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CONSTRUCTIVE APPROXIMATION

Greedy Bases for Besov Spaces

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Abstract We prove that the Banach space $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, which is isomorphic to certain Besov spaces, has a greedy basis whenever $1 \le p \le \infty$ and $1 < q < \infty$. Furthermore, the Banach spaces $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_1}$, with $1 < p \le \infty$, and $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{c_0}$, with $1 \le p < \infty$, do not have a greedy basis. We prove as well that the space $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$ has a 1-greedy basis if and only if $1 \le p = q \le \infty$.

Keywords Greedy bases · Besov spaces

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1 Introduction

Let X be a Banach space, and let (x_i) be a Schauder basis for X with biorthogonal sequence (x_i^*). For $x \in X$ and $n \geq 1$, the error in the best n-term approximation to x (using (x_i)) is given by

$$\sigma_n(x) := \inf\left\{\left\|x - \sum_{i \in A} a_i x_i\right\| : (a_i) \subset \mathbb{R}, |A| \leq n\right\}.$$

Let $A_n(x) \subset \mathbb{N}$ be the indices corresponding to a choice of n largest coefficients of x in absolute value, i.e., $A_n(x)$ satisfies

$$\min\left\{\left|e_i^*(x)\right| : i \in A_n(x)\right\} \geq \max\left\{\left|e_i^*(x)\right| : i \in \mathbb{N} \setminus A_n(x)\right\}.$$

Then $G_n(x) := \sum_{i \in A_n(x)} x_i^*(x) x_i$ is called an nth *greedy approximant* to x. We say that (x_i) is *greedy* with constant C if

$$\left\|x - G_n(x)\right\| \leq C\sigma_n(x) \quad (x \in X, n \geq 1).$$

If $C = 1$, then (x_i) is said to be 1-greedy. Temlyakov [18] proved that the Haar system for $L_p[0,1]^d$ $(1 < p < \infty, d \geq 1)$ is greedy, which provides an important theoretical justification for the thresholding procedure used in data compression. Subsequently, Konyagin and Temlyakov [13] gave a very useful abstract characterization of greedy bases. To state their result, we recall that (x_i) is *unconditional* with constant K if, for all choices of signs, we have

$$\left\|\sum_{i=1}^{\infty} \pm x_i^*(x) x_i\right\| \leq K\|x\| \quad (x \in X).$$

We say that (x_i) is *democratic* with constant Δ if, for all finite $A, B \subset \mathbb{N}$ with $|A| = |B|$, we have

$$\left\|\sum_{i \in A} x_i\right\| \leq \Delta \left\|\sum_{i \in B} x_i\right\|.$$

Theorem A [13] *Suppose that (x_i) is unconditional with constant K and democratic with constant Δ. Then (x_i) is greedy with constant $K + K^3\Delta$. Conversely, if (x_i) is greedy with constant C, then (x_i) is suppression-unconditional with constant C and democratic with constant C.*

Theorem A was used in [20, 21] to prove that $L_p[0,1]$ $(p \neq 2)$ has a greedy basis that is not equivalent to a subsequence of the Haar basis, and in [5] to prove that ℓ_p and $L_p[0,1]$ $(p \neq 2)$ have a continuum of mutually nonequivalent greedy bases. It was also used in [7] to study duality for greedy bases, and a similar theorem was proved in [7] to characterize the larger class of *almost greedy* bases (see also [6]).

Some examples of greedy bases are given in [21]. In most cases these bases are greedy simply because they are *symmetric* (e.g., Riesz bases for a Hilbert space, which are equivalent to the unit vector basis of ℓ_2, or good wavelet bases for the Besov spaces $B^p_{\alpha,p}(\mathbb{R})$, which are equivalent to the unit vector basis of ℓ_p), or because they are equivalent to the Haar basis (e.g., good wavelet bases for $L_p(\mathbb{R}^d)$) or to a subsequence of the Haar basis (e.g., generalized Haar systems [12]). In [10] certain wavelet bases in the Triebel-Lizorkin spaces $\mathfrak{f}^s_{p,r}$ are shown to be greedy. In [1] it is proved that 1-symmetric bases (e.g., the unit vector bases of Orlicz and Lorentz sequence spaces) are in fact 1-greedy. On the other hand, there are examples of spaces with an unconditional basis but no democratic unconditional basis, and hence no greedy basis, e.g., certain spaces with a unique unconditional basis up to permutation [2], the spaces $\ell_p \oplus \ell_q$ and $\ell_p \oplus c_0$ for $1 \le p < q < \infty$ [8], and the original Tsirelson space T^* [19]. Wojtaszczyk [22] proved that the L_p spaces $(1 < p < \infty)$ are the only rearrangement-invariant function spaces on [0, 1] for which the Haar system is greedy.

Using Theorem A we prove that for every $1 \le p \le \infty$ and $1 < q < \infty$ the Banach space $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ has a greedy basis. Furthermore, we show that the Banach space $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_1}$ does not have a greedy basis whenever $1 < p \le \infty$. This answers a question posed by P. Wojtaszczyk, who asked when such spaces have a greedy basis. The problem of finding a greedy basis for Banach spaces of the form $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ is particularly pertinent from the approximation theoretical standpoint as such spaces are isomorphic to certain Besov spaces on the circle [17]. As shown in [17, Theorem 2] (see also [16, p. 255]), the Besov space $B^{\alpha,m}_{p,q}[0,1]$, where $1 \le p \le \infty$, $1 \le q < \infty$, $m \in \{-1, 0, 1, 2, \dots\}$, and $\alpha \in (1, m+1+1/p)$, is isomorphic to $\ell_1^{m+2} \oplus (\bigoplus_{n=0}^\infty \ell_p^{2^n})_{\ell_q}$ which is easily seen to be isomorphic to $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$.

The greedy bases which we construct in Theorem 1 differ from the examples discussed above in that they are neither subsymmetric (see [14, p. 114] for this notion) nor equivalent to a subsequence of the Haar basis.

The following result completely characterizes for which pairs (p, q) the space $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ has a greedy basis.

Theorem 1 *Let* $1 \le p, q \le \infty$.

(a) *If* $1 < q < \infty$, *then the Banach space* $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ *has a greedy basis.*
(b) *The spaces* $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_1}$, *with* $1 < p \le \infty$, *and* $(\bigoplus_{n=1}^\infty \ell_p^n)_{c_0}$, *with* $1 \le p < \infty$, *do not have greedy bases.*

The following result yields that only in the trivial case that $p = q$ does $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ have a 1-greedy basis:

Theorem 2 *Let* $1 \le p \le \infty$, *and let* $(E_n)_{n=1}^\infty$ *be a sequence of finite dimensional Banach spaces. If* $(x_i)_{i=1}^\infty$ *is a normalized* 1*-greedy basis for the space* $(\bigoplus_{n=1}^\infty E_n)_{\ell_p}$, *then* $(x_i)_{i=1}^\infty$ *is* 1*-equivalent to the standard unit vector basis for* ℓ_p *(as usual, if* $p = \infty$ *we consider the* c_0*-sum).*

As the cases $\ell_p \oplus \ell_q$ and $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ are settled, the following spaces might be interesting to consider:

Problem 3 *Assume* $1 < p \ne q < \infty$. *Does* $\ell_q(\ell_p) = (\bigoplus \ell_p)_{\ell_q}$ *have a greedy basis?*

2 Proof of Theorems 1 and 2

Part (a) of Theorem 1 will follow easily from the following lemma, whose proof will require some work:

Lemma 4 *Let* $1 \le p \le \infty$ *and* $1 < q < \infty$, *and let* $\varepsilon > 0$. *There is a constant* $1 \le K < \infty$ *such that for all* $N \in \mathbb{N}$ *there exist* $M = M_N$ *and a finite normalized sequence* $(x_i)_{i=1}^M \subset \ell_q(\ell_p^N)$ *such that*

(a) $(x_i)_{i=1}^M$ *is* 1-*unconditional*,
(b) $(1-\varepsilon)|A| \le \|\sum_{i\in A} x_i\|^q \le (1+\varepsilon)|A|$ *for all* $A \subset \{1,\dots,M\}$,
(c) *the span of* $(x_i)_{i=1}^M$ *is* K-*complemented in* $\ell_q(\ell_p^N)$, *and*
(d) ℓ_p^N *is isometric to a* K-*complemented subspace of the span of* $(x_i)_{i=1}^M$.

Using the lemma, we give a quick proof of the first part of Theorem 1.

Proof of Theorem 1 (a) Let $1 \le p \le \infty$ and $1 < q < \infty$. It will be more convenient for us to work with the space $X := (\bigoplus_{N=1}^\infty \ell_q(\ell_p^N))_{\ell_q}$ instead of $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$. That these spaces are isomorphic follows from *Pełczyński's Decomposition Method* [15], which says that if two Banach spaces are complementably embedded in each other and one of them is isomorphic to the (countably infinite) ℓ_r-sum, $1 \le r < \infty$, or c_0-sum of itself, then they are isomorphic. It is easy to observe that X and $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ are 1-complemented in each other and that X is isometric to its ℓ_q sum.

We let $\varepsilon > 0$ and choose, for each $N \in \mathbb{N}$, a sequence $(x_i^{(N)})_{i=1}^{M_N}$ in the Nth coordinate of $X = (\bigoplus_{N=1}^\infty \ell_q(\ell_p^{(N)}))_{\ell_q}$ which satisfies Lemma 4. From the conditions (a) and (b) in Lemma 4 it follows that $(x_i^{(N)})_{N\in\mathbb{N},1\le i\le M_N}$ is a 1-unconditional and $\frac{1+\varepsilon}{1-\varepsilon}$-democratic sequence in X. As the span of $(x_i^{(N)})_{i=1}^{M_N}$ is K-complemented in $\ell_q(\ell_p^N)$, for each $N \in \mathbb{N}$, it follows that the closed span of $(x_i^{(N)})_{N\in\mathbb{N},1\le i\le M_N}$ is K-complemented in X. Furthermore, as ℓ_p^N is K-complemented in the span of $(x_i^{(N)})_{i=1}^{M_N}$ for each $N \in \mathbb{N}$, it follows that $(\bigoplus_{N=1}^\infty \ell_p^N)_{\ell_q}$ is K-complemented in the closed span of $(x_i^{(N)})_{N\in\mathbb{N},1\le i\le M_N}$. Thus, by the Pełczyński decomposition theorem, X is isomorphic to the closed span of $(x_i^{(N)})_{N\in\mathbb{N},1\le i\le M_N}$. Hence X, and thus $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_q}$, has a greedy basis. □

Proof of Lemma 4 For $\varepsilon > 0$, we choose numbers $\varepsilon_i \searrow 0$ such that $\prod_{i=1}^\infty (1+\varepsilon_i) < 1+\varepsilon$ and $\prod_{i=1}^\infty (1-\varepsilon_i) > 1-\varepsilon$. For each $n \in \mathbb{N}$, we denote by $(e_{(i,n)})_{i=1}^N$ the unit vector basis for the nth coordinate of $\ell_q(\ell_p^N)$, and we denote by $(e_{(i,n)}^*)_{i=1}^N$ their biorthogonal functionals. Thus the norm on $\ell_q(\ell_p^N)$ is calculated by

$$\left\| \sum a_{(i,n)} e_{(i,n)} \right\| = \left(\sum_{n=1}^\infty \left(\sum_{i=1}^N |a_{(i,n)}|^p \right)^{q/p} \right)^{1/q} \quad \text{for } (a_{(i,n)} : 1 \le i \le N, n \in \mathbb{N}) \subset \mathbb{R}.$$

If $x \in \ell_q(\ell_p^N)$, then we denote the support of x by $\operatorname{supp}(x) = \{(i,n) | e_{(i,n)}^*(x) \ne 0\}$.

Before proceeding, we fix two sequences of integers $(m_i), (k_i) \in [\mathbb{N}]^\omega$ with $m_1 = k_1 = 1$ and which satisfy the following inequalities for all $i > 1$:

(i) $1/\varepsilon_i < m_i$,
(ii) $m_i^{1/q} + 1 < (1+\varepsilon_i)^{1/q} m_i^{1/q}$,
(iii) $(1 + (m_i/k_i)^{1/q})^q < 1 + \varepsilon_i$, and
(iv) $1 - \varepsilon_i < (1 - (m_i/k_i)^{1/q})^q$.

The above inequalities can be easily guaranteed by first choosing m_i large enough to satisfy (i) and (ii), and then choosing k_i large enough to satisfy (iii) and (iv). For the sake of convenience we define $n_j = \prod_{i=1}^{j} k_i$ for all $j \in \mathbb{N}$. We define the finite family $(x_{(i,j)})_{1\le i\le N, 1\le j\le n_N/n_i}$ by

$$x_{(i,j)} = \frac{1}{n_i^{1/q}} \sum_{s=1}^{n_i} e_{(i,s+(j-1)n_i)} \quad \text{for all } 1 \le i \le N \text{ and } 1 \le j \le n_N/n_i.$$

It is clear that $(x_{(i,j)})$ is a normalized and 1-unconditional basic sequence as the sequence has pairwise disjoint support. Also, $\bigcup_{i\le N, j\le n_N/n_i} \operatorname{supp}(x_{(i,j)}) = \{1,2,\ldots,N\} \times \{1,2,\ldots,n_N\}$. For each integer $1 \le \ell \le N$ and subset

$$A \subset \left\{(i,j) \in \mathbb{N}^2 | 1 \le i \le \ell \text{ and } 1 \le j \le n_N/n_i\right\},$$

we will prove by induction on ℓ that

$$|A| \prod_{i=2}^{\ell} (1-\varepsilon_i) \le \Big\| \sum_{(i,j)\in A} x_{(i,j)} \Big\|^q \le |A| \prod_{i=2}^{\ell} (1+\varepsilon_i). \tag{1}$$

First note that if $\ell = 1$ and $A \subset \{(1,j) \in \mathbb{N}^2 | 1 \le j \le n_N\}$, then

$$\Big\| \sum_{(1,j)\in A} x_{(1,j)} \Big\|^q = \Big\| \sum_{(1,j)\in A} e_{(1,j)} \Big\|^q = |A|.$$

Thus (1) is trivially satisfied. We now assume that (1) is satisfied for a given $1 \le \ell < N$, and we will prove it for $\ell + 1$. We first partition the set $\Omega = \{(i,j) \in \mathbb{N}^2 : 1 \le i \le \ell+1 \text{ and } 1 \le j \le n_N/n_i\}$ into sets $\Omega_1, \Omega_2, \ldots, \Omega_{n_N/n_{\ell+1}}$ defined for each $1 \le r \le n_N/n_{\ell+1}$ by

$$\Omega_r := \left\{(i,j) \in \mathbb{N}^2 | 1 \le i \le \ell+1, (r-1)n_{\ell+1}/n_i + 1 \le j \le r n_{\ell+1}/n_i\right\}.$$

Observe that for $1 \le r \le n_N/n_{\ell+1}$, $1 \le i \le \ell+1$, and $1 \le j \le n_N/n_i$,

$$\begin{aligned}
(i,j) \in \Omega_r \quad &\iff \quad (r-1)\frac{n_{\ell+1}}{n_i} + 1 \le j \le r\frac{n_{\ell+1}}{n_i} \\
&\iff \quad \operatorname{supp}(x_{(i,j)}) = \{i\} \times \left[(j-1)n_i + 1, j n_i\right] \\
&\qquad\qquad \subset \{i\} \times \left[(r-1)n_{\ell+1} + 1, r n_{\ell+1}\right].
\end{aligned}$$

Since $\operatorname{supp}(x_{(\ell+1,r)}) = \{\ell+1\} \times [(r-1)n_{\ell+1} + 1, r n_{\ell+1}]$, it follows that

$$(i,j) \in \Omega_r \quad \iff \quad \left\{(\ell+1, s) : (i,s) \in \operatorname{supp}(x_{(i,j)})\right\} \subset \operatorname{supp}(x_{(\ell+1,r)}).$$

Given the set $A \subset \Omega$, we partition A into $A_1, A_2, \dots, A_{n_N/n_{\ell+1}}$, by defining $A_r = A \cap \Omega_r$, for all $1 \le r \le n_N/n_{\ell+1}$. We note that

$$\Big\| \sum_{(i,j)\in A} x_{(i,j)} \Big\|^q = \sum_{r=1}^{n_N/n_{\ell+1}} \Big\| \sum_{(i,j)\in A_r} x_{(i,j)} \Big\|^q$$

and that $(x_{(i,j)})_{(i,j)\in\Omega_1}$ is 1-equivalent to $(x_{(i,j)})_{(i,j)\in\Omega_r}$ for all $1 \le r \le n_N/n_{\ell+1}$. Thus, to prove the inequality (1), we just need to consider the case $A = A_1$. We first note that if $(\ell+1, 1) \notin A_1$, then the inequality (1) is immediately true by the induction hypothesis. Thus we now assume that $(\ell+1, 1) \in A_1$ and $A_1 \setminus \{(\ell+1, 1)\} \neq \emptyset$.

Roughly speaking, we will argue that either $|A_1|$ is large enough so that $\sum_{(i,j)\in A_1} x_{(i,j)}$ can be replaced by $\sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} x_{(i,j)}$, or $|A_1|$ is so small that a large part of the support of $x_{(\ell+1,1)}$ is disjoint from

$$B = \big\{n\colon (i,n) \in \operatorname{supp}(x_{(i,j)}) \text{ for some } (i,j) \in A_1 \setminus \big\{(\ell+1,1)\big\}\big\},$$

and we can approximate $x_{(\ell+1,1)}$ by its projection onto $\operatorname{span}(e_{(\ell+1,n)} : n \in \{1, 2, \dots, N\} \setminus B)$. The first case we consider is that $|A_1| \ge m_{\ell+1}$. This assumption, together with the inequality $m_{\ell+1} > 1/\varepsilon_{\ell+1}$, yields

$$\frac{|A_1|}{|A_1| - 1} \le \frac{m_{\ell+1}}{m_{\ell+1} - 1} < \frac{1}{1 - \varepsilon_{\ell+1}}.$$

This allows us to obtain the desired lower estimate. Indeed,

$$\begin{aligned} \Big\| \sum_{(i,j)\in A_1} x_{(i,j)} r \Big\|^q &\ge \Big\| \sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} x_{(i,j)} \Big\|^q \\ &\ge \big(|A_1| - 1\big) \prod_{i=2}^{\ell} (1 - \varepsilon_i) \ge |A_1| \prod_{i=2}^{\ell+1} (1 - \varepsilon_i). \end{aligned}$$

To prove the upper estimate in (1), we use that $|A_1| \ge m_{\ell+1}$ together with (ii) to get

$$\frac{(|A_1| - 1)^{1/q} + 1}{|A_1|^{1/q}} < \frac{m_{\ell+1}^{1/q} + 1}{m_{\ell+1}^{1/q}} < (1 + \varepsilon_{\ell+1})^{1/q}.$$

Thus,

$$\begin{aligned} \Big\| \sum_{(i,j)\in A_1} x_{(i,j)} \Big\| &\le \Big\| \sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} x_{(i,j)} \Big\| + 1 \\ &\le \big(|A_1| - 1\big)^{1/q} \left(\prod_{i=2}^{\ell} (1 + \varepsilon_i) \right)^{1/q} + \left(\prod_{i=2}^{\ell} (1 + \varepsilon_i) \right)^{1/q} \\ &< |A_1|^{1/q} \left(\prod_{i=2}^{\ell+1} (1 + \varepsilon_i) \right)^{1/q}. \end{aligned}$$

This completes the proof of (1) for $\ell+1$ in the case that $|A_1| \geq m_{\ell+1}$. We now assume that $|A_1| < m_{\ell+1}$. The size of the support of each $x_{(i,j)}$ is given by $|\operatorname{supp}(x_{(i,j)})| = n_i$ for all $1 \leq i \leq \ell+1$ and $1 \leq j \leq n_N/n_i$. We thus have a simple estimate for the size of the union of the supports

$$\Bigg|\bigcup_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} \operatorname{supp}(x_{(i,j)})\Bigg| = \sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} \big|\operatorname{supp}(x_{(i,j)})\big| < |A_1| n_\ell. \tag{2}$$

We define sets

$$B_1 := \Bigg\{(\ell+1,n)\in\mathbb{N}^2 : (m,n)\in \bigcup_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} \operatorname{supp}(x_{(i,j)}) \text{ for some } 1\leq m\leq \ell\Bigg\}$$

and

$$B_2 := \operatorname{supp}(x_{(\ell+1,1)}) \setminus B_1.$$

The inequality (2) gives that $|B_1| < |A_1| n_\ell$. For $i=1,2$, we define $P_{B_i} x_{(\ell+1,1)}$ by $P_{B_i} x_{(\ell+1,1)} = \frac{1}{n_{\ell+1}^{1/q}} \sum_{(\ell+1,j)\in B_i} e_{(\ell+1,j)}$. We may estimate the value $\|P_{B_1} x_{(\ell+1,1)}\|$ by

$$\|P_{B_1} x_{(\ell+1,1)}\| = \frac{1}{n_{\ell+1}^{1/q}} |B_1|^{1/q} < \left(\frac{|A_1| n_\ell}{n_{\ell+1}}\right)^{1/q} < \left(\frac{m_{\ell+1} n_\ell}{n_{\ell+1}}\right)^{1/q} = \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}. \tag{3}$$

We use this to obtain the following estimate:

$$\begin{aligned}
\Bigg\|\sum_{(i,j)\in A_1} x_{(i,j)}\Bigg\|^q &= \Bigg\| P_{B_2} x_{(\ell+1,1)} + P_{B_1} x_{(\ell+1,1)} + \sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} x_{(i,j)}\Bigg\|^q \\
&= \|P_{B_2} x_{(\ell+1,1)}\|^q + \Bigg\| P_{B_1} x_{(\ell+1,1)} + \sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}} x_{(i,j)}\Bigg\|^q \\
&\leq 1 + \left[\left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} + \left(\big(|A_1|-1\big)\prod_{i=1}^{\ell}(1+\varepsilon_i)\right)^{1/q}\right]^q \\
&= 1 + \left[1 + \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \left(\big(|A_1|-1\big)\prod_{i=1}^{\ell}(1+\varepsilon_i)\right)^{-1/q}\right]^q \\
&\quad \times \big(|A_1|-1\big)\prod_{i=1}^{\ell}(1+\varepsilon_i) \\
&\leq \left[1 + \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \left(\prod_{i=1}^{\ell}(1+\varepsilon_i)\right)^{-1/q}\right]^q \prod_{i=1}^{\ell}(1+\varepsilon_i)
\end{aligned}$$

$$+\left[1+\left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}\left(\big(|A_1|-1\big)\prod_{i=1}^{\ell}(1+\varepsilon_i)\right)^{-1/q}\right]^q$$
$$\times\big(|A_1|-1\big)\prod_{i=1}^{\ell}(1+\varepsilon_i)$$
$$\le\left[1+\left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}\left(\prod_{i=1}^{\ell}(1+\varepsilon_i)\right)^{-1/q}\right]^q|A_1|\prod_{i=1}^{\ell}(1+\varepsilon_i)$$
$$\le|A_1|\prod_{i=1}^{\ell+1}(1+\varepsilon_i)\quad\text{by (iii).}$$

For proving the remaining lower inequality in (1), we will use the following estimate for $\|P_{B_2}x_{(\ell+1,1)}\|$ which follows from (3):

$$\|P_{B_2}x_{(\ell+1,1)}\|\ge 1-\|P_{B_1}x_{(\ell+1,1)}\|>1-\left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}.$$

This yields

$$\left\|\sum_{(i,j)\in A_1}x_{(i,j)}\right\|^q=\|P_{B_2}x_{(\ell+1,1)}\|^q+\left\|P_{B_1}x_{(\ell+1,1)}+\sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}}x_{(i,j)}\right\|^q$$
$$\ge\|P_{B_2}x_{(\ell+1,1)}\|^q+\left(-\|P_{B_1}x_{(\ell+1,1)}\|+\left\|\sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}}x_{(i,j)}\right\|\right)^q$$
$$=\|P_{B_2}x_{(\ell+1,1)}\|^q$$
$$+\left(1-\|P_{B_1}x_{(\ell+1,1)}\|\cdot\left\|\sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}}x_{(i,j)}\right\|^{-1}\right)^q$$
$$\times\left\|\sum_{(i,j)\in A_1\setminus\{(\ell+1,1)\}}x_{(i,j)}\right\|^q$$
$$\ge\left(1-\left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}\right)^q+\left(1-\left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}\right)^q\big(|A_1|-1\big)$$
$$\times\prod_{i=1}^{\ell}(1-\varepsilon_i)$$
$$\ge\left(1-\left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}\right)^q|A_1|\prod_{i=1}^{\ell}(1-\varepsilon_i)$$
$$>\prod_{i=1}^{\ell+1}(1-\varepsilon_i)|A_1|\quad\text{by (iv).}$$

Thus we have proven the inequalities (1) in all cases. It remains to prove that there exists a constant $1 \le K < \infty$, independent of $N \in \mathbb{N}$, such that $X := \operatorname{span}(x_{(i,j)})$ is K-complemented in $\ell_q(\ell_p^N)$ and ℓ_p^N is isometric to a K-complemented subspace of X. For each $1 \le i \le N$, we define the vector y_i as

$$y_i := \frac{n_i^{1/q}}{n_N^{1/q}} \sum_{j=1}^{n_N/n_i} x_{(i,j)} = \frac{1}{n_N^{1/q}} \sum_{j=1}^{n_N} e_{(i,j)}.$$

It should be clear that $(y_i)_{i=1}^N$ is 1-equivalent to the unit vector basis for ℓ_p^N. Indeed, if $(a_i) \in \ell_p^N$, then

$$\left\| \sum_{i=1}^N a_i y_i \right\| = \left(\sum_{j=1}^{n_N} \left(\sum_{i=1}^N \frac{|a_i|^p}{n_N^{p/q}} \right)^{q/p} \right)^{1/q} = \left(\frac{1}{n_N} \sum_{j=1}^{n_N} \left(\sum_{i=1}^N |a_i|^p \right)^{q/p} \right)^{1/q}$$
$$= \left(\sum_{i=1}^N |a_i|^p \right)^{1/p}.$$

We let $Y = \operatorname{span}(y_i)$ and define projections $T_X : \ell_q(\ell_p^N) \to X$ and $T_Y : \ell_q(\ell_p^N) \to Y$ by

$$T_X\Big(\sum a_{(i,j)} e_{(i,j)}\Big) = \sum_{i=1}^N \sum_{j=1}^{n_N/n_i} \sum_{s=1}^{n_i} \left(\frac{1}{n_i} \sum_{k=1}^{n_i} a_{(i,k+(j-1)n_i)} \right) e_{(i,s+(j-1)n_i)}$$
$$= \sum_{i=1}^N \sum_{j=1}^{n_N/n_i} \left(\frac{1}{n_i^{(q-1)/q}} \sum_{k=1}^{n_i} a_{(i,k+(j-1)n_i)} \right) x_{(i,j)} \quad \text{and}$$
$$T_Y\Big(\sum a_{(i,j)} e_{(i,j)}\Big) = \sum_{i=1}^N \sum_{s=1}^{n_N} \left(\frac{1}{n_N} \sum_{k=1}^{n_N} a_{(i,k)} \right) e_{(i,s)} = \sum_{i=1}^N \left(\frac{1}{n_N^{(q-1)/q}} \sum_{k=1}^{n_N} a_{(i,k)} \right) y_i.$$

It is simple to check that T_X and T_Y are projections of $\ell_q(\ell_p^N)$ onto X and Y, respectively. As Y is a subspace of X, we have that T_Y restricted to X is a projection of X onto Y. Thus we just need to prove that there exists a uniform constant K such that $\|T_X\|, \|T_Y\| \le K$. We note that $T_X = T_Y$ if, considering a more general case, $n_i = n_N$ for all $1 \le i \le N$. Thus proving there exists a constant K independent of $(n_i)_{i=1}^N$ and N such that $\|T_X\| \le K$ will prove the inequality for $\|T_Y\|$ as well. We first consider the case that $p = q$. In this case, the basis $(e_{(i,j)})$ for the space $\ell_q(\ell_q^N)$ is symmetric. The operator T_X is then an averaging operator on a space with a symmetric basis, and hence has norm one.

Now if $q < p < \infty$, then $\ell_q(\ell_p^N)$ is an interpolation space for the spaces $\ell_q(\ell_q^N)$ and $\ell_q(\ell_\infty^N)$ [3]. Then by the vector-valued Riesz–Thorin interpolation theorem [3] (see also [4]) , if $\theta = \frac{q}{p}$, then

$$\|T_X\|_{\ell_q(\ell_p^N)} \le \|T_X\|_{\ell_q(\ell_q^N)}^{\theta} \|T_X\|_{\ell_q(\ell_\infty^N)}^{1-\theta} \le \|T_X\|_{\ell_q(\ell_\infty^N)}.$$

Thus if we prove that there exists a uniform constant K such that $\|T_X\|_{\ell_q(\ell_\infty^N)} \leq K$, then the result will follow as well for all $1 < q < p < \infty$. On the other hand, if $1 \leq p < q < \infty$, then $1 < q' < p' \leq \infty$, with $q' = q/(q-1)$ and $p' = p/(p-1)$. It is simple to check that our operator $T_X : \ell_q(\ell_p^N) \to \ell_q(\ell_p^N)$ has adjoint $T_X^* = T_X : \ell_{q'}(\ell_{p'}^N) \to \ell_{q'}(\ell_{p'}^N)$. We thus have that $\|T_X\|_{\ell_q(\ell_p^N)} = \|T_X^*\|_{\ell_{q'}(\ell_{p'}^N)} \leq K$.

All that remains is to prove is that there exists a uniform constant K such that $\|T_X\|_{\ell_q(\ell_\infty^N)} \leq K$. This constant K will come from a *discretization* of the classical Hardy–Littlewood maximal operator [11], which is defined as

$$n_u(g)(x) = \sup_{y<x<z} \frac{1}{z-y} \int_y^z \left|g(t)\right| dt \quad \text{for } x \in \mathbb{R} \text{ and } g \in L^1_{\text{loc}}(\mathbb{R}). \tag{4}$$

It is known that the operator n_u is of strong type (q,q) for $1 < q \leq \infty$, and for a proof of this see [9, Theorem 8.9.1 and Corollary 8.9.1]. In other words, there exists a constant $1 \leq K < \infty$ such that

$$\left(\int \left|n_u(g)(x)\right|^q dx\right)^{1/q} \leq K\left(\int \left|g(x)\right|^q dx\right)^{1/q} \quad \text{for all } g \in L_q(\mathbb{R}).$$

By applying this to step functions whose discontinuities are contained in $\mathbb{N}$, we get the following inequality for ℓ_q:

$$\left(\sum_{j\in\mathbb{N}}\left(\sup_{m\leq j\leq n} \frac{1}{n-m+1}\sum_{k=m}^{n}|a_k|\right)^q\right)^{1/q} \leq K\left(\sum_{j\in\mathbb{N}}|a_j|^q\right)^{1/q} \quad \text{for all } (a_j) \in \ell_q. \tag{5}$$

We now prove that $\|T_X\|_{\ell_q(\ell_\infty^N)} \leq K$. Note that T_X is zero on all vectors of the form $\sum_{k>n_N} a_{(i,k)} e(i,k)$. It follows that the operator T_X attains its norm at an extreme point of the unit ball $B_{\ell_q^{n_N}(\ell_\infty^N)}$ of the finite-dimensional subspace $\ell_q^{n_N}(\ell_\infty^N) \subset \ell_q(\ell_\infty^N)$. The set of extreme points of $B_{\ell_q^{n_N}(\ell_\infty^N)}$ is given by

$$\operatorname{Ext}(B_{\ell_q(\ell_\infty^{n_N})}) = \left\{\sum_{j=1}^{n_N} a_j \sum_{i=1}^{N} \varepsilon_{(i,j)} e_{(i,j)} : \varepsilon_{(i,j)} = \pm 1, (a_j) \in S_{\ell_q^{n_N}}\right\}.$$

Thus there exist constants $\varepsilon_{(i,j)} = \pm 1$ and a sequence $(a_j) \in S_{\ell_q^{n_N}}$ such that

$$\begin{aligned}
\|T_X\|_{\ell_q(\ell_\infty^N)} &= \left\| T_X\left(\sum_{j=1}^{n_N} a_j \sum_{i=1}^{N} \varepsilon_{(i,j)} e_{(i,j)}\right)\right\| \\
&= \left\| \sum_{i=1}^{N} \sum_{j=1}^{n_N/n_i} \sum_{s=1}^{n_i} \left(\frac{1}{n_i}\sum_{k=1}^{n_i} \varepsilon_{(i,k+(j-1)n_i)} a_{k+(j-1)n_i}\right) e_{(i,s+(j-1)n_i)}\right\| \\
&\leq \left\| \sum_{i=1}^{N} \sum_{j=1}^{n_N/n_i} \sum_{s=1}^{n_i} \left(\frac{1}{n_i}\sum_{k=1}^{n_i} |a_{k+(j-1)n_i}|\right) e_{(i,s+(j-1)n_i)}\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \left\| \sum_{i=1}^{N} \sum_{j=1}^{n_N/n_i} \sum_{s=1}^{n_i} \left(\sup_{m \le s+(j-1)n_i \le n} \frac{1}{n+1-m} \sum_{k=m}^{n} |a_k| \right) e_{(i,s+(j-1)n_i)} \right\| \\
&\le \left\| \sum_{i=1}^{N} \sum_{j=1}^{n_N} \left(\sup_{m \le j \le n} \frac{1}{n+1-m} \sum_{k=m}^{n} |a_k| \right) e_{(i,j)} \right\| \\
&= \left(\sum_{j=1}^{n_N} \left(\sup_{m \le j \le n} \frac{1}{n+1-m} \sum_{k=m}^{n} |a_k| \right)^q \right)^{1/q} \\
&\le K \left(\sum_{j=1}^{n_N} |a_j|^q \right)^{1/q} \qquad \text{by (5)} \\
&= K. \qquad \square
\end{aligned}
$$

Theorem 5 *The Banach space $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_1}$ does not have a greedy basis whenever $1 < p \le \infty$.*

We recall that Bourgain, Casazza, Lindenstrauss, and Tzafriri proved that the spaces $(\bigoplus_{n=1}^{\infty} \ell_2^n)_{\ell_1}$ and $(\bigoplus_{n=1}^{\infty} \ell_\infty^n)_{\ell_1}$ each have unconditional bases which are unique up to permutation [2]. In particular, these spaces cannot have a greedy basis, as the unconditional basis of each of these spaces (which is unique up to permutation) fails to be democratic. Thus we need only to prove Theorem 5 for the case $1 < p < \infty$. This is important for us as ℓ_p has nontrivial type and cotype when $1 < p < \infty$. We rely on the following proposition, which was used in [2] to prove, among other uniqueness results, that $(\bigoplus_{n=1}^{\infty} \ell_2^n)_{\ell_1}$ has a unique unconditional basis up to permutation.

Proposition B [2, Proposition 2.1] *Let V be a Banach lattice of type p and cotype q for some $1 \le p \le q < \infty$, and let $C_c, C_u \ge 1$ be constants. There exists a uniform constant $K \ge 1$ which satisfies the following statement. If Z is a C_c-complemented subspace of the direct sum $X = (\bigoplus_{n=1}^{\infty} V)_{c_0}$ and Z has a normalized basis $(z_n)_{n=1}^{k}$ with unconditional constant C_u, then there exists a partition of the integers $\{1, 2, \ldots, k\}$ into mutually disjoint subsets $\{\tau_s\}_{s=1}^{r}$ so that, for any choice of scalars $\{\alpha_n\}_{n=1}^{k}$, we have*

$$
K^{-1} \max_{1 \le s \le r} \left(\sum_{n \in \tau_s} |\alpha_n|^q \right)^{1/q} \le \left\| \sum_{n=1}^{k} \alpha_n z_n \right\| \le K \max_{1 \le s \le r} \left(\sum_{n \in \tau_s} |\alpha_n|^p \right)^{1/p}.
$$

Using Proposition B, we are now prepared to give a proof of Theorem 5.

Proof of Theorem 5 Let $1 < p < \infty$. To reach a contradiction, we assume that $X = (\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_1}$ has a normalized basis $(x_i)_{i=1}^{\infty}$ which is C_d-democratic and C_u-unconditional for some constants $C_d, C_u \ge 1$. Let $(x_i^*) \subset (\bigoplus_n \ell_p^n)_{\ell_1}^* = (\bigoplus_n \ell_q^n)_{\ell_\infty}$, with $\frac{1}{p} + \frac{1}{q} = 1$, be the biorthogonal functionals. We let $(e_{(i,n)})_{1 \le i \le n < \infty}$ be the unit vector basis for X with biorthogonal functionals $(e^*_{(i,n)})_{1\le i\le n<\infty}$. By a standard perturbation argument we may assume that

$$\mathrm{supp}(x_j) = \left\{(i,n) : n \in \mathbb{N}, i \le n, e^*_{(i,n)}(x_j) \ne 0\right\}$$

is finite for each $j \in \mathbb{N}$.

We now fix $N \in \mathbb{N}$. There exists $M_N \in \mathbb{N}$ such that $(x_j)_{j=1}^N \subset (\bigoplus_{n=1}^{M_N} \ell_p^n)_{\ell_1}$, i.e., $e^*_{(i,n)}(x_j) = 0$ for all $n > M_n$ and $i \le n$. Define $(z^*_{(N,j)})_{j=1}^N \subset (\bigoplus_{n=1}^{M_N} \ell_p^n)^*_{\ell_1}$ by $z^*_{(N,j)} = x_j^*|_{(\bigoplus_{n=1}^{M_N} \ell_p^n)_{\ell_1}}$ for all $1 \le j \le N$. As $(x_j)_{j=1}^\infty$ has basis constant at most C_u, it is easy to show both that $(x_j^*)_{j=1}^N$ is C_u-equivalent to $(z^*_{(N,j)})_{j=1}^N$ and that the span of $(z^*_{(N,j)})_{j=1}^N$ is C_u-complemented in $(\bigoplus_{n=1}^{M_N} \ell_p^n)^*_{\ell_1}$. Indeed, let $J : \mathrm{span}(x_i)_{i=1}^N \to X$ be the inclusion map. Then $J^* : X^* \to \mathrm{span}(x_i^*)_{i=1}^N$ is a quotient map of norm 1. Let $H : \mathrm{span}(x_i^*)_{i=1}^N \to \mathrm{span}(z^*_{(N,i)})_{i=1}^N$ be the isomorphism defined by $H(x_i^*) = z^*_{(N,i)}$, which has norm at most C_u. Then $H \circ J^* : X^* \to \mathrm{span}(z^*_{(N,i)})_{i=1}^N$ is a quotient map of norm at most C_u. We just need to check that $H \circ J^*(z^*_{(N,i)}) = z^*_{(N,i)}$. Indeed, for $x \in \mathrm{span}(x_j : j \le N)$, $J^*(z^*_{(N,i)})(x) = z^*_{(N,i)}(x) = x_i^*(x)$. Hence, $J^*(z^*_{(N,i)}) = x_i^*$ and $H \circ J^*(z^*_{(N,i)}) = z^*_{(N,i)}$. Thus $H \circ J^*$ is a projection of norm at most C_u.

We will be applying Proposition B for the space $V = \ell_q$ and consider the spaces $\mathrm{span}(z^*_{(N,i)} : i \le N)$, $N \in \mathbb{N}$, (in the natural way) to be C_u-complemented subspaces of $c_0(\ell_q)$. For the sake of convenience, we denote $\underline{q} = \min(q,2)$ and $\bar{q} = \max(q,2)$. We have thus defined $\underline{q}$ and $\bar{q}$ exactly so that ℓ_q has type $\underline{q}$ and cotype $\bar{q}$. By Proposition B, there exists a constant K independent of $N \in \mathbb{N}$, and there exists for all $N \in \mathbb{N}$ a partition of the integers $\{1, 2, \dots, N\}$ into mutually disjoint subsets $\{\tau_s^N\}_{s=1}^{r_N}$ so that, for any choice of scalars $\{\alpha_n\}_{n=1}^N$, we have

$$K^{-1} \max_{1\le s\le r_N} \left(\sum_{n\in\tau_s^N} |\alpha_n|^{\bar{q}}\right)^{1/\bar{q}} \le \left\|\sum_{n=1}^N \alpha_n z^*_{(N,n)}\right\| \le K \max_{1\le s\le r_N} \left(\sum_{n\in\tau_s^N} |\alpha_n|^{\underline{q}}\right)^{1/\underline{q}}.$$

We first consider the case that $\sup_{N\in\mathbb{N}} \max_{1\le s\le r_N} |\tau_s^N| < \infty$. In this case we have that if $N \in \mathbb{N}$ and $(\alpha_i)_{i=1}^N \subset \mathbb{R}$, then

$$\left\|\sum_{i=1}^N \alpha_i x_i^*\right\| \le C_u \left\|\sum_{i=1}^N \alpha_i z^*_{(N,i)}\right\| \le C_u K \Big(\sup_{M\in\mathbb{N}} \max_{1\le s\le r_M} \left|\tau_s^M\right|\Big) \max_{1\le i\le N} |\alpha_i|.$$

Hence (x_i^*) is equivalent to the unit vector basis of c_0, which implies that (x_i) is equivalent to the unit vector basis of ℓ_1. This is a contradiction, as (x_i) is a basis for $X = (\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_1}$ and X is not isomorphic to ℓ_1 as $1 < p < \infty$. We now consider the remaining case that $\sup_{N\in\mathbb{N}} \max_{1\le s\le r_N} |\tau_s^N| = \infty$.

First note that there exists a subsequence of (x_i) which is equivalent to the unit vector basis of ℓ_1. Indeed, there is a subsequence (x'_j) which converges in the w^*-topology (considering X as the dual of $X_* = (\bigoplus_{n=1}^\infty \ell_q^n)_{c_0}$) to some $x \in X$. If $x \ne 0$, then we may choose $y \in (\bigoplus_{n=1}^\infty \ell_q^n)_{c_0}$, with $\|y\| = 1$, such that $\langle y, x\rangle \ge \|x\|/2$. (Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $X_* \times X$.) Since (x'_j) is w^*-convergent to x, we may assume (by passing to a further subsequence if necessary) that $\langle y, x'_j\rangle \geq \|x\|/4$ for all $j \geq 1$. By unconditionality of (x'_j), for all scalars (a'_j), we have

$$C_u \Big\| \sum a'_j x'_j \Big\| \geq \Big\| \sum |a'_j| x'_j \Big\| \geq \Big\langle y, \sum |a'_j| x'_j \Big\rangle \geq \frac{\|x\|}{4} \sum |a'_j|,$$

and hence (x'_j) is equivalent to the unit vector basis of ℓ_1. On the other hand, if $x = 0$, then we can find an arbitrarily small norm perturbation (z'_j) of a further subsequence of (x'_j) such that for each $n \in \mathbb{N}$ there is at most one j such that the ℓ_q^n component of z'_j is nonzero. Note that (z'_j) is then isometrically equivalent to the unit vector basis of ℓ_1. Hence any sufficiently small norm perturbation of (z'_j) is equivalent to the unit vector basis of ℓ_1. In particular, (x'_j) is equivalent to the unit vector basis of ℓ_1.

Thus, as (x_i) is democratic and has a subsequence equivalent to the unit vector basis of ℓ_1, there exists $C \geq 1$ such that $C\|\sum_{i\in A} x_i\| > |A|$ for all finite nonempty sets $A \subset \mathbb{N}$. We choose $N \in \mathbb{N}$ and $1 \leq s \leq r_N$ such that $|\tau_s^N|^{1/\bar{q}} > 2KCC_u^2$. Thus,

$$\begin{aligned}
\Big\| \sum_{i\in\tau_s^N} x_i \Big\| &\leq 2C_u \Big(\sum_{i\in\tau_s^N} b_i x_i^* \Big)\Big(\sum_{i\in\tau_s^N} x_i \Big) \quad \text{for some } (b_i) \in c_{00}, \text{ with} \Big\| \sum_{i\in\tau_s^N} b_i x_i^* \Big\| = 1 \\
&\leq 2C_u^2 \frac{\sum_{i\in\tau_s^N} |b_i|}{\| \sum_{i\in\tau_s^N} b_i z^*_{(N,i)} \|} \\
&\leq 2KC_u^2 \frac{\sum_{i\in\tau_s^N} |b_i|}{(\sum_{i\in\tau_s^N} |b_i|^{\bar{q}})^{1/\bar{q}}} \\
&\leq 2KC_u^2 |\tau_s^N|^{1-1/\bar{q}} \quad \text{by Hölder's inequality.}
\end{aligned}$$

Combining this result with the inequality $|\tau_s^N|^{1/\bar{q}} > 2KCC_u^2$ gives the following contradiction:

$$2KCC_u^2 \big|\tau_s^N\big|^{1-1/\bar{q}} < \big|\tau_s^N\big| < C \Big\| \sum_{i\in\tau_s^N} x_i \Big\| \leq 2KCC_u^2 \big|\tau_s^N\big|^{1-1/\bar{q}}.$$

As both possible cases result in a contradiction, we see that $(\bigoplus_{n=1}^\infty \ell_p^n)_{\ell_1}$ cannot have a greedy basis when $1 < p < \infty$. □

Finally, the case $q = \infty$ and $1 \leq p < \infty$ in Theorem 1 is easy to handle.

Proposition 6 *For $1 \leq p < \infty$ the space $(\bigoplus \ell_p^n)_{c_0}$ does not have a greedy basis.*

Proof Assume that (x_n) is a greedy basis of $(\bigoplus \ell_p^n)_{c_0}$. Let (x_n^*) be the biorthogonal sequence in $(\bigoplus \ell_q^n)_{\ell_1}$. Then (x_n^*) is unconditional since (x_n) is unconditional. By the argument given above (which is valid for any semi-normalized unconditional basis sequence in $(\bigoplus \ell_q^n)_{\ell_1}$) (x_n^*) has a subsequence $(x_n'^*)$ equivalent to the unit vector basis of ℓ_1. By unconditionality the natural projection of the closed linear span of (x_n^*) onto the closed linear span of $(x_n'^*)$ is bounded. Since $(x_n'^*)$ is equivalent to the unit vector basis of ℓ_1, it follows that (x'_n) is equivalent to the unit vector basis of c_0. Since (x_n) is democratic and contains a subsequence (x'_n) which is equivalent to the unit vector basis of c_0, it follows that for some constant $C \geq 1$, we have

$$\left\| \sum_{n \in A} x_n \right\| \leq C \quad \text{for all finite } A \subset \mathbb{N}.$$

This together with the unconditionality of (x_n) implies that (x_n) is equivalent to the unit vector basis of c_0, which is a contradiction since $(\bigoplus \ell_p^n)_{c_0}$ is not isomorphic to c_0. □

We have now finished the proof of Theorem 1 and determined which spaces of the form $(\bigoplus_{N=1}^{\infty} \ell_p^N)_{\ell_q}$ have a greedy basis. We now turn to the proof of Theorem 2.

We rely on the concept of greedy permutations developed by Albiac and Wojtaszczyk [1], which we recall here. Let $M(x)$ denote the subset of the support of x consisting of the largest coordinates of x in absolute value. We will say that a one-to-one map $\pi : \text{supp}(x) \to \mathbb{N}$ is a greedy permutation of x if $\pi(j) = j$ for all $j \in \text{supp}(x) \setminus M(x)$ and if $j \in M(x)$ then, either $\pi(j) = j$ or $\pi(j) \notin \text{supp}(x)$.

Definition A basic sequence (e_n) is defined to have property (A) if for any $x \in \text{span}(e_i)$ we have

$$\left\| \sum_{n \in \text{supp}(x)} e_n^*(x) e_n \right\| = \left\| \sum_{n \in \text{supp}(x)} \theta_{\pi(n)} e_n^*(x) e_{\pi(n)} \right\|$$

for all greedy permutations π of x and all sequences of signs (θ_k) with $\theta_{\pi(n)} = 1$ if $\pi(n) = n$.

We recall that (e_n) is called C-*suppression unconditional*, for some $C \geq 1$, if for any $(a_i) \subset c_{00}$ and any $A \subset \mathbb{N}$,

$$\left\| \sum_{i \in A} a_i e_i \right\| \leq C \left\| \sum_{i \in \mathbb{N}} a_i e_i \right\|.$$

Theorem C [1, Theorem 3.4] *A basic sequence (e_n) is 1-greedy if and only if (e_n) is 1-suppression unconditional and satisfies property (A).*

Proof of Theorem 2 We first consider the case that $1 < p < \infty$. We show that if $A \subset \mathbb{N}$ is any finite set, then $\| \sum_{i \in A} a_i x_i \| = (\sum_{i \in A} |a_i|^p)^{1/p}$ for all $(a_i) \in c_{00}$. This is trivial if $|A| = 1$ as (x_i) is normalized. We now assume that the equality holds for $|A| \leq k$ for some $k \geq 1$. Let $(a_i) \in c_{00}$ and $A \subset \mathbb{N}$ such that $|A| = k + 1$. Choose $N \in A$ such that $|a_N| = \max_{i \in A} |a_i|$. We define $\pi_j : A \to \mathbb{N}$ by $\pi_j(N) = j$ and $\pi_j(n) = n$ for all $n \neq N$. The map π_j is a greedy permutation whenever $j \notin A$, and hence by Theorem C we have the following equalities:

$$\begin{aligned}
&\Big\|\sum_{i\in A} a_i x_i\Big\|\\
&\quad=\Big\|\sum_{i\in A, i\neq N} a_i x_i + a_N x_j\Big\| \quad \text{for all } j\notin A\\
&\quad=\lim_{j\to\infty}\Big\|\sum_{i\in A, i\neq N} a_i x_i + a_N x_j\Big\|\\
&\quad=\Big(\Big\|\sum_{i\in A, i\neq N} a_i x_i\Big\|^p + |a_N|^p\Big)^{1/p}, \quad \text{as } (x_j) \text{ is normalized and weakly null}\\
&\quad=\Big(\sum_{i\in A}|a_i|^p\Big)^{1/p} \quad \text{by the induction hypothesis.}
\end{aligned}$$

This finishes the proof of the induction step, and, thus, the proof of our claim.

The case $p=\infty$, in which case we consider the c_0-sum of the E_n, works similarly, as every normalized unconditional sequence in $(\sum E_n)_{c_0}$ must be weakly null.

We now consider the ℓ_1 case. Let (x_i) be a 1-greedy basis for $(\sum E_n)_{\ell_1}$. If (x_i) is w^*-null with respect to the w^* topology given by $(\sum E_n^*)_{\ell_\infty}$, then the proof that (x_i) is 1-equivalent to the unit vector basis for ℓ_1 is the same as the previous case $1<p<\infty$. If (x_i) is not w^*-null, then (x_i) has a subsequence (x_{k_i}) which converges w^* to some nonzero $x\in(\sum E_n)_{\ell_1}$. Hence $(x_{k_i}-x)$ is w^*-null. This implies that $\lim_{i\to\infty}\|y+x_{k_i}-x\|=\|y\|+\lim_{i\to\infty}\|x_{k_i}-x\|$ for all $y\in(\sum E_n)_{\ell_1}$. We use this to achieve the following equalities:

$$\begin{aligned}
\lim_{n\to\infty}\lim_{i\to\infty}\|x_{k_n}-x_{k_i}\| &= \lim_{n\to\infty}\lim_{i\to\infty}\big\|(x_{k_n}-x)-(x_{k_i}-x)\big\|\\
&=\lim_{n\to\infty}\|x_{k_n}-x\|+\lim_{i\to\infty}\|x_{k_i}-x\| = 2\lim_{i\to\infty}\|x_{k_i}-x\|.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\lim_{n\to\infty}\lim_{i\to\infty}\|x_{k_n}+x_{k_i}\| &= \lim_{n\to\infty}\lim_{i\to\infty}\big\|(x_{k_n}-x)+(x_{k_i}-x)+2x\big\|\\
&=\lim_{n\to\infty}\|x_{k_n}-x+2x\|+\lim_{i\to\infty}\|x_{k_i}-x\|\\
&=2\|x\|+\lim_{n\to\infty}\|x_{k_n}-x\|+\lim_{i\to\infty}\|x_{k_i}-x\|\\
&=2\|x\|+2\lim_{i\to\infty}\|x_{k_i}-x\|.
\end{aligned}$$

As (x_i) is 1-greedy, we must have, by Theorem C, that $\lim_{n\to\infty}\lim_{i\to\infty}\|x_{k_n}-x_{k_i}\| = \lim_{n\to\infty}\lim_{i\to\infty}\|x_{k_n}+x_{k_i}\|$. This however implies that $\|x\|=0$, which is a contradiction with our assumption that x is nonzero. □

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References

1. Albiac, F., Wojtaszczyk, P.: Characterization of 1-greedy bases. J. Approx. Theory **138**(1), 65–86 (2006)
2. Bourgain, J., Casazza, P.G., Lindenstrauss, J., Tzafriri, L.: Banach spaces with a unique unconditional basis, up to permutation. Mem. Am. Math. Soc. **322** (1985)
3. Benedek, A., Panzone, R.: The space L_p, with mixed norm. Duke Math. J. **28**, 301–324 (1961)
4. Calderón, A.P.: Intermediate spaces, and interpolation. Stud. Math. **24**, 113–190 (1964)
5. Dilworth, S.J., Hoffmann, Mark, Kutzarova, D.: Non-equivalent greedy and almost-greedy bases in ℓ_p. J. Funct. Spaces Appl. **318**, 692–706 (2006)
6. Dilworth, S.J., Kalton, N.J., Kutzarova, D.: On the existence of almost greedy bases in Banach spaces. Stud. Math. (Pełczyński Anniversary Volume) Stud. Math. **159**, 67–101 (2003)
7. Dilworth, S.J., Kalton, N.J., Kutzarova, D., Temlyakov, V.N.: The thresholding greedy algorithm, greedy bases, and duality. Constr. Approx. **19**(4), 575–595 (2003)
8. Edelstein, I.S., Wojtaszczyk, P.: On projections and unconditional bases in direct sums of Banach spaces. Stud. Math. **56**(3), 263–276 (1976)
9. Garling, D.J.H.: Inequalities: A Journey into Linear Analysis. Cambridge University Press, Cambridge (2007)
10. Garrigós, G., Hernández, E.: Sharp Jackson and Bernstein inequalities for N-term approximation in sequence spaces with applications. Indiana Univ. Math. J. **53**, 1739–1762 (2004)
11. Hardy, G.H., Littlewood, J.E.: A maximal theorem with function-theoretic applications. Acta Math. **54**, 81–116 (1930)
12. Kamont, A.: General Haar systems and greedy approximation. Stud. Math. **145**, 165–184 (2001)
13. Konyagin, S.V., Temlyakov, V.N.: A remark on greedy approximation in Banach spaces. East J. Approx. **5**(3), 365–379 (1999)
14. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I: Sequence Spaces. Springer, Berlin-Heidelberg-New York (1977)
15. Pełczyński, A.: Projections in certain Banach spaces. Stud. Math. **19**, 209–228 (1960)
16. Pietsch, A.: Eigenvalues and s-Numbers. Cambridge Studies in Advanced Mathematics, vol. 13. Cambridge University Press, Cambridge (1987)
17. Ropela, S.: Spline bases in Besov spaces. Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. **24**, 319–325 (1976)
18. Temlyakov, V.N.: The best m-term approximation and greedy algorithms. Adv. Comput. Math. **8**, 249–265 (1998)
19. Tsirelson, B.S.: Not every Banach space contains ℓ_p or c_0. Funct. Anal. Appl. **8**, 138–141 (1974)
20. Wojtaszczyk, P.: Greedy algorithm for general biorthogonal systems. J. Approx. Theory **107**, 293–314 (2000)
21. Wojtaszczyk, P.: Greedy type bases in Banach spaces. In: Constructive Theory of Functions, pp. 136–155. DARBA, Sofia (2003)
22. Wojtaszczyk, P.: Greediness of the Haar system in rearrangement invariant spaces. In: Approximation and Probability. Banach Center Publ., vol. 72, pp. 385–395. Polish Acad. Sci., Warsaw (2006)