DISCRETE MATH: LECTURE 15

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1. Chapter 6.1 Set Theory: Definitions and the Element Method of Proof

Recall that a set is a collection of elements.

Some examples of sets of numbers are:

- $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is the set of integers.
- $\mathbb{R}$ is the set of all real numbers.
- $\mathbb{Q} = \{m \in \mathbb{R} \,|\, \exists p, q \in \mathbb{Z}$ with $q \neq 0$ and $m = \frac{p}{q}\}$ is the set of rational numbers.

If P is a property, then we write $P(x)$ to mean that x satisfies property P. If S is a set, then the following denotes the subset of S consisting of all elements of S which satisfy property P.

$$A = \{x \in S \,|\, P(x)\}$$

1.1. Subsets.

Subsets. If A and B are sets, then we say that A is a subset of B if every element of A is an element of B. That is:

$$A \subseteq B \Leftrightarrow \forall x \in A, x \in B.$$

A is not a subset of B is given by:

$$A \not\subseteq B \Leftrightarrow \exists x \in A, x \notin B.$$

A is called a proper subset of B if

(1) $A \subseteq B$,
(2) $\exists y \in B$, $y \notin A$.

How to prove that one set is a subset of the other: The Element Argument

Given sets X and Y, the following shows that $X \subseteq Y$.

(1) Let x be an element of X.
(2) Prove that x is an element of Y.
(3) Conclude that $X \subseteq Y$.

Example: Let A and B be the following sets,

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{m \in \mathbb{Z} \mid m = 3s \text{ for some } s \in \mathbb{Z}\}$$

(1) Prove that $A \subseteq B$

(2) Prove that A is a proper subset of B.

Given sets A and B. A equals B, written $A = B$, if and only if, every element of A is an element of B and every element of B is an element of A. In other word,

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$$

Example: Let A and B be the following sets,

$$A = \{m \in \mathbb{Z} \mid m \bmod 2 = 0\}$$
$$B = \{m \in \mathbb{Z} \mid 2m \bmod 4 = 0\}$$

(1) Prove that $A = B$

In class work: Let A and B be the following sets,

$$A = \{m \in \mathbb{Z} \mid m = 2s \text{ for some } s \in \mathbb{Z}\}$$
$$B = \{m \in \mathbb{Z} \mid m = 2r - 2 \text{ for some } r \in \mathbb{Z}\}$$

(1) Prove that $A = B$

In class work: Let A and B be the following sets,

$$A = \{m \in \mathbb{Z} \,|\, m = 6r - 5 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{m \in \mathbb{Z} \,|\, m = 3s + 1 \text{ for some } s \in \mathbb{Z}\}$$

(1) Prove that $A \subset B$

(2) Prove that A is a proper subset of B.

1.2. **Operations on Sets.** Let A and B be subsets of a set U.

(1) The **union** of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.

$$A \cup B = \{x \in U \,|\, x \in A \text{ or } x \in B\}$$

(2) The **intersection** of A and B, denoted $A \cap B$, is the set of all elements that are both in A and in B.

$$A \cap B = \{x \in U \,|\, x \in A \text{ and } x \in B\}$$

(3) The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.

$$A^c = \{x \in U \,|\, x \notin A\}$$

(4) The **set difference**, B minus A, (or **relative complement** of A in B), denoted $B - A$, is the set of all elements that are in B and not A.

$$B - A = \{x \in B \,|\, x \notin A\}$$

Venn Diagrams are visual representations of sets

$A \cup B$ $A \cap B$ A^c $B - A$

Unions and Intersections of an Indexed Collection of Sets:
Given subsets $A_0, A_1, A_2, ...$ of U.

$$\cup_{i=0}^{n} A_i = \{x \in U \,|\, x \in A_i \text{ for at least one } i = 0, 1, 2, ..., n\}$$
$$\cup_{i=0}^{\infty} A_i = \{x \in U \,|\, x \in A_i \text{ for at least one non-negative integer } i\}$$
$$\cap_{i=0}^{n} A_i = \{x \in U \,|\, x \in A_i \text{ for all } i = 0, 1, 2, ..., n\}$$
$$\cap_{i=0}^{\infty} A_i = \{x \in U \,|\, x \in A_i \text{ for all non-negative integers } i\}$$

Examples:

- $\cup_{i=0}^{n}[i, i+1]$
- $\cap_{i=1}^{n}[-1-\frac{1}{i}, 1+\frac{1}{i}]$
- $\cup_{i=1}^{\infty}[-1+\frac{1}{i}, 1-\frac{1}{i}]$
- $\cap_{i=1}^{\infty}[-1-\frac{1}{i}, 1+\frac{1}{i}]$

The **empty set** (or **null set**) is the set consisting of no elements, and is denoted by $\emptyset$.

Two sets are called **disjoint** if they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset$$

Examples: Which of the following pairs of sets are disjoint?

(1) $[0,3]$ and $[4,5]$

(2) $[0,3]$ and $[2,4]$

(3) $[0,3]$ and $[3,4]$

(4) $[0,3]$ and $(3,4]$

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint**) if

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$$

A finite or infinite collection of nonempty sets $\{A_1, A_2, \ldots\}$ is a **partition** of a set A if

(1) A is the union of all the A_i

(2) The sets $A_1, A_2, A_3, ..$ are mutually disjoint.

Examples:

(1) Define the following:

$$T_0 = \{n \in \mathbb{Z} \,|\, n \bmod 3 = 0\}$$
$$T_1 = \{n \in \mathbb{Z} \,|\, n \bmod 3 = 1\}$$
$$T_2 = \{n \in \mathbb{Z} \,|\, n \bmod 3 = 2\}$$

(2) Define $I_n = [n, n+1)$ for all $n \in \mathbb{Z}$.

Given a set A, the **power set** of A is the set of all subsets of A, and is denoted $\mathcal{P}(A)$
Examples:

(1) $A = \{x\}$

(2) $B = \{x, y, \}$

(3) $C = \{x, y, z\}$

If A contains n elements then how many elements does $P(A)$ contain?