it is enough to show that  $Z_T$  is reflexive. Note that we already know  $(z_n^T)$  is a shrinking basis for  $Z_T$ .

By Theorem 9(c) it is enough to show that  $W_T$  is weakly compact. This is proved in [12, Lemma 5.18]. Let  $T_2 : Z_T \rightarrow C(2^{\mathbb{N}})$ , be as in Theorem 9(b). Set  $K = T_2^{-1}(E_T)$  (note that  $T_2^{-1}$  is well defined on  $E_T$ ). Since  $E_T$  is weakly compact and  $T_2$  is weak-weak continuous,  $K$  is a weakly compact subset of  $Z_T$ . For  $k \in \mathbb{N}$  let  $Q_k : Z_T \rightarrow \text{span}\{z_n^T : n \leq k\}$  be the natural projection. Since  $(z_n^T)_{n=1}^{\infty}$  is shrinking we may use [11, Lemma 2] (also see [12, Lemma B.10]) to conclude that

$$K' = K \cup \bigcup_{k \in \mathbb{N}} Q_k(K)$$

is weakly compact. Note that  $T_2(K')$  is also weakly compact and

$$T_2(K') = E_T \cup \bigcup_{k \in \mathbb{N}} T_2(Q_k(K)) = E_T \cup \bigcup_{k \in \mathbb{N}} P_k(E_T) = \overline{\bigcup_{k \in \mathbb{N}} P_k(E_T)} = W_T.$$

This completes the proof.  $\square$

**Corollary 21.** *Suppose  $Z$  is a complemented subspace of  $C(2^{\mathbb{N}})$  and  $\mathcal{A} \subset \mathcal{L}(X, Z)$  be an analytic collection of weakly compact operators (resp. a collection of operators whose adjoints have separable range). Then there is a reflexive space (resp. space with separable dual)  $Z_{\mathcal{A}}$  such that each  $T \in \mathcal{A}$  factors through  $Z_{\mathcal{A}}$ .*

#### 4. ANALYTIC COLLECTIONS OF SPACES

In this section we present generalizations of Theorems 1 and 2. Our goal is to uniformly factor sets of operators of the form  $T : X \rightarrow Y$ , where  $X$  and  $Y$  are allowed to vary. Our previous results relied on the fact that both the set of separable Banach spaces and the set of bounded operators between two fixed separable Banach spaces can be naturally considered as standard Borel spaces. However, the set of operators between separable Banach spaces which are allowed to vary is not immediately realized as a standard Borel space. To get around this, we will code operators using sequences.

Let  $X, Y \in SB$  and define  $\mathcal{C}_{X,Y} \subset C(2^{\mathbb{N}})^{\mathbb{N}}$  by

$$\begin{aligned} (w_n)_{n \in \mathbb{N}} \in \mathcal{C}_{X,Y} &\iff w_k \in Y \ \forall k \in \mathbb{N} \text{ and } (\forall n, m, l \in \mathbb{N} \ \forall q, r \in \mathbb{Q}) \\ &\quad d_n(X) = qd_m(X) + pd_l(X) \implies w_n = qw_m + pw_l \text{ and} \\ &\quad (\exists K \in \mathbb{N} \ \forall (a_i)_i \in \mathbb{Q}^{<\mathbb{N}} \ \|\sum_i a_i w_i\| \leq K \|\sum_i a_i d_i(X)\|). \end{aligned}$$

The map defined by  $\mathcal{C}_{X,Y} \ni (w_k)_{k \in \mathbb{N}} \mapsto T \in \mathcal{L}(X, Y)$ , where  $T$  is the unique operator  $Td_n(X) := w_n$  for each  $n \in \mathbb{N}$ , is an isomorphism. Define  $\mathcal{L} \subset SB \times SB \times C(2^{\mathbb{N}})^{\mathbb{N}}$  by

$$(X, Y, (w_k)) \in \mathcal{L} \iff (w_k)_{k \in \mathbb{N}} \in \mathcal{C}_{X,Y}.$$

Recall that Bossard [8] proved that the set  $\{(Y, y) : y \in Y\}$  is Borel in  $SB \times C(2^{\mathbb{N}})$ . Therefore, by counting quantifiers, the definition of  $\mathcal{C}_{X,Y}$  yields that  $\mathcal{L}$  is a Borel subset of  $SB \times SB \times C(2^{\mathbb{N}})^{\mathbb{N}}$ . Thus  $\mathcal{L}$  is a Standard Borel space.

**Proposition 22.** *The following subsets of  $\mathcal{L}$  are coanalytic.*

$\mathcal{W} = \{(X, Y, (w_k)) \in \mathcal{L} : \text{the operator } T \in \mathcal{L}(X, Y) \text{ defined by}$

$$Td_k(X) = w_k \text{ for all } k \in \mathbb{N}, \text{ is weakly compact}\}$$

$\mathcal{SR} = \{(X, Y, (w_k)) \in \mathcal{L} : \text{the adjoint of the operator } T \in \mathcal{L}(X, Y) \text{ defined by}$

$$Td_k(X) = w_k \text{ for all } k \in \mathbb{N} \text{ has separable range}\}$$

*Proof.* In [6] it is proved that an operator  $T : X \rightarrow Y$  is weakly compact if for every bounded sequence  $(x_n)$  in  $B_X$  the image  $(Tx_n)$  does not dominate the summing basis of  $c_0$ . Let  $[\mathbb{N}]$  denote the set of all infinite increasing sequences in  $\mathbb{N}$ . This gives us the following characterization of  $\mathcal{W}$

$$\begin{aligned} (X, Y, (w_k)) \in \mathcal{W} &\iff \forall (k_i)_{i \in \mathbb{N}} \in [\mathbb{N}], \ \forall n \in \mathbb{N} \ \exists (a_i) \in \mathbb{Q}^{<\mathbb{N}}, \\ &\quad \left\| \sum_{i \in \mathbb{N}} a_i w_{k_i} \right\| < \frac{1}{n} \sup_{k \in \mathbb{N}} \left| \sum_{i \geq k} a_i \right|. \end{aligned}$$

Therefore  $\mathcal{W}$  is coanalytic.

It remains to show that  $\mathcal{SR}$  is coanalytic. As before it suffices to consider triples  $(X, Y, (w_k)_{n \in \mathbb{N}})$  whose corresponding operators are norm at most 1. Call this collection  $\mathcal{SR}_1$ . The proof follows the proof of Proposition 3 after making the following changes to accommodate the triples  $(X, Y, (w_k)) \in \mathcal{SR}_1$ . Let  $y^* \in B_{Y^*}$  and  $(X, Y, (w_k)) \in \mathcal{SR}_1$ . Define

$$f_{(X,Y,(w_k)),y^*} = \left( \frac{y^*(w_n)}{\|d_n(X)\|} \right)_{n=1}^{\infty} \in B_{\ell_{\infty}}.$$

and

$$K_{(X,Y,(w_k))} = \{f_{(X,Y,(w_k)),y^*} : y^* \in B_{Y^*}\}.$$

Finally, define  $\mathcal{D} \subset \mathcal{L} \times B_{\ell_{\infty}}$  by

$$((X, Y, (w_k)), f) \in \mathcal{D} \iff f \in K_{(X,Y,(w_k))}.$$

As before,  $\mathcal{D}$  is Borel and the map  $\Phi : \mathcal{L} \rightarrow K(B_{\ell_\infty})$  defined by  $\Phi((X, Y, (w_k))) = K_{(X, Y, (w_k))}$  is Borel with

$$(X, Y, (w_k)) \in \mathcal{SR} \iff \Phi((X, Y, (w_k))) = \Sigma.$$

Thus,  $\mathcal{SR}$  is coanalytic.  $\square$

**Notation 2.** In this new setting we make set the following notation. Note that in most cases we are simply replacing  $T$  by  $(X, Y, (w_k))$ .

- (a) Let  $(X, Y, (w_k)) \in \mathcal{L}$ . Denote by  $(y_n^{(X, Y, (w_k))})_{n \in \mathbb{N}}$  a normalized basis of  $Y$  that depends on  $(X, Y, (w_k))$  and for  $m \in \mathbb{N}$ , let  $P_m^{(X, Y, (w_k))} : Y \rightarrow [y_n^{(X, Y, (w_k))} : n \leq m]$  be the natural projection.
- (b) Let  $y_0 \in C(2^{\mathbb{N}})$  be the sum of a function that separates points and the constant function 1. Define

$$E_{(X, Y, (w_k))} := \overline{\text{co}(\{w_k\}_{k \in \mathbb{N}} \cup \{\pm y_0\})}.$$

- (c) Define

$$W_{(X, Y, (w_k))} = \overline{\bigcup_{m \in \mathbb{N}} P_m^{(X, Y, (w_k))}(E_{(X, Y, (w_k))})}.$$

The set  $W_{(X, Y, (w_k))}$  is closed, bounded, convex and symmetric. Also,  $P_k^{(X, Y, (w_k))}(W_{(X, Y, (w_k))}) \subset W_{(X, Y, (w_k))}$  for each  $k \in \mathbb{N}$ .

- (d) Let

$$Z_{(X, Y, (w_k))} = \{z \in Y : \sum_{m=1}^{\infty} \|z\|_{W_{(X, Y, (w_k))}^m}^2 < \infty\}$$

$$\|z\|_{(X, Y, (w_k))} = \left( \sum_{m=1}^{\infty} \|z\|_{W_{(X, Y, (w_k))}^m}^2 \right)^{\frac{1}{2}}.$$

The next lemmas, which we state without proof, are analogous to Lemmas 10 and 13.

**Lemma 23.** *Let  $\mathcal{B} \subset \mathcal{L}$  be Borel and suppose the map  $\mathcal{B} \ni (X, Y, (w_k)) \mapsto (y_n^{(X, Y, (w_k))})_{n \in \mathbb{N}} \in C(2^{\mathbb{N}})^{\mathbb{N}}$  is Borel. Then the following hold:*

- (a) *The map  $\mathcal{B} \ni (X, Y, (w_k)) \mapsto E_{(X, Y, (w_k))} \in F(C(2^{\mathbb{N}}))$  is Borel.*
- (b) *The map  $\mathcal{B} \ni (X, Y, (w_k)) \mapsto W_{(X, Y, (w_k))} \in F(C(2^{\mathbb{N}}))$  is Borel. Moreover, for each  $m \in \mathbb{N}$  the map  $\mathcal{B} \ni (X, Y, (w_k)) \mapsto W_{(X, Y, (w_k))}^m \in F(C(2^{\mathbb{N}}))$  is Borel.*
- (c) *The map  $\mathcal{B} \times Y \ni ((X, Y, (w_k)), y) \mapsto \|y\|_{W_{(X, Y, (w_k))}^m}$  is Borel.*

**Lemma 24.** *Let  $\mathcal{B} \subset \mathcal{L}$  be Borel and  $\mathcal{B} \ni (X, Y, (w_k)) \mapsto (y_n^{(X, Y, (w_k))})_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  be a Borel map. The set*

$$\mathcal{Z} = \{((X, Y, (w_k)), E) \in \mathcal{B} \times SB : E \text{ is isometric to } Z_{(X, Y, (w_k))}\}.$$

*is analytic in  $\mathcal{L} \times SB$ .*

A quick note concerning this lemma is in order. The first step in the proof is to consider the the set  $\mathcal{F}_u$  defined below

$$\mathcal{F}_u = \{((X, Y, (w_k)), (x_n)) \in \mathcal{B} \times C(2^{\mathbb{N}})^{\mathbb{N}} : (z_n^{(X, Y, (w_k))}) \text{ is 1-equivalent to } (x_n)\}.$$

Reasoning as in the proof of Lemma 13,  $\mathcal{F}_u$  is Borel. To see that  $\mathcal{Z}$  is analytic observe that

$$(2) \quad \begin{aligned} ((X, Y, (w_k)), E) \in \mathcal{Z} &\iff \exists (x_n) \in C(2^{\mathbb{N}}) \text{ with } [x_n] = E \\ &\text{and } ((X, Y, (w_k)), (x_n)) \in \mathcal{F}_u. \end{aligned}$$

We can now state and prove our main theorem of this section.

**Theorem 25.** *Set*

$$\mathcal{W}_{C(2^{\mathbb{N}})} = \{(X, Y, (w_k)) \in \mathcal{W} : Y \text{ is isomorphic to } C(2^{\mathbb{N}})\}$$

$$\mathcal{SR}_{C(2^{\mathbb{N}})} = \{(X, Y, (w_k)) \in \mathcal{SR} : Y \text{ is isomorphic to } C(2^{\mathbb{N}})\}$$

*Suppose that  $\mathcal{A}$  is an analytic subset of  $\mathcal{W}_{C(2^{\mathbb{N}})}$  (resp.  $\mathcal{SR}_{C(2^{\mathbb{N}})}$ ). Then there is a separable reflexive Banach space with a basis (resp. space with a shrinking basis)  $Z$  such that for each  $(X, Y, (w_k)) \in \mathcal{A}$  the operator  $T$  defined by  $Td_n(X) = w_n$  for each  $n \in \mathbb{N}$ , factors through  $Z$ .*

*Proof.* We will sketch the proof for  $\mathcal{W}_{C(2^{\mathbb{N}})}$ , the proof in the case of  $\mathcal{SR}_{C(2^{\mathbb{N}})}$  is analogous. Let  $\mathcal{A} \subset \mathcal{W}_{C(2^{\mathbb{N}})}$  be analytic. Proposition 22 and Lusin's theorem [22, Lemma 18.1] together tell us that  $\mathcal{W}_{C(2^{\mathbb{N}})}$  is coanalytic and that there is a Borel subset  $\mathcal{B}$  of  $\mathcal{W}_{C(2^{\mathbb{N}})}$  such that  $\mathcal{A} \subset \mathcal{B}$ . The goal is to apply Lemma 24. Following along the same route we tracked out in the proof of Theorem 2 we can find for each  $(X, Y, (w_k)) \in \mathcal{W}_{C(2^{\mathbb{N}})}$  a basis  $(y_n^{(X, Y, (w_k))})_{n \in \mathbb{N}}$  of  $C(2^{\mathbb{N}})$  such that the map  $(X, Y, (w_k)) \mapsto (y_n^{(X, Y, (w_k))})_{n \in \mathbb{N}}$  is Borel, as desired by Lemma 24. Again, using the same argument, we claim that for  $(X, Y, (w_k)) \in \mathcal{W}_{C(2^{\mathbb{N}})}$  the space  $Z_{(X, Y, (w_k))}$  (defined above) is reflexive with a basis. Applying Lemma 24 yields that

$$\mathcal{Z}_{\mathcal{B}} = \{Z \in \mathcal{B} : \exists (X, Y, (w_k)) \in \mathcal{B}, Z_{(X, Y, (w_k))} = Z\}.$$

is analytic. Therefore, using the same procedure as in the proof of Proposition 14 we obtain a reflexive space  $Z_{\mathcal{B}}$  such that every operator  $T$  coded by a triple  $(X, Y, (w_k)) \in \mathcal{B}$  factors through  $Z_{\mathcal{B}}$ .  $\square$

## 5. APPLICATIONS

x In this section we provide several consequences of our uniform factorization results. In [6] several examples are given of Banach spaces  $X$  and  $Y$  such that the space of weakly compact operators from  $X$  to  $Y$  is coanalytic but not analytic. For example, let  $U$  be the separable Banach space of Pełczyński which contains complemented copies of every Banach space with a basis. It is shown in [6] that the set of weakly compact operators on  $U$  is coanalytic but not Borel. In terms of factorization, we have the following.



**Proposition 26.** *There does not exist a separable reflexive space  $Z$  such that every weakly compact operator from  $U$  to  $C(2^{\mathbb{N}})$  factors through  $Z$ . In particular, the set of weakly compact operators from  $U$  to  $C(2^{\mathbb{N}})$  is not analytic.*

*Proof.* Let  $\xi$  be a countable ordinal and let  $X_\xi$  be the Tsirelson space of order  $\xi$ . For our purposes we just need that  $X_\xi$  is a reflexive Banach space with a basis and has Szlenk index  $\omega^{\xi\omega}$  [28]. We consider  $X_\xi$  as a complemented subspace of  $U$  and let  $P_\xi : U \rightarrow X_\xi$  be a bounded projection from  $U$  onto  $X_\xi$ . Let  $i_\xi : X_\xi \rightarrow C(2^{\mathbb{N}})$  be an embedding of  $X_\xi$ . The operator  $i_\xi$  is weakly compact as  $X_\xi$  is reflexive, and hence the operator  $T_\xi := i_\xi \circ P_\xi$  is weakly compact. If there was a Banach space  $Z$  with separable dual such that for all countable ordinals  $\xi$  the operator  $T_\xi$  factored through  $Z$ , then, since  $i_\xi$  is an isometry,  $Z$  would contain an isomorphic copy of  $X_\xi$  for all countable ordinals  $\xi$ . This would imply that the Szlenk index of  $Z$  is uncountable which contradicts that  $Z$  has separable dual [29].  $\square$

**Proposition 27.** *There exists a Banach space  $Y$  with a shrinking basis such that there does not exist a separable reflexive Banach space  $Z$  so that every weakly compact operator on  $Y$  factors through  $Z$ . In particular, the set of weakly compact operators on  $Y$  is not analytic.*

*Proof.* Consider the collection  $\mathcal{A}_{\omega^\omega}$  of all shrinking basic sequences whose closed linear spaces in  $C(2^{\mathbb{N}})^{\mathbb{N}}$  have and Szlenk index less than or equal to  $\omega^\omega$ . It is shown in [8] that  $\mathcal{A}_{\omega^\omega}$  is an analytic subset of  $C(2^{\mathbb{N}})^{\mathbb{N}}$ . Using Theorem 8 there is a Banach space  $Y$  with a shrinking basis such that for each  $X$  in  $\mathcal{A}_{\omega^\omega}$  there is a complemented subspace of  $Y$  isomorphic to  $X$ . Now let  $\xi$  be a countable ordinal and let  $X_\xi$  be the Tsirelson space of order  $\xi$ . For this proof, we just need that  $X_\xi$  is a reflexive Banach space with a basis and, Szlenk index  $\omega^{\xi\omega}$  and that  $X_\xi^*$  has Szlenk index at most  $\omega^\omega$  [28]. Thus  $X_\xi^*$  is isomorphic to a complemented subspace of  $Y$ . We consider  $X_\xi^*$  as a complemented subspace of  $Y$  and let  $P_\xi : Y \rightarrow X_\xi^*$  be a bounded projection from  $Y$  onto  $X_\xi^*$ . Let  $i_\xi : X_\xi^* \rightarrow Y$  be the identity on  $X_\xi^*$ . The operator  $i_\xi$  is weakly compact as  $X_\xi^*$  is reflexive, and hence the operator  $T_\xi := i_\xi \circ P_\xi$  is weakly compact. If there was a reflexive Banach space  $Z$  such that for all countable ordinals  $\xi$  the operator  $T_\xi$  factored through  $Z$ , then, since  $i_\xi$  is an isometry,  $Z$  would contain an isomorphic copy of  $X_\xi^*$  for all countable ordinals  $\xi$ . Thus,  $X_\xi$  would be a quotient of  $Z^*$  for all countable ordinals  $\xi$ . This would imply that the Szlenk index of  $Z^*$  is uncountable which contradicts that  $Z$  is reflexive [29].  $\square$

In contrast to the negative results of Proposition 26 and Proposition 27, we have the following theorem.

**Theorem 28.** *Let  $X$  be a Banach space with a shrinking basis such that  $X^{**}$  is separable. The set of weakly compact operators on  $X$  is a Borel subset of  $\mathcal{L}(X)$ . In particular, there exists a reflexive Banach space  $Z$  such that every weakly compact operator on  $X$  factors through  $Z$ .*

*Proof.* Let  $(x_k)_{k=1}^\infty$  be a shrinking basis for  $X$  with biorthogonal functionals  $(x_k^*)_{k=1}^\infty$ , and let  $D \subset X^{**}$  be dense. We denote the set of weakly compact operators on  $X$  by  $\mathcal{W}(X)$ . By Gantmacher's Theorem, an operator  $T \in \mathcal{L}(X)$  is weakly compact if and only if  $T^{**}(X^{**}) \subseteq X$ . In particular,

$$(3) \quad T \in \mathcal{W}(X) \Leftrightarrow T^{**}f \in X \forall f \in D.$$

Since  $(x_k)_{k=1}^\infty$  is a  $w^*$ -basis for  $X^{**}$ ,

$$(4) \quad f = w^* - \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^*(f)x_i \quad \text{for all } f \in X^{**}.$$

Thus, we have for all  $f \in X^{**}$  that

$$(5) \quad T^{**}f \in X \Leftrightarrow T^{**}f = \|\cdot\| - \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k^*(T^{**}f)x_k \Leftrightarrow \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \left\| \sum_{k=M}^N x_k^*(T^{**}f)x_k \right\| = 0$$

Note that  $T^{**}(x) = T(x)$  for all  $x \in X$ . Since  $T^{**}$  is  $w^*$  to  $w^*$  continuous,

$$T^{**}f = w^* - \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^*(f)T(x_i) \quad \text{for all } f \in X^{**}.$$

Hence, for all  $k \in \mathbb{N}$ , we have that

$$x_k^*(T^{**}f)x_k = \lim_{n \rightarrow \infty} x_k^* \left( \sum_{i=1}^n x_i^*(f)T(x_i) \right) x_k.$$

Substituting into (5) gives,

$$T^{**}f \in X \Leftrightarrow \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} \left\| \sum_{k=M}^N x_k^* \left( \sum_{i=1}^n x_i^*(f)T(x_i) \right) x_k \right\| = 0$$

Thus,  $\mathcal{W}(X)$  is Borel by 3. □

Let  $J$  be the quasi-reflexive space of James [19]. Laustsen [24, Theorem 4.3] proved the following result by constructing the required space. As  $J$  has a shrinking basis and  $J^{**}$  is separable, we obtain it as a corollary of Theorem 28.

**Proposition 29.** *There is a reflexive space  $Z$  such that every weakly compact operator on  $J$  factors through  $Z$ .*

In [25], Lindenstrauss showed that for each separable Banach space  $X$  there is a separable Banach space  $Y$  such that  $Y^{**}/Y$  is isomorphic to  $X$ . In particular, the space  $Y$  has separable bidual. Therefore, Theorem 28 yields that whenever  $Y$  has a shrinking basis, every weakly compact operator on  $Y$  factors through a single reflexive space.

**Proposition 30.** *Let  $X$  and  $Y$  be separable Banach spaces. Then every closed norm-separable set  $S$  of weakly compact operators is Borel in the strong operator topology.*

*Proof.* Let  $(T_k)_{k \in \mathbb{N}}$  be a dense subset of  $S$  and  $(d_k)_{k \in \mathbb{N}}$  be dense in  $B_X$ . Then

$$T \in S \iff \forall m \in \mathbb{N}, \exists k \in \mathbb{N} \text{ such that } \forall j \in \mathbb{N}, \|(T - T_k)d_j\| < \frac{1}{m}.$$

From this characterization it follows that  $S$  is Borel.  $\square$

One corollary of Proposition 30 is that if  $X^*$  or  $Y$  has the approximation property, then the set of compact operators from  $X$  to  $Y$  is Borel. In [20] Johnson proved that there is a space  $Z_K$  such that every operator which is the uniform limit of finite rank operators (independent of the spaces  $X$  and  $Y$ ) factors through  $Z_K$ . In particular this implies that whenever either  $X^*$  or  $Y$  has the approximation property every compact operator from  $X$  to  $Y$  factors through  $Z_K$ . Johnson and Szankowski [21] proved that there is no separable Banach space such that every compact operator factors through it. The following result follows from Proposition 30 and Theorem 1 and is a weaker version of Johnson's Theorem.

**Corollary 31.** *If  $Y$  is Banach space with a shrinking basis or is isomorphic to  $C(2^{\mathbb{N}})$  then there exists a reflexive space  $Z$  such that if  $X$  is a separable Banach space with the approximation property then every compact operator from  $X$  to  $Y$  factors through  $Z$ .*

*Proof.* Let  $Y$  either be a Banach space with a shrinking basis or be isomorphic to  $C(2^{\mathbb{N}})$ . By Proposition 30 and Theorem 1, there exists a reflexive Banach space  $Z$  such that every compact operator from  $U$  to  $Y$  factors through  $Z$ . If  $X$  is a separable Banach space with the approximation property then  $X$  is isomorphic to a complemented subspace of  $U$ . Every compact operator from  $X$  to  $Y$  has a compact factorization through  $U$  and hence factors through  $Z$  as well.  $\square$

**Proposition 32.** *There exists a separable hereditarily indecomposable Banach space  $X$ , with HI dual and non-separable bidual, and a reflexive Banach space  $Z$  such that every weakly compact operator on  $X$  factors through  $Z$ .*

*Proof.* In [1] the authors construct an HI space  $X$  with a shrinking basis such that  $X^*$  is HI and  $X^{**}$  is non-separable and on which every operator is a scalar multiple of the identity plus a weakly compact operator. Once again it suffices to show that the set of weakly compact operators on  $X$  is Borel. In [1] they prove that each weakly compact operator on  $X$  is strictly singular. It is shown in [4] that when the strictly singular operators have codimension-one in  $\mathcal{L}(X)$  they are a Borel subset. It follows that the set of weakly compact operators on  $X$  is a Borel subset of  $\mathcal{L}(X)$ . Hence, we may apply Theorem 1.  $\square$

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