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Continuous Schauder Frames for Banach Spaces

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Abstract

We introduce the notion of a continuous Schauder frame for a Banach space. This is both a generalization of continuous frames for Hilbert spaces and a generalization of unconditional Schauder frames for Banach spaces. Furthermore, we generalize the properties shrinking and boundedly complete to the continuous Schauder frame setting, and prove that many of the fundamental James theorems still hold in this general context.

Keywords Schauder frames · Continuous frames · Shrinking · Boundedly complete

Mathematics Subject Classification 42C15 · 81R30 · 46B10

1 Introduction

Frames and orthonormal bases give discrete ways to represent vectors in a Hilbert space using series, and continuous frames and coherent states give continuous ways to represent vectors using integrals.$(x_j)_{j\in J} \subset H$ for which there exists constants $0 < A \leq B$ such that for any $x \in H$, $A\|x\|^2 \leq \sum_{j\in J} |\langle x, x_j\rangle|^2 \leq B\|x\|^2$. Given any frame $(x_j)_{j\in J}$ for a Hilbert space H, there exists a frame $(f_j)_{j\in J}$ for H, called a *dual frame*, such that

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$$x = \sum_{j \in J} \langle x, f_j \rangle x_j \quad \text{for all } x \in H. \tag{1.1}$$

The equality in (1.1) allows the reconstruction of any vector x in the Hilbert space from the sequence of coefficients $(\langle x, f_j \rangle)_{j \in J}$. Continuous frames and coherent states are a generalization of frames in that instead of summing over a discrete set, we integrate over a measure space. Coherent states were invented by Schrödinger [27] and were generalized to continuous frames by Ali et al. [2]. The short time Fourier transform and the continuous wavelet transform are two particularly important examples of continuous frames. Let (M, Σ, μ) be a σ-finite measure space and let H be a separable Hilbert space. A weakly measurable function $\psi : M \to H$ is a *continuous frame* of H with respect to μ if there exists constants $A, B > 0$ such that for all $x \in H$, $A\|x\|^2 \leq \int_M |\langle x, \psi(t) \rangle|^2 d\mu(t) \leq B\|x\|^2$. If $A = B = 1$, then the continuous frame is called a *Parseval frame*. As is the case with frames, any continuous frame may be used to reconstruct vectors using a dual frame. That is, if $\psi : M \to H$ is a continuous frame, then there exists a *dual frame* $\phi : M \to H$ such that

$$x = \int_M \langle x, \phi(t) \rangle \psi(t) d\mu(t) \quad \text{for all } x \in H. \tag{1.2}$$

Equation (1.2) involves integrating vectors in a Hilbert space, and is defined weakly using the Pettis Integral. We will define the Pettis Integral and discuss it further in Sect. 2.

Frames for Hilbert spaces have been generalized to Banach spaces in multiple ways, such as atomic decompositions [14], Banach frames [19], framings [9], and Schauder frames [10]. Given a Banach space X with dual X^*, a sequence of pairs $(x_j, f_j)_{j=1}^{\infty} \subseteq X \times X^*$ is called a *Schauder frame* of X if

$$x = \sum_{j=1}^{\infty} f_j(x) x_j \quad \text{for all } x \in X. \tag{1.3}$$

Thus, Schauder frames are direct generalizations of the reconstruction formula (1.1) for frames in Hilbert spaces. A Schauder frame is called *unconditional* or a *framing* if the series in (1.3) converges in every order.

Coherent states and continuous frames for Hilbert spaces have long been studied and play important roles in mathematical physics and harmonic analysis. Continuous frames have been generalized to the Banach space setting in multiple ways such as through coorbit theory [14–17], Banach frames [1], and p-frames for complemented subspaces of L_p [13]. However, a continuous version of Schauder frames has not been previously considered. Our formulation of a continuous Schauder frame will be defined solely in terms of a reconstruction formula analogous to (1.2). The other generalizations of continuous frames to the Banach space setting each make use of some additional structure such as groups for coorbit spaces and admissible function spaces for Banach frames.

Given a Banach space X with dual X^* and a measure space (M, Σ, μ), we call a (w, w^*) measurable map $t \mapsto (x_t, f_t) \in X \times X^*$ a *continuous Schauder frame* of X if for all $x \in X$,

$$x = \int_M f_t(x) x_t d\mu(t). \tag{1.4}$$

As with continuous frames for Hilbert spaces, the integral in Eq. (1.4) involves integrating vectors and is defined weakly using the Pettis Integral which we define in Sect. 2. Unlike series, there is no order for integration, and so all continuous Schauder frames are by necessity unconditional. In the case that the measure space (M, μ) is simply the natural numbers with counting measure, then $(x_n, f_n)_{n\in\mathbb{N}}$ is a continuous Schauder frame if and only if it is an unconditional Schauder frame. Thus, continuous Schauder frames are indeed generalizations of unconditional Schauder frames.

In the case that the Banach space is a Hilbert space H, then a continuous Schauder frame $(x_t, f_t)_{t\in M}$ of H is called a *reproducing pair*. Note that continuous frames are reproducing pairs for the case $x_t = f_t$ for all $t \in M$. Many of the important properties of continuous frames extend to reproducing pairs, but reproducing pairs allow for more flexibility in their construction and do not require each map individually to satisfy frame bounds [4,28,29].

In [7] and [22], they define the properties shrinking and boundedly complete for Schauder frames and prove that many of James' classic theorems [21] on shrinking and boundedly complete Schauder bases can be extended to Schauder frames. In particular, a Schauder frame $(x_j, f_j)_{j=1}^{\infty}$ for a Banach space X is shrinking if and only if $(f_j, x_j)_{j=1}^{\infty}$ is a Schauder frame for X^*, and if $(x_j, f_j)_{j=1}^{\infty}$ is shrinking and boundedly complete then X is reflexive. On the other hand, in [6] they prove that every infinite dimensional Banach space which has a Schauder frame also has a non-shrinking Schauder frame, so unlike for Schauder bases the converse of the previous theorem for Schauder frames does not hold. In [8], they prove that an unconditional Schauder frame is shrinking if and only if the Banach space does not contain ℓ_1, and that an unconditional Schauder frame is shrinking and boundedly complete if and only if the Banach space is reflexive. In Sect. 3 we define what it means for a continuous Schauder frame to be shrinking or boundedly complete, and we prove theorems on when the previous stated theorems are true for continuous Schauder frames as well. In particular, for the case of separable Banach spaces, we prove the following.

Theorem 1.1 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame of a separable Banach space X with respect to a measure space (M, Σ, μ) such that $t \mapsto f_t$ is w-measurable. Then the following are equivalent,*

(i) *The continuous Schauder frame $(x_t, f_t)_{t\in M}$ is shrinking.*
(ii) *The dual frame $(f_t, x_t)_{t\in M} \subseteq X^* \times X^{**}$ is a continuous Schauder Frame for X^*.*
(iii) *X does not contain an isomorphic copy of ℓ_1.*

We consider the case where X may be non-separable as well. In that case, we must make an additional assumption about either the structure of X or the structure of the map $t \mapsto f_t$. In Sect. 2 we define what it means for a map to be semi-discrete, and in Sect. 3 we prove that if $t \mapsto f_t$ is semi-discrete then (i), (ii), and (iii) in Theorem 1.1 are all equivalent. Furthermore, if every $x^{**} \in X^{**}$ is the w^*-limit limit of a sequence in X, then (i), (ii), and (iii) in Theorem 1.1 are all true. Similarly, the following Theorem is a generalization of the classical James Theorem characterizing unconditional bases for reflexive Banach spaces.

Theorem 1.2 *Let $(x_t, f_t)_{t\in M}$ be continuous Schauder frame for a Banach space X such that either: X is separable, $(f_t)_{t\in M}$ is semi-discrete, or every $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X. Then the following are equivalent:*

(1) *$(x_t, f_t)_{t\in M}$ is shrinking and boundedly complete,*
(2) *X does not contain an isomorphic copy of c_0 or ℓ_1,*
(3) *X is reflexive.*

Lemma 1 in [5] and Lemma 5.3 in [18] show that every continuous frame for a separable Hilbert space can be approximated by a continuous frame with countable range. This gives a method for extending results about frames for Hilberts spaces to results about continuous frames for Hilbert spaces. However, the analogous result for continuous Schauder frames for separable Banach spaces is not necessarily true. The key reason for this is that if H is a separable Hilbert space then the space of rank one operators on H is norm separable, but if X is a separable Banach space with non-separable dual then the space of rank one operators on X is not norm separable. Because of this, the above results for continuous Schauder frames require new proofs and do not follow directly from the corresponding results for discrete Schauder frames given in [7,8], and [22].

We recommend [30] for a reference on Pettis integrals in Banach spaces and [12] for a reference on bases in Banach spaces. We thank the referees for their advice which has led to significant improvements of this paper over the previous versions.

2 The Pettis Integral and Continuous Schauder Frames

For the sake of simplicity, we will be assuming that all Banach spaces we consider are over $\mathbb{R}$. However, all of our proofs and results are still valid for Banach spaces over $\mathbb{C}$.

We will be using the Pettis integral to integrate vector valued functions. The main concept of the Pettis integral is to integrate vector valued functions by considering the corresponding Lebesgue integrals of the real valued functions formed by composition with linear functionals. This method allows one to transfer many of the fundamental properties of Lebesgue integration to the Banach space setting.

Definition 2.1 Let (M, Σ, μ) be a measure space and let X be a Banach Space. A weakly measurable map $F : M \to X$ is said to be *μ-Pettis integrable* (or *Pettis integrable* if context is understood) if for any $E \in \Sigma$ there exists $x_E \in X$ such that $f(x_E) = \int_E f(F)d\mu$ for all $f \in X^*$ (where this latter integral is Lebesgue). Then we say $\int_E F d\mu = x_E$ and, in particular, $\int F d\mu = x_M$.

If the vector valued map takes values in a dual space X^* then one can instead consider just using the weak*-continuous linear functionals.

Definition 2.2 Let (M, Σ, μ) be a measure space and X be a Banach Space with dual X^*. A w^*-measurable map $G : M \to X^*$ is said to be *μ-Pettis* integrable* (or *Pettis* integrable* if context is understood) if for any $E \in \Sigma$ there exists $f_E \in X^*$ such that $f_E(x) = \int_E G(x)d\mu$ for all $x \in X$. Then we say $\int_E^* G d\mu = f_E$ and, in particular, $\int^* G d\mu = f_M$.

Recall that we use the Pettis integral to define continuous Schauder frames, and we will use the Pettis* integral to define continuous* Schauder frames.

Definition 2.3 Given a Banach space X with dual X^* and a measure space (M, Σ, μ), a (w, w^*)-measurable function $t \mapsto (x_t, f_t) \in X \times X^*$ is called a *continuous Schauder frame of X with respect to* (M, Σ, μ) if

$$x = \int_M f_t(x) x_t d\mu(t) \qquad \text{for all } x \in X. \tag{2.1}$$

The dual map $t \mapsto (f_t, x_t) \in X^* \times X$ is called a *continuous* Schauder frame of* X^* *with respect to* (M, Σ, μ) if

$$f = \int_M^* f(x_t) f_t d\mu(t) \qquad \text{for all } f \in X^*. \tag{2.2}$$

We call a continuous Schauder frame $(x_t, f_t)_{t\in M}$ *bounded* if $\sup_{t\in M} \|x_t\|\|f_t\| < \infty$. The following lemma allows us to use a change of measure to convert any continuous Schauder frame into a bounded continuous Schauder frame.

Lemma 2.4 *Let* (M, Σ, μ) *be a measure space and X be a Banach space. Suppose that* $t \mapsto x_t$ *is a w-measurable map from M to X and that* $t \mapsto f_t$ *is a* w^**-measurable map from M to* X^* *such that* $\|f_t\|\|x_t\| \neq 0$ *for all* $t \in M$. *If we let* $\nu(t) = \|x_t\|\|f_t\|\mu(t)$ *for all* $t \in M$ *then*

(1) (M, Σ, ν) *is a measure space, and both* $\nu \ll \mu$ *and* $\mu \ll \nu$.
(2) $t \mapsto x_t/\|x_t\|$ *is w-measurable, and* $t \mapsto f_t/\|f_t\|$ *is* w^**-measurable.*

In particular, if $(x_t, f_t)_{t\in M}$ *is a continuous Schauder frame of X with respect to the measure space* (M, Σ, μ) *then* $(x_t/\|x_t\|, f_t/\|f_t\|)_{t\in M}$ *is a bounded continuous Schauder frame of X with respect to the measure space* (M, Σ, ν).

Proof Let Σ_w be the Borel σ-algebra generated by the weak topology on X and let Σ_{w^*} be the Borel σ-algebra generated by the w^* topology on X^*. Note that the unit ball of X is w-closed, and hence $\{x \in X : a \leq \|x\| \leq b\} \in \Sigma_w$ for all real numbers $a < b$. Thus, $x \to \|x\|$ is a measurable map from the measurable space (X, Σ_w) to $\mathbb{R}$. Likewise, $f \to \|f\|$ is a measurable map from (X^*, Σ_{w^*}) to $\mathbb{R}$. By composition, $t \mapsto \|x_t\|$ and $t \mapsto \|f_t\|$ are measurable maps from (M, Σ) to $\mathbb{R}$. Thus, $\nu(t) = \|x_t\|\|f_t\|\mu(t)$ defines a measure on (M, Σ) which is absolutely continuous with respect to μ. Likewise, $\mu(t) = \|x_t\|^{-1}\|f_t\|^{-1}\nu(t)$ as $\|x_t\|\|f_t\| \neq 0$ for all $t \in M$ and hence μ is absolutely continuous with respect to ν.

We have that $t \mapsto x_t$ is w-measurable and that $t \mapsto \|x_t\|^{-1}$ is measurable and well defined. Thus, $t \mapsto (x_t, \|x_t\|^{-1}) \in X \oplus \mathbb{R}$ is w-measurable. The map from $X \oplus \mathbb{R}$ to X given by $(x, s) \mapsto sx$ is weakly continuous. Thus by composition, $t \mapsto \|x_t\|^{-1} x_t$ w-measurable. The same argument gives that $t \mapsto \|f_t\|^{-1} f_t$ is w^*-measurable. □

The natural numbers are used to index Schauder bases and Schauder frames, and so it is very easy to work with properties using limits. Continuous frames however are indexed by arbitrary measure spaces, and so we will have to work with nets over a directed set instead. This is a significant difference as many fundamental theorems for integrals such as the dominated convergence theorem apply to limits for sequences but do not hold for limits of nets.

Definition 2.5 Let (M, Σ, μ) be a measure space and X be a Banach space. Suppose that $t \mapsto x_t$ is a w-measurable map from M to X and that $t \mapsto f_t$ is a w^*-measurable map from M to X^*. We introduce

$$\mathcal{D} = \Big\{E \in \Sigma : \int_E \|f_t\|\|x_t\| d\mu(t) < \infty\Big\}.$$

We make $\mathcal{D}$ a directed set by defining $E \preceq F$ whenever $E \subseteq F$, and we refer to the elements of $\mathcal{D}$ as *absolutely finite sets*.

Recall that by Lemma 2.4, if $(x_t, f_t)_{t\in M}$ is a continuous Schauder frame of a Banach space X with respect to a measure space (M, Σ, μ) then $(x_t/\|x_t\|, f_t/\|f_t\|)_{t\in M}$ is a continuous Schauder frame of X with respect to the measure space (M, Σ, ν) where $\nu(t) = \|x_t\|\|f_t\|\mu(t)$. Note that $\nu(E) = \int_E \|f_t\|\|x_t\| d\mu(t)$ for all $E \in \Sigma$ and hence the absolutely finite sets for $(x_t, f_t)_{t\in M}$ with respect to (M, Σ, μ) are just the finite measure sets in (M, Σ, ν). The first step of many of our proofs will be to apply Lemma 2.4 to assume that $\|x_t\| = \|f_t\| = 1$ for all $t \in M$ and that $\mathcal{D}$ is the set of finite measure sets in Σ.

If $(x_n)_{n=1}^\infty$ is a Schauder basis, the basis projections are the operators $P_n : X \to X$ defined by $P_n(\sum_{j=1}^\infty a_j x_j) = \sum_{j=1}^n a_j x_j$. We let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame of a Banach space X with respect to the measure space (M, Σ, μ). For $E \in \Sigma$, we define an operator $P_E : X \to X$ by

$$P_E(x) = \int_E f_t(x) x_t d\mu(t) \qquad \text{for all } x \in X.$$

We refer to P_E as a *restriction operator*. The restriction operators P_E for $E \in \mathcal{D}$ will play a similar role for continuous Schauder frames as the basis projections play for Schauder bases.

We will prove two fundamental properties about the restriction operators $(P_E)_{E\in\mathcal{D}}$. In Theorem 2.11 we will prove that the restriction operators $(P_E)_{E\in\mathcal{D}}$ can be used to locally approximate the identity operator. Secondly, in Theorem 2.13 we will prove that for each $E \in \mathcal{D}$, the restriction operator P_E is compact. However, we first use the restriction operators to prove the following which shows that every continuous Schauder frame and every continuous* Schauder frame satisfies an unconditionality inequality.

Lemma 2.6 *Let either $(x_t, f_t)_{t\in M} \in X \times X^*$ be a continuous Schauder frame of a Banach space X or $(f_t, x_t)_{t\in M} \in X^* \times X$ be a continuous* Schauder frame for X^*. Then, there exists a constant $B_u > 0$ such that for every $x \in X$ and $f \in X^*$ we have that*

$$\int |f_t(x) f(x_t)| d\mu(t) \leq B_u \|x\|\|f\|. \tag{2.3}$$

Proof Suppose that either $(x_t, f_t) \in X \times X^*$ is a continuous Schauder frame of X or $(f_t, x_t) \in X^* \times X$ is a continuous* Schauder frame for X^*. In either case, we have for all $x \in X$ and $f \in X^*$ that

$$|f(x)| = \left| \int f_t(x) f(x_t) d\mu(t) \right| \leq \int |f_t(x) f(x_t)| d\mu(t) < \infty. \tag{2.4}$$

We now fix $f \in X^*$. For all $E \in \mathcal{D}$ we let $F_{E,f} : X \to L_1(M)$ be the operator $F_{E,f}(x) = (f_t(x) f(x_t))_{t \in M}$. We have for all $E \in \mathcal{D}$ that

$$\|F_{E,f}(x)\| = \int_E |f_t(x) f(x_t)| d\mu \leq \|x\| \|f\| \int_E \|f_t\| \|x_t\| d\mu.$$

Thus, $F_{E,f} : X \to L_1(M)$ is a bounded operator for all $E \in \mathcal{D}$. For each $x \in X$ we have that

$$\sup_{E \in \mathcal{D}} \|F_{E,f} x\| = \sup_{E \in \mathcal{D}} \int_E |f_t(x) f(x_t)| d\mu = \int |f_t(x) f(x_t)| d\mu < \infty \qquad \text{by (2.4)}.$$

By the uniform boundedness principle we have for all $f \in X^*$ that there exists $D_f > 0$ such that $\|F_{E,f}\| \leq D_f$ for all $E \in \mathcal{D}$. Thus, for all $x \in X$ we have that

$$\int |f_t(x) f(x_t)| d\mu = \sup_{E \in \mathcal{D}} \int_E |f_t(x) f(x_t)| d\mu \leq D_f \|x\|. \tag{2.5}$$

By switching the roles of $x \in X$ and $f \in X^*$ we have for all $x \in X$ there exists $D_x > 0$ so that for all $f \in X^*$,

$$\int |f_t(x) f(x_t)| d\mu = \sup_{E \in \mathcal{D}} \int_E |f_t(x) f(x_t)| d\mu \leq D_x \|f\|. \tag{2.6}$$

Let $F : X \oplus X^* \to L_1(M)$ be defined by $F((x, f)) = (f_t(x) f(x_t))_{t \in M}$ for all $(x, f) \in X \oplus X^*$. By (2.5) and (2.6) we have that F is continuous in each coordinate, and hence F is continuous. Thus, there exists a constant B_u so that for all $(x, f) \in X \oplus X^*$ we have that

$$\|F(x, y)\|_{L_1(M)} = \int |f_t(x) f(x_t)| d\mu \leq B_u \|x\| \|f\|.$$

□

We call the least constant B_u to satisfy Lemma 2.6 the *unconditionality constant* of $(x_t, f_t)_{t \in M}$. Likewise, the *suppression unconditionality constant* of $(x_t, f_t)_{t \in M}$ is the least constant B_s to satisfy $|\int_E f_t(x) f(x_t) d\mu(t)| \leq B_s \|x\| \|f\|$ for all $(x, f) \in X \times X^*$ and all measurable $E \subseteq M$. In other words, $B_s = \sup_E \|P_E\|$. Just like for unconditional bases, the following proposition shows that these constants satisfy $B_s \leq B_u \leq 2B_s$.

Proposition 2.7 *Let either $(x_t, f_t)_{t\in M} \in X \times X^*$ be a continuous Schauder frame of a Banach space X or $(f_t, x_t)_{t\in M} \in X^* \times X$ be a continuous* Schauder frame for X^*. If B_u is the unconditionality constant and B_s is the suppression unconditionality constant, then $B_s \le B_u \le 2B_s$.*

Proof By Lemma 2.6, we have that the unconditionality constant B_u exists where

$$\int |f_t(x)f(x_t)|d\mu(t) \le B_u\|x\|\|f\| \qquad \text{for all } x \in X \text{ and } f \in X^*.$$

Let $E \subseteq M$ be measurable. Then for all $x \in X$ and $f \in X^*$ we have that

$$\left|\int_E f_t(x)f(x_t)d\mu(t)\right| \le \int |f_t(x)f(x_t)|d\mu(t) \le B_u\|x\|\|f\|.$$

Thus, we have that $B_s \le B_u$.

For $x \in X$ and $f \in X^*$ we let $E_{x,f} = \{t \in M : f(x_t)f_t(x) \ge 0\}$. We now have that

$$\begin{aligned}\int |f_t(x)f(x_t)|d\mu(t) &= \left|\int_{E_{x,f}} f_t(x)f(x_t)d\mu(t)\right| + \left|\int_{E^c_{x,f}} f_t(x)f(x_t)d\mu(t)\right| \\ &\le 2\sup_{E\subseteq M}\left|\int_E f_t(x)f(x_t)d\mu(t)\right|\end{aligned}$$

Thus, we have that $B_u \le 2B_s$. □

By the definition of the Pettis integral, $t \mapsto (x_t, f_t) \in X \times X^*$ is a continuous Schauder frame of X if and only if for all $x \in X$, $f \in X^*$, and $E \in \Sigma$ there exists $x_E \in X$ such that $x = x_M$ and

$$f(x_E) = \int_E f_t(x)f(x_t)d\mu(t). \tag{2.7}$$

We are primarily interested in the representation of x as $x = \int_M f_t(x)x_t d\mu(t)$, and so it may feel tedious to check Eq. (2.7) for all measurable sets $E \in \Sigma$. However, the following example shows that it is necessary to check (2.7) for all $E \in \Sigma$ and not just $E = M$.

Example 2.8 Let $(e_j)_{j\in\mathbb{N}}$ be the unit vector basis for c_0 with biorthogonal functionals $(e_j^*)_{j\in\mathbb{N}}$. Consider the following sequence of pairs in $c_0 \times \ell_1$,

$$\begin{aligned}(x_n, f_n)_{n=1}^{\infty} = (e_1, e_1^*), (e_1, -e_1^*), (e_1, e_1^*), (e_2, e_2^*),\\ (e_2, -e_1^*), (e_2, e_1^*), (e_3, e_3^*), (e_3, -e_1^*), (e_3, e_1^*), \ldots\end{aligned}$$

If we consider $\mathbb{N}$ with counting measure, then for all $x \in c_0$ and $f \in \ell_1$, $f(x) = \int_{\mathbb{N}} f_n(x)f(x_n)$. However, $(x_n, f_n)_{n\in\mathbb{N}}$ is not a continuous Schauder frame. Indeed, let $x = e_1$ and suppose $x_{3\mathbb{N}} \in c_0$ is such that for all $f \in \ell_1$, $f(x_{3\mathbb{N}}) = \int_{3\mathbb{N}} f_n(e_1)f(x_n)$.

Then for all $n \in \mathbb{N}$, $e_n^*(x_{3\mathbb{N}}) = 1$. Thus, $x_{3\mathbb{N}} = (1, 1, 1, ...) \notin c_0$ which is a contradiction.

Many results for Schauder bases for Banach spaces may be used to build intuition for continuous Schauder frames. However, there are often surprising differences. A sequence of vectors $(x_j)_{j=1}^\infty$ is a Schauder basis for a Banach space X, if and only if the biorthogonal functionals $(x_j^*)_{j=1}^\infty$ are a w^*-Schauder basis for X^*. That is, for all $f \in X^*$, the series $\sum_{j=1}^\infty f(x_j)x_j^*$ converges w^* to f. What is particularly interesting about Example 2.8 is that although $(x_n, f_n)_{n\in\mathbb{N}}$ is not a continuous Schauder frame of c_0, we do have that the dual $(f_n, x_n)_{n\in\mathbb{N}}$ is a continuous* Schauder frame of ℓ_1. Indeed, if $f \in \ell_1$ and $E \subseteq \mathbb{N}$ then

$$f_E = \sum_{n\in E\cap(3\mathbb{N}-2)} f(e_{(n+2)/3})e^*_{(n+2)/3} + \sum_{n\in E\cap 3\mathbb{N}} f(e_{n/3})e_1^* - \sum_{n\in E\cap(3\mathbb{N}-1)} f(e_{(n+1)/3})e_1^*.$$

As $f \in \ell_1$, we have that all the above series converge in norm to an element of ℓ_1. Thus, in contrast to the case for Schauder bases, we have that it is possible to have a continuous* Schauder frame $(f_t, x_t)_{t\in M}$ for a dual space X^* such that $(x_t, f_t)_{t\in M}$ is not a continuous Schauder frame for X. However, we will prove in Lemma 2.9 that the converse still holds for continuous Schauder frames. That is, if $(x_t, f_t)_{t\in M}$ is a continuous Schauder frame for X then $(f_t, x_t)_{t\in M}$ is a continuous* Schauder frame for X^*. As duality techniques are ubiquitous in functional analysis, this result will be very useful for us. In Sect. 3 we will go further and characterize when $(f_t, x_t)_{t\in M}$ is a continuous Schauder frame of X^* and not just a continuous* Schauder frame.

Lemma 2.9 *Let (M, Σ, μ) be a measure space and X be a Banach space. Suppose that $t \mapsto x_t$ is a w-measurable map from M to X and that $t \mapsto f_t$ is a w^*-measurable map from M to X^*. If $(x_t, f_t)_{t\in M}$ is a continuous Schauder frame for X then the dual frame $(f_t, x_t)_{t\in M}$ is a continuous* Schauder frame for X^*.*

Proof We assume that $(x_t, f_t)_{t\in M}$ is a continuous Schauder frame of X. Fix $f \in X^*$ and let $E \in \Sigma$. We have by Lemma 2.6 that the map $x \mapsto \int_E f_t(x)f(x_t)d\mu(t)$ defines a bounded linear functional on X. Thus, there exists $f_E \in X^*$ such that $f_E(x) = \int_E f_t(x)f(x_t)d\mu(t)$ for all $x \in X$. Furthermore, $f(x) = \int f_t(x)f(x_t)d\mu(t)$ for all $x \in X$ and hence $f_M = f$. This proves that $(f_t, x_t)_{t\in M}$ is a continuous* Schauder Frame for X^*. □

The following lemma will be essential in proving approximation properties of continuous frames.

Lemma 2.10 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X over a measure space (M, Σ, μ). For all $\varepsilon > 0$ and $x \in X$, there exists $F \in \mathcal{D}$ such that if $G \in \mathcal{D}$ with $G \cap F \neq \emptyset$ then*

$$|f(P_G(x))| = \left| \int_G f_t(x)f(x_t)d\mu(t) \right| \leq \varepsilon\|f\| \qquad \textit{for all } f \in X^*.$$

Proof For the sake of contradiction, we assume that for some $\varepsilon > 0$ there exists a sequence of disjoint sets $(E_n)_{n=1}^{\infty} \subseteq \mathcal{D}$ and a sequence of norm one functionals $(g_n)_{n=1}^{\infty} \subseteq X^*$ such that $\int_{E_n} f_t(x) g_n(x_t) d\mu(t) \geq \varepsilon$ for all $n \in \mathbb{N}$. After passing to a subsequence, we may assume that there exists $(a_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{E_k} f_t(x) g_n(x_t) d\mu(t) = a_k \qquad \text{for all } k \in \mathbb{N}.$$

After passing to a further subsequence we may assume that

$$\left| a_k - \int_{E_k} f_t(x) g_n(x_t) d\mu(t) \right| < 2^{-k}\varepsilon \qquad \text{for all } k < n. \tag{2.8}$$

As $\int |f_t(x) g_n(x_t)| d\mu(t) < \infty$ for all $n \in \mathbb{N}$ and $(E_k)_{k=1}^{\infty}$ is a sequence of disjoint sets, we may assume after passing to a further subsequence that

$$\left| \int_{E_k} f_t(x) g_n(x_t) d\mu(t) \right| < 2^{-k}\varepsilon \qquad \text{for all } n < k. \tag{2.9}$$

For all $n \in \mathbb{N}$ let $h_n := g_{2n} - g_{2n-1}$. By (2.8) we have for all $k < n$ that

$$\begin{aligned} \left| \int_{E_{2k}} f_t(x) h_n(x_t) d\mu \right| &\leq \left| a_{2k} - \int_{E_{2k}} f_t(x) g_{2n}(x_t) d\mu \right| \\ &\quad + \left| a_{2k} - \int_{E_{2k}} f_t(x) g_{2n-1}(x_t) d\mu \right| < 2 \cdot 2^{-2k}\varepsilon. \end{aligned}$$

Likewise, by (2.9) we have for all $k > n$ that

$$\begin{aligned} \left| \int_{E_{2k}} f_t(x) h_n(x_t) d\mu \right| &\leq \left| \int_{E_{2k}} f_t(x) g_{2n}(x_t) d\mu \right| \\ &\quad + \left| \int_{E_{2k}} f_t(x) g_{2n-1}(x_t) d\mu \right| < 2 \cdot 2^{-2k}\varepsilon. \end{aligned}$$

In the case, $n = k$, we have that

$$\begin{aligned} \int_{E_{2n}} f_t(x) h_n(x_t) d\mu &\geq \left| \int_{E_{2n}} f_t(x) g_{2n}(x_t) d\mu \right| \\ &\quad - \left| \int_{E_{2n}} f_t(x) g_{2n-1}(x_t) d\mu \right| > \varepsilon - 2^{-2n}\varepsilon. \end{aligned}$$

Thus, by combining the three above inequalities, we have for all $N \in \mathbb{N}$ and $n \geq N$ that

$$h_n(P_{\cup_{k\geq N} E_{2k}} x) = \int_{\cup_{k\geq N} E_{2k}} f_t(x) h_n(x_t) d\mu > \varepsilon - 2\sum_{k=N}^{\infty} 2^{-2k}\varepsilon \geq \varepsilon/3. \tag{2.10}$$

As $\|h_n\| \leq 2$ for all $n \in \mathbb{N}$, there exists a w^*-accumulation point $h \in X^*$ of $(h_n)_{n=1}^{\infty}$. By (2.10) we have that $\int_{\cup_{k\geq N} E_{2k}} f_t(x)h(x_t)d\mu \geq \varepsilon/3$ for all $N \in \mathbb{N}$. On the other hand, as $(E_{2k})_{k=1}^{\infty}$ is a sequence of disjoint sets, we have that $\lim_{N\to\infty} \int_{\cup_{k\geq N} E_{2k}} f_t(x)h(x_t)d\mu = 0$. Thus we have a contradiction. □

The Pettis integral is a weak integral in the sense that it is defined in terms of linear functionals. Thus by definition, a continuous Schauder frame is a coordinate system for X in the weak topology. The following theorem can be interpreted as stating that a continuous Schauder frame is also a coordinate system for X in the norm topology. Or in other words, the restriction operators $(P_E)_{E\in\mathcal{D}}$ can be used to locally approximate the identity operator in the norm topology. This is significant as weak properties for Pettis integrals in infinite dimensional Banach spaces often do not imply norm properties [26].

Theorem 2.11 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame of a Banach space X over a measure space (M, Σ, μ). Then*

$$\lim_{E\in\mathcal{D}} \|x - P_E x\| = \lim_{E\in\mathcal{D}} \|P_{E^c} x\| = 0 \textit{ for all } x \in X.$$

Proof By Lemma 2.4 we may assume that $\mathcal{D}$ is the set of finite measure sets in M. Let $x \in X$ and $\varepsilon > 0$. By Lemma 2.10 there exists $F \in \mathcal{D}$ such that $|f(P_G(x))| \leq \varepsilon\|\mathrm{f}\|$ for all $f \in X^*$ and all $G \in \mathcal{D}$ with $G \cap F = \emptyset$. Let $H \in \mathcal{D}$ with $H \succeq F$. There exists $f \in X^*$ with $\|f\| = 1$ and $f(x - P_H(x)) = \|x - P_H(x)\|$. We have that

$$\begin{aligned}
\|x - P_H(x)\| &= f(x - P_H(x)) \\
&= f(P_{H^c}(x)) \\
&= \int_{H^c} f_t(x) f(x_t) d\mu \\
&= \lim_{E\in\mathcal{D}} \int_{E\cap H^c} f_t(x) f(x_t) d\mu \qquad \text{as } \mathcal{D} \text{ is the set of finite measure subsets of } M, \\
&= \lim_{E\in\mathcal{D}} f(P_{E\cap H^c}(x)) \\
&\leq \lim_{E\in\mathcal{D}} \varepsilon\|f\| \qquad \text{as } (E \cap H^c) \cap F = \emptyset, \\
&= \varepsilon
\end{aligned}$$

Thus, $\|x - P_H x\| \leq \varepsilon$ for all $H \succeq F$. This proves that $\lim_{E\in\mathcal{D}} \|x - P_E x\| = 0$. □

The following corollary is a useful way to express Theorem 2.11.

Corollary 2.12 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame of a Banach space X over a measure space (M, Σ, μ). For all $\varepsilon > 0$ and $x \in X$, there exists $E \in \mathcal{D}$ such that*

$$\int_{E^c} |f_t(x) f(x_t)| d\mu(t) \leq \varepsilon\|f\| \qquad \textit{for all } f \in X^*.$$

Proof Let $\varepsilon > 0$ and $x \in X$. By Theorem 2.11, there exists $E \in \mathcal{D}$ such that $\|x - P_F x\| < \varepsilon$ for all $F \succeq E$. Let $f \in X^*$. We denote $M^+ = \{t \in M : f_t(x)f(x_t) > 0\}$ and $M^- = \{t \in M : f_t(x)f(x_t) < 0\}$.

$$\begin{aligned}\int_{E^c} |f_t(x)f(x_t)|d\mu(t) &= \int_{E^c\cap M^+} f_t(x)f(x_t)d\mu(t) + \int_{E^c\cap M^-} -f_t(x)f(x_t)d\mu(t)\\ &= \left(\sup_{F\succeq E}\int_{F^c} f_t(x)f(x_t)d\mu(t)\right) + \left(\sup_{F\succeq E}\int_{F^c} -f_t(x)f(x_t)d\mu(t)\right)\\ &\le \left(\sup_{F\succeq E}\|x - P_F\|\|f\|\right) + \left(\sup_{F\succeq E}\|x - P_F\|\|f\|\right)\\ &\le 2\varepsilon\|f\|\end{aligned}$$

We previously showed that the restriction operators $(P_E)_{E\in\mathcal{D}}$ can be used to locally approximate the identity operator in the norm topology. For the case of Schauder frames and Schauder bases, the restriction operators are used to locally approximate the identity operator using finite rank operators. The following theorem shows that a continuous Schauder frame may be used to locally approximate the identity operator using compact operators. We note that the corresponding theorem for continuous Banach frames is a fundamental result in [1].

Theorem 2.13 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X over a measure space (M, Σ, μ). Then P_E is a compact operator for every $E \in \mathcal{D}$.*

Proof By Lemma 2.4, we may assume that $\|x_t\| = \|f_t\| = 1$ for all $t \in M$ and that $\mu(E) < \infty$ for all $E \in \mathcal{D}$.

For the sake of contradiction, we assume that P_E is not compact for some $E \in \mathcal{D}$. As P_E is not a compact operator, there exists a normalized sequence $(y_n)_{n=1}^\infty$ in X and $\varepsilon > 0$ such that $\|P_E(y_n - y_m)\| \ge \varepsilon$ for all $n \ne m$. By Rosenthal's ℓ_1 Theorem, we may assume after passing to a subsequence that either $(P_E y_n)_{n=1}^\infty$ is w-Cauchy or that $(P_E y_n)_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1.

We first consider the case that $(P_E y_n)_{n=1}^\infty$ is w-Cauchy. For all $t \in E$, we have that $|f_t(y_n - y_m)| \le \|f_t\|(\|y_n\| + \|y_m\|) = 2$. As $\mu(E) < \infty$, we may use the dominated convergence theorem to obtain the following.

$$\begin{aligned}\limsup_{m,n\ge N}\|P_E(y_n - y_m)\| &= \limsup_{m,n\ge N}\left\|\int_E f_t(y_n - y_m)x_t d\mu(t)\right\|\\ &\le \limsup_{m,n\ge N}\int_E |f_t(y_n - y_m)|\|x_t\|d\mu(t)\\ &= \limsup_{m,n\ge N}\int_E |f_t(y_n - y_m)|d\mu(t) = 0 \text{ by dominated convergence.}\end{aligned}$$

This contradicts that $\|P_E(y_n - y_m)\| \ge \varepsilon$ for all $n \ne m$.

We now consider the case that $(P_E y_n)_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1. There exists $C > 0$ such that $C^{-1}\sum|a_j| \le \|\sum a_j P_E y_j\|$ for all $(a_j)_{j=1}^\infty \in \ell_1$. Let

$(\gamma_j)_{j=1}^{\infty}$ be a sequence of independent ± 1 symmetric random variables. Khintchine's inequality states that there exists a uniform constant $B_1 > 0$ such that expectation satisfies

$$\mathbb{E}\left|\sum_{j=1}^{n}\gamma_j a_j\right| \leq B_1\left(\sum_{j=1}^{n}|a_j|^2\right)^{1/2} \qquad \text{for all } n\in\mathbb{N} \text{ and } (a_j)_{j=1}^{n}\in\ell_1^n.$$

For each $n \in \mathbb{N}$ we define a random vector z_n by $z_n = \frac{1}{n}\sum_{j=1}^{n}\gamma_j y_j$. Note that every realization of z_n satisfies that $C^{-1} \leq \|P_E z_n\|$. We now calculate the following expectation.

$$\begin{aligned}
\mathbb{E}\|P_E z_n\| &= \mathbb{E}\left\|\int_E f_t(z_n)x_t d\mu\right\| \\
&= n^{-1}\mathbb{E}\left\|\int_E\left(\sum_{j=1}^{n}\gamma_j f_t(y_j)\right)x_t d\mu\right\| \\
&\leq n^{-1}\int_E \mathbb{E}\left|\sum_{j=1}^{n}\gamma_j f_t(y_j)\right| \|x_t\| d\mu \\
&\leq n^{-1}B_1\int_E\left(\sum_{j=1}^{n}|f_t(y_j)|^2\right)^{1/2}\|x_t\| d\mu \qquad \text{by Khintchine's inequality.} \\
&\leq n^{-1}B_1\int_E\left(\sum_{j=1}^{n}\|y_j\|^2\right)^{1/2} d\mu \qquad \text{as } \|x_t\| = \|f_t\| = 1. \\
&= n^{-1}B_1 n^{1/2}\mu(E) \qquad \text{as } \|y_j\| = 1.
\end{aligned}$$

Hence, we have that $\lim_{n\to\infty}\mathbb{E}\|P_E z_n\| = 0$. This contradicts that for all $n \in \mathbb{N}$, every realization of z_n satisfies that $C^{-1} \leq \|P_E z_n\|$. As we have a contradiction in both cases, P_E must be compact. □

We now have the following immediate corollary.

Corollary 2.14 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X over a measure space (M, Σ, μ). Then either X is finite dimensional or $\int_M \|x_t\|\|f_t\|d\mu(t) = \infty$.*

Proof If $\int_M \|x_t\|\|f_t\|d\mu(t) = \infty$ then $M \in \mathcal{D}$. Hence $P_X = I_X$ is compact by Theorem 2.13 and consequently X is finite dimensional by Riesz's Lemma. □

Let (M, Σ, μ) be a measure space and let X be a Banach space. The measure space (M, Σ, μ) is called *semi-finite* if for all $F \in \Sigma$ with $\mu(F) \neq 0$ there exists $E \in \Sigma$ with $E \subseteq F$ and $0 < \mu(E) < \infty$. Analogously, we say that a map $\Psi : M \to X$ is *semi-discrete with respect to* (M, Σ, μ) if for all $\varepsilon > 0$ and all $F \in M$ with $0 < \mu(F)$

there exists $E \in \Sigma$ with $E \subseteq F$ and $0 < \mu(E)$ so that $\|\Psi(s) - \Psi(t)\| < \varepsilon$ for all $s, t \in E$. If X is separable then every measurable map $\Psi : M \to X$ is semi-discrete. However, it is possible for X to be separable, but for a measurable map $\Psi : M \to X^*$ to not be semi-discrete. If μ is a Baire measure on a locally compact Hausdorff space M and $\Psi : M \to X$ is continuous then Ψ is semi-discrete. This case covers most of the interesting examples of continuous frames, and in general it may be convenient to think of semi-discrete as being a measure theoretic version of a map being continuous almost everywhere.

We will be using the property of a map being semi-discrete to prove results for non-separable Banach spaces. The following shows that the change of measure technique from Lemma 2.4 preserves semi-discrete.

Lemma 2.15 *Let (M, Σ, μ) be a measure space and X be a Banach space. Suppose that $t \mapsto x_t$ is a w-measurable map from M to X and that $t \mapsto f_t$ is a w^*-measurable map from M to X^* such that $\|f_t\|\|x_t\| \neq 0$ for all $t \in M$. Let $\nu(t) = \|x_t\|\|f_t\|\mu(t)$ for all $t \in M$. If $t \mapsto x_t$ is semi-discrete with respect to (M, Σ, μ) then $t \mapsto \|x_t\|^{-1}x_t$ is semi-discrete with respect to (M, Σ, ν). Likewise, if $t \mapsto f_t$ is semi-discrete with respect to (M, Σ, μ) then $t \mapsto \|f_t\|^{-1}f_t$ is semi-discrete with respect to (M, Σ, ν).*

Proof We assume that $t \mapsto x_t$ is semi-discrete with respect to (M, Σ, μ). Let $F \in \Sigma$ with $\nu(F) > 0$ and let $\varepsilon > 0$. We have that $t \mapsto \|x_t\|$ is measurable by Lemma 2.4. Thus, there exists $\delta > 0$ and $F_\delta \subseteq F$ such that $\nu(F_\delta) > 0$ and $\|x_t\| \geq \delta$ for all $t \in F_\delta$. We have that $\mu(F_\delta) > 0$ as ν is absolutely continuous with respect to μ. As $t \mapsto x_t$ is semi-discrete with respect to (M, Σ, μ), there exists $E \subseteq F_\delta$ such that $0 < \mu(E)$ and $\|x_t - x_s\| < \varepsilon\delta$ for all $s, t \in E$. Note that $0 < \nu(E)$ as μ is absolutely continuous with respect to ν. We have for all $s, t \in E$ that

$$\begin{aligned}\big\| \|x_t\|^{-1}x_t - \|x_s\|^{-1}x_s \big\| &\leq \big\| \|x_t\|^{-1}x_t - \|x_s\|^{-1}x_t \big\| + \big\| \|x_s\|^{-1}x_t - \|x_s\|^{-1}x_s \big\| \\ &= \big| \|x_t\|^{-1} - \|x_s\|^{-1} \big| \|x_t\| + \|x_s\|^{-1}\|x_t - x_s\| \\ &= \big| \|x_s\| - \|x_t\| \big| \|x_s\|^{-1} + \|x_s\|^{-1}\|x_t - x_s\| \\ &< (\varepsilon\delta)\delta^{-1} + \delta^{-1}(\varepsilon\delta) = 2\varepsilon.\end{aligned}$$

This proves that $t \mapsto \|x_t\|^{-1}x_t$ is semi-discrete with respect to (M, Σ, ν). The same argument gives that if $t \mapsto f_t$ is semi-discrete with respect to (M, Σ, μ) then $t \mapsto \|f_t\|^{-1}f_t$ is semi-discrete with respect to (M, Σ, ν). □

The following is a key result that gives us conditions on when we can calculate P_E^{**} for $E \in \mathcal{D}$. As every measurable map into a separable Banach space is semi-discrete, we have that the following lemma applies to every Banach space X such that X^* is separable.

Lemma 2.16 *Let $(x_t, f_t)_{t \in M}$ be a continuous Schauder frame for a Banach space X. If either $(f_t)_{t \in M}$ is semi-discrete or every element of X^{**} is the w^* limit of a sequence in X then for all $x^{**} \in X^{**}$ and $E \in \mathcal{D}$ we have that*

$$P_E^{**}x^{**} = \int_E x^{**}(f_t)x_t d\mu(t). \tag{2.11}$$

Proof Let $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$ and let $E \in \mathcal{D}$. By Lemma 2.4 we may assume that $\|x_t\| = \|f_t\| = 1$ for all $t \in E$ and that $\mu(E) < \infty$. As P_E is compact, $P_E^{**}x^{**} \in X$. To show (2.11) we thus need to prove that $P_E^{**}x^{**}(f) = \int_E x^{**}(f_t)f(x_t)d\mu(t)$ for all $f \in X^*$.

We first assume that there exists a sequence $(x_n)_{n=1}^{\infty}$ in X which w^* converges to x^{**}. As $(x_n)_{n=1}^{\infty}$ is w^*-convergent, we have by the uniform boundedness principle that there exists a constant $B > 0$ such that $\|x_n\| \leq B$ for all $n \in \mathbb{N}$. Let $f \in X^*$. We have for all $n \in \mathbb{N}$ and $t \in E$ that

$$|f_t(x_n)f(x_t)| \leq \|f_t\|\|x_n\|\|f\|\|x_t\| \leq B\|f\|. \tag{2.12}$$

Note that $\lim_{n\to\infty} f_t(x_n)f(x_t) = x^{**}(f_t)f(x_t)$ for all $t \in E$. As $\mu(E) < \infty$ and $|f_t(x_n)f(x_t)| \leq B\|f\|$, we may use the dominated convergence theorem in the following argument.

$$\begin{aligned} P_E^{**}x^{**}(f) &= x^{**}(P_E^* f) \\ &= \lim_{n\to\infty} P_E^* f(x_n) \\ &= \lim_{n\to\infty} \int_E f_t(x_n)f(x_t)d\mu(t) \\ &= \int_E x^{**}(f_t)f(x_t)d\mu(t) \qquad \text{by (2.12) and dominated convergence.} \end{aligned}$$

Thus we have proven that $P_E^{**}x^{**}(f) = \int_E x^{**}(f_t)f(x_t)d\mu(t)$.

We now consider the case that $(f_t)_{t\in M}$ is semi-discrete. Let $\varepsilon > 0$. Consider a maximal collection $(E_j)_{j\in J}$ of disjoint subsets of E such that for all $j \in J$, $\mu(E_j) > 0$ and $\|f_t - f_s\| < \varepsilon$ for all $s, t \in E_j$. Note that J must be countable as $\mu(E) < \infty$. If $\mu(E \setminus \cup_{j\in J} E_j) \neq 0$ then as $(f_t)_{t\in M}$ is semi-discrete there exists $E_0 \subseteq E \setminus \cup_{j\in J} E_j$ with $\mu(E_0) > 0$ and $\|f_t - f_s\| < \varepsilon$ for all $t, s \in E_0$. This would contradict that $(E_j)_{j\in J}$ is maximal. Hence, $\mu(E \setminus \cup_{j\in J} E_j) = 0$. As J is countable, there exists a finite collection $(E_j)_{j=1}^N$ such that $\mu(E \setminus \cup_{j=1}^N E_j) < \varepsilon$. For each $1 \leq j \leq N$ we choose some $t_j \in E_j$.

Let $f \in X^*$. As X is w^*-dense in X^{**} there exists $x \in X$ with $\|x\| \leq \|x^{**}\| = 1$ such that $|x^{**}(P_E^* f) - P_E^* f(x)| < \varepsilon\|f\|$ and $|x^{**}(f_{t_j}) - f_{t_j}(x)| < \varepsilon$ for all $1 \leq j \leq N$. If $1 \leq j \leq N$ and $s \in E_j$ then

$$\begin{aligned} |x^{**}(f_s) - f_s(x)| &\leq |x^{**}(f_s) - x^{**}(f_{t_j})| \\ &\quad + |x^{**}(f_{t_j}) - f_{t_j}(x)| + |f_{t_j}(x) - f_s(x)| \\ &\leq \|x^{**}\|\|f_s - f_{t_j}\| + |x^{**}(f_{t_j}) - f_{t_j}(x)| + \|f_{t_j} - f_s\|\|x\| \\ &< \varepsilon + \varepsilon + \varepsilon \end{aligned}$$

Thus, we have that

$$|x^{**}(f_s) - f_s(x)| < 3\varepsilon \qquad \text{for all } s \in \cup_{j=1}^N E_j. \tag{2.13}$$

We now have the following estimate.

$$
\begin{aligned}
&\big|P_E^* f(x) - \int_E x^{**}(f_t) f(x_t) d\mu(t)\big| \\
&\leq \int_{\cup_{j=1}^N E_j} |f_t(x) - x^{**}(f_t)||f(x_t)| d\mu(t) + \int_{E\setminus\cup_{j=1}^N E_j} |f_t(x) - x^{**}(f_t)||f(x_t)| d\mu(t) \\
&\leq \int_{\cup_{j=1}^N E_j} 3\varepsilon \|f\| d\mu(t) + \int_{E\setminus\cup_{j=1}^N E_j} 2\|f\| d\mu(t) \qquad \text{by (2.13),} \\
&< 3\varepsilon\mu(E)\|f\| + 2\varepsilon\|f\|
\end{aligned}
$$

As $|x^{**}(P_E^* f) - P_E^* f(x)| < \varepsilon\|f\|$ we have that

$$
|P_E^{**} x^{**}(f) - \int_E x^{**}(f_t) f(x_t) d\mu(t)| < \varepsilon\|f\| + 3\varepsilon\mu(E)\|f\| + 2\varepsilon\|f\|.
$$

As $\varepsilon > 0$ is arbitrary we have that $P_E^{**} x^{**}(f) = \int_E x^{**}(f_t) f(x_t) d\mu(t)$. Hence, $P_E^{**} x^{**} = \int_E x^{**}(f_t) x_t d\mu(t)$. □

3 Shrinking and Boundedly Complete Continuous Schauder Frames

The properties shrinking and boundedly complete play fundamental roles in the theory and application of Schauder bases. The properties are extended to atomic decompositions and Schauder frames in [7,8] and [22], and they prove that many of the fundamental James theorems for bases extend to Schauder frames. The goal for this section is to extend these results to continuous Schauder frames as well. The natural numbers are used to index Schauder bases and Schauder frames, and so it is very easy to work with properties using limits. Continuous frames however are indexed by arbitrary measure spaces. We extend the definitions of shrinking and boundedly complete in [7] by using limits over the net $\mathcal{D}$.

Definition 3.1 A continuous Schauder Frame $(x_t, f_t)_{t\in M}$ for a Banach space X is called *shrinking* if $\lim_{E\in\mathcal{D}} \|P_{E^c}^* f\| = 0$ for all $f \in X^*$.

Lemma 3.2 *Let (M, Σ, μ) be a measure space and let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame of a Banach space X. If $(x_t, f_t)_{t\in M}$ is shrinking then for all $H \in \Sigma$ we have that*

$$
\lim_{E\in\mathcal{D}} \|P_{(H\cup E)^c}^* f\| = 0 \quad \textit{for all } f \in X^*.
$$

Proof Let $f \in X^*$. Assume that $(x_t, f_t)_{w\in M}$ is shrinking. Thus, for every $\varepsilon > 0$ there exists $E_\varepsilon \in \mathcal{D}$ such that $\|P_{E^c}^* f\| < \varepsilon$ for all $E \in \mathcal{D}$ with $E_\varepsilon \subseteq E$. Let $x \in X$. We consider the two sets

$$
F_x^+ = \{t \in M : f_t(x) f(x_t) > 0\} \quad \text{and} \quad F_x^- = \{t \in M : f_t(x) f(x_t) < 0\}.
$$

As $\int f_t(x)f(x_t)d\mu$ is integrable, both F_x^+ and F_x^- are σ-finite. Choose increasing sequences $(F_{x,n}^+)_{n\in\mathbb{N}}, (F_{x,n}^-)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that $\cup F_{x,n}^+ = F_x^+$ and $\cup F_{x,n}^- = F_x^-$. We now have the following estimate for all $\varepsilon > 0$.

$$\begin{aligned}\int_{E_\varepsilon^c} |f_t(x)f(x_t)|d\mu &= \int_{(E_\varepsilon\cup F_x^-)^c} f_t(x)f(x_t)d\mu - \int_{(E_\varepsilon\cup F_x^+)^c} f_t(x)f(x_t)d\mu \\ &= \lim_{n\to\infty}\int_{(E_\varepsilon\cup F_{x,n}^-)^c} f_t(x)f(x_t)d\mu - \lim_{n\to\infty}\int_{(E_\varepsilon\cup F_{x,n}^+)^c} f_t(x)f(x_t)d\mu \\ &\leq \lim_{n\to\infty}\|P^*_{(E_\varepsilon\cup F_{x,n}^-)^c}f\|\|x\| + \lim_{n\to\infty}\|P^*_{(E_\varepsilon\cup F_{x,n}^+)^c}f\|\|x\| \leq 2\varepsilon\|x\|\end{aligned}$$

Thus, for all $x \in X$ and $\varepsilon > 0$ we have that $\int_{E_\varepsilon^c}|f_t(x)f(x_t)|d\mu < 2\varepsilon\|x\|$. Let $H \in M$ and $E \in \mathcal{D}$ with $E_\varepsilon \subseteq E$.

$$\|P^*_{(E\cup H)^c}f\| = \sup_{\|x\|=1}\left|\int_{E^c\cap H^c} f_t(x)f(x_t)d\mu\right| \leq \sup_{\|x\|=1}\int_{E_\varepsilon^c}|f_t(x)f(x_t)|d\mu \leq 2\varepsilon$$

Thus, $\lim_{E\in\mathcal{D}}\|P^*_{(H\cup E)^c}f\| = 0$. □

Before continuing, we recall some definitions about basic sequences in Banach spaces. If $(x_n)_{n=1}^\infty$ is a sequence in a Banach space, then sequences of the form $(\sum_{j=m_n}^{m_{n+1}-1} b_jx_j)_{n=1}^\infty$ are called *block sequences* of $(x_n)_{n=1}^\infty$ where $(m_n)_{n=1}^\infty$ is an increasing sequence of natural numbers and $(b_j)_{j=1}^\infty$ is a sequence of scalars. If $(x_j)_{j=1}^\infty$ is a sequence in a Banach space $(X, \|\cdot\|_X)$ and $(y_j)_{j=1}^\infty$ is a sequence in a possibly different Banach space $(Y, \|\cdot\|_Y)$, then $(x_j)_{j=1}^\infty$ and $(y_j)_{j=1}^\infty$ are called *C-equivalent* where $1 \leq C < \infty$, if $C^{-1}\|\sum a_jx_j\| \leq \|\sum a_jy_j\| \leq C\|\sum a_jx_j\|$ for all scalars $(a_j)_{j=1}^\infty \in c_{00}$. We give the following well known lemma which shows that being equivalent to the unit vector basis of ℓ_1 is preserved by block sequences.

Lemma 3.3 *Let $(x_j)_{j=1}^\infty$ be a sequence of unit vectors in a Banach space X which is C-equivalent to the unit vector basis of ℓ_1. Then every normalized block sequence of $(x_j)_{j=1}^\infty$ is C-equivalent to the unit vector basis of ℓ_1.*

Proof Let $(\sum_{j=m_n}^{m_{n+1}-1} b_jx_j)_{n=1}^\infty$ be a normalized block sequence of $(x_n)_{n=1}^\infty$ and let $(a_n)_{j=1}^\infty \in c_{00}$ be a sequence of scalars. We have that,

$$\left\|\sum_{n=1}^\infty a_n\sum_{j=m_n}^{m_{n+1}-1} b_jx_j\right\| \leq \sum_{n=1}^\infty |a_n|\left\|\sum_{j=m_n}^{m_{n+1}-1} b_jx_j\right\| = \sum_{n=1}^\infty |a_n|.$$

For the other direction, we have that

$$\left\|\sum_{n=1}^\infty a_n\sum_{j=m_n}^{m_{n+1}-1} b_jx_j\right\| = \left\|\sum_{n=1}^\infty\sum_{j=m_n}^{m_{n+1}-1} a_nb_jx_j\right\|$$

$$\geq C^{-1}\sum_{n=1}^{\infty}\sum_{j=m_n}^{m_{n+1}-1}|a_n b_j|$$
$$= C^{-1}\sum_{n=1}^{\infty}|a_n|\sum_{j=m_n}^{m_{n+1}-1}|b_j|$$
$$\geq C^{-1}\sum_{n=1}^{\infty}|a_n|\Big\|\sum_{j=m_n}^{m_{n+1}-1}b_j x_j\Big\|$$
$$= C^{-1}\sum_{n=1}^{\infty}|a_n|$$

□

A Schauder basis for a Banach space X is shrinking if and only if its biorthogonal functionals form a Schauder basis for the dual X^*, and if the Schauder basis is unconditional then shrinking is also equivalent to ℓ_1 not embedding into X [21]. These characterizations of shrinking hold for Schauder frames as well [7,22,26]. The following theorem extends these results to continuous Schauder frames for separable Banach spaces. Our proof is fundamentally different from the discrete case, and will instead use a theorem of Odell and Rosenthal that states that if X is a separable Banach space then ℓ_1 does not embed into X if and only if every $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X [25]. As the following theorem considers when $(f_t, x_t)_{t\in M}$ is a continuous Schauder frame of X^*, we will require that $t \mapsto f_t$ is weak measurable instead of just w^*-measurable.

Theorem 3.4 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame of a separable Banach space X with respect to a measure space (M, Σ, μ) such that $t \mapsto f_t$ is w-measurable. Then the following are equivalent,*

(i) *The continuous Schauder frame $(x_t, f_t)_{t\in M}$ is shrinking.*
(ii) *The dual frame $(f_t, x_t)_{t\in M} \subseteq X^* \times X^{**}$ is a continuous Schauder frame for X^*.*
(iii) *ℓ_1 does not embed isomorphically into X.*
(iv) *Every $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X.*

Proof Note that $(iii) \Leftrightarrow (iv)$ is the theorem of Odell and Rosenthal [25].

We first prove $(ii) \Rightarrow (i)$. Let $\varepsilon > 0$ and $f \in X^*$. Assume that $(f_t, x_t)_{t\in M} \subseteq X^* \times X^{**}$ is a continuous Schauder frame for X^*. For all $H \in \Sigma$ and $x \in X$,

$$(P_H^* f)(x) = f(P_H x) = \int_H f_t(x) f(x_t) d\mu(t).$$

Thus, P_H^* is the restriction operator for the continuous Schauder frame $(f_t, x_t)_{t\in M}$ of X^*. By Theorem 2.11, $\lim_{E\in\mathcal{D}} \|P_{E^c}^* f\| = 0$. Hence $(x_t, f_t)_{t\in M}$ is shrinking.

We now prove $(i) \Rightarrow (iii)$ by contrapositive and assume that ℓ_1 embeds into X. As ℓ_1 is not distortable, there exists a sequence of unit vectors $(y_n)_{n=1}^{\infty}$ in X which is 2-equivalent to the unit vector basis of ℓ_1. That is, $2^{-1}\sum |a_n| \leq \|\sum a_n y_n\| \leq \sum |a_n|$

for all $(a_n)_{n=1}^{\infty} \in \ell_1$. Note that we automatically obtain that $\|\sum a_n y_n\| \leq \sum |a_n|$ by the triangle inequality because $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Let $0 < \varepsilon < 1/3$. We will inductively choose a sequence of disjoint sets $(E_n)_{n=1}^{\infty} \subseteq \mathcal{D}$ and a block sequence $(z_j)_{j=1}^{\infty}$ of $(y_j)_{j=1}^{\infty}$ such that $\|z_n - P_{E_n} y_n\| < \varepsilon$ for all $n \in \mathbb{N}$.

We start with $z_1 = y_1$ and choose $E_1 \in \mathcal{D}$ such that $\|z_1 - P_{E_1}(z_1)\| < \varepsilon$. We now let $k \in \mathbb{N}$ and assume that $(E_n)_{n=1}^{k}$ and $(z_j)_{j=1}^{k}$ have been chosen. Choose $M \in \mathbb{N}$ such that $(z_j)_{j=1}^{k}$ is a block sequence of $(y_n)_{n=1}^{M}$. We have that $P_{\cup_{j=1}^{k} E_j}$ is a compact operator by Theorem 2.13. Thus, there exists a unit vector $z_{k+1} \in span(y_n)_{n=M+1}^{\infty}$ such that $\|P_{\cup_{j=1}^{k} E_j} z_{k+1}\| < \varepsilon/2$. Choose $E \succeq \cup_{j=1}^{k} E_j$ such that $\|z_{k+1} - P_E z_{k+1}\| < \varepsilon/2$. Let $E_{k+1} = E \setminus \cup_{j=1}^{k} E_j$. We have that

$$\|z_{k+1} - P_{E_{k+1}} z_{k+1}\| \leq \|z_{k+1} - P_E z_{k+1}\| + \|P_{\cup_{j=1}^{k} E_j} z_{k+1}\| < \varepsilon/2 + \varepsilon/2.$$

This finishes our induction. As $(z_j)_{j=1}^{\infty}$ is a normalized block sequence of $(y_j)_{j=1}^{\infty}$, $(z_j)_{j=1}^{\infty}$ is 2-equivalent to the unit vector basis of ℓ_1 by Lemma 3.3. The constant 1 vector $g \in \ell_\infty$ satisfies $\|g\| = 1$ and $g(e_j) = 1$ for all $j \in \mathbb{N}$ where $(e_j)_{j=1}^{\infty}$ is the unit vector basis of ℓ_1. Thus, as $(z_j)_{j=1}^{\infty}$ is 2-equivalent to $(e_j)_{j=1}^{\infty}$ there exists $f \in X^*$ such that $\|f\| \leq 2$ and $f(z_j) = 1$ for all $j \in \mathbb{N}$. For all $n \in \mathbb{N}$,

$$\begin{aligned} P_{E_n}^* f(z_n) &= f(P_{E_n} z_n) \\ &= f(z_n) - f(z_n - P_{E_n} z_n) \\ &\geq f(z_n) - \|f\| \|z_n - P_{E_n} z_n\| \\ &> 1 - 2\varepsilon \\ &> 1/3 \qquad \text{as } \varepsilon < 1/3. \end{aligned}$$

Thus, $\|P_{E_n}^* f\| > 1/3$ for all $n \in \mathbb{N}$. As $(E_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{D}$, we have that $\lim_{E \in \mathcal{D}} P_{E^C}^* f$ does not converge in norm to 0. Thus, $(x_t, f_t)_{t \in M}$ is not shrinking.

We now prove $(iv) \Rightarrow (ii)$ directly. We assume that every $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X. Note that this assumption also implies that ℓ_1 does not embed into X. By Lemma 2.4, we may assume that $\|x_t\| = \|f_t\| = 1$ for all $t \in M$ and that $\mathcal{D}$ is the set of finite measure sets in M. Recall that $(f_t, x_t)_{t \in M}$ is a continuous* Schauder frame for X^* by Lemma 2.9. In particular, for all $f \in X^*$ and $H \in \Sigma$, we have that

$$P_H^* f(x) = f(P_H x) = \int_H f_t(x) f(x_t) d\mu(t) \qquad \text{for all } x \in X. \tag{3.1}$$

To prove that $(f_t, x_t)_{t \in M}$ is a continuous Schauder frame for X^* we need to prove for all $f \in X^*$ and $H \in \Sigma$ that

$$x^{**}(P_H^* f) = P_H^{**} x^{**}(f) = \int_H x^{**}(f_t) f(x_t) d\mu(t) \qquad \text{for all } x^{**} \in X^{**}. \tag{3.2}$$

Note that equation (3.2) holds for all $H \in \mathcal{D}$ by Lemma 2.16. Let $H \in \Sigma$, $f \in X^*$, and $x^{**} \in X^{**}$ with $\|f\| = \|x^{**}\| = 1$. We now claim that $x^{**}(P_H^* f) = \lim_{E\in\mathcal{D}} x^{**}(P_{E\cap H}^* f)$. Assuming the claim is true, we have the following.

$$\begin{aligned} x^{**}(P_H^* f) &= \lim_{E\in\mathcal{D}} x^{**}(P_{E\cap H}^* f) \\ &= \lim_{E\in\mathcal{D}} \int_{H\cap E} x^{**}(f_t) f(x_t) d\mu(t) \qquad \text{by Lemma 2.16,} \\ &= \int_H x^{**}(f_t) f(x_t) d\mu(t) \end{aligned}$$

Thus proving the claim will complete our proof. Let $(y_n)_{n=1}^\infty$ be a sequence of unit vectors in X which w^*-converges to x^{**}. By Corollary 2.12, there exists a nested sequence $F_1 \subseteq F_2 \subseteq F_3 \ldots$ in $\mathcal{D}$ such that

$$\int_{F_m^c} |f_t(y_n) f(x_t)| d\mu < 2^{-m} \qquad \text{for all } n \leq m. \tag{3.3}$$

We have that $(f_t(y_j))_{j=1}^\infty$ is convergent for all $t \in M$, $|f_t(y_i - y_j) f(x_t)| \leq 2$ for all $t \in M$ and $i, j \in \mathbb{N}$, and $\mu(E) < \infty$ for all $E \in \mathcal{D}$. We may thus apply the Dominated Convergence Theorem and pass to a subsequence of $(y_n)_{n=1}^\infty$ so that

$$\int_{F_m} |f_t(y_i - y_j) f(x_t)| d\mu < 2^{-m} \qquad \text{for all } i, j > m. \tag{3.4}$$

Let $\varepsilon > 0$. After possibly passing to a subsequence, we may assume by Ramsey's Theorem that one of the following two mutually exclusive properties holds.

(1) $\int_{F_m^c} |f_t(y_n) f(x_t)| d\mu < 2\varepsilon$ for all $m < n$.
(2) $\int_{F_m^c} |f_t(y_n) f(x_t)| d\mu \geq 2\varepsilon$ for all $m < n$.

We will prove that (1) implies our claim and that (2) leads to a contradiction. We first assume that (1) holds. As $y_n \to_{w^*} x^{**}$ we may choose $N \in \mathbb{N}$ such that $|x^{**}(P_H^* f) - P_H^* f(y_n)| < \varepsilon$ for all $n \geq N$. Let $E \succeq F_N$ and choose $n \geq N$ such that $|x^{**}(P_{H\cap E}^* f) - P_{H\cap E}^* f(y_n)| < \varepsilon$. We now have that

$$\begin{aligned} &|x^{**}(P_H^* f) - x^{**}(P_{H\cap E}^* f)| \\ &\quad \leq |x^{**}(P_H^* f) - P_H^* f(y_n)| + |P_H^* f(y_n) - P_{H\cap E}^* f(y_n)| \\ &\qquad + |P_{H\cap E}^* f(y_n) - x^{**}(P_{H\cap E}^* f)| \\ &\quad < \varepsilon + |P_H^* f(y_n) - P_{H\cap E}^* f(y_n)| + \varepsilon \\ &\quad = \Big| \int_{H\cap E^c} f_t(y_n) f(x_t) d\mu \Big| + 2\varepsilon \\ &\quad \leq \int_{H\cap E^c} |f_t(y_n) f(x_t)| d\mu + 2\varepsilon \end{aligned}$$

$$\leq \int_{F_N^c} |f_t(y_n) f(x_t)| d\mu + 2\varepsilon < 2\varepsilon + 2\varepsilon \qquad \text{by (1).}$$

Thus, we have that $\lim_{E \in \mathcal{D}} x^{**}(P^*_{H \cap E} f) = x^{**}(P^*_H f)$ which proves our claim.

We now assume that (2) holds. We will obtain a contradiction by constructing a block sequence of $(y_n)_{n=1}^{\infty}$ which is equivalent to the unit vector basis of ℓ_1. Let $F_0 = \emptyset$ and $E_n = F_n \setminus F_{n-1}$ for all $n \in \mathbb{N}$. Choose $N_0 \in \mathbb{N}$ such that $2^{-N_0+2} < \varepsilon/2$. For all $n \geq N_0$ we apply (2) and (3.3) to obtain

$$\int_{E_n} |f_t(y_n) f(x_t)| d\mu = \int_{F_{n-1}^c} |f_t(y_n) f(x_t)| d\mu - \int_{F_n^c} |f_t(y_n) f(x_t)| d\mu > 2\varepsilon - \varepsilon/2. \tag{3.5}$$

Let $z_n = y_{2n} - y_{2n-1}$ for all $n \geq N_0$. We will prove that $(z_n)_{n=N_0}^{\infty}$ is equivalent to the unit vector basis of ℓ_1. Indeed, let $(a_n)_{n=N_0}^{\infty} \in \ell_1$ with $\sum |a_n| = 1$ and let B_u be the unconditionality constant of $(x_t, f_t)_{t \in M}$. Then,

$$\begin{aligned}
B_u \Big\| \sum_{n=N_0}^{\infty} a_n z_n \Big\| &\geq \int \Big| f_t\Big(\sum_{n=N_0}^{\infty} a_n z_n \Big) f(x_t) \Big| d\mu \\
&\geq \sum_{m=N_0}^{\infty} \int_{E_{2m}} \Big| f_t\Big(\sum_{n=N_0}^{\infty} a_n z_n \Big) f(x_t) \Big| d\mu \\
&\geq \sum_{m=N_0}^{\infty} \Big(\int_{E_{2m}} \Big| f_t\Big(a_m y_{2m} + \sum_{n=m+1}^{\infty} a_n z_n \Big) f(x_t) \Big| d\mu \\
&\quad - \int_{F_{2m-1}^c} \Big| f_t\Big(a_m y_{2m-1} + \sum_{n=N_0}^{m-1} a_n z_n \Big) f(x_t) \Big| d\mu \Big) \\
&\geq \sum_{m=N_0}^{\infty} \Big(\int_{E_{2m}} \Big| f_t\Big(a_m y_{2m} + \sum_{n=m+1}^{\infty} a_n z_n \Big) f(x_t) \Big| d\mu - 2^{-m}\Big(|a_m| + \sum_{n=N_0}^{m} 2|a_n| \Big) \Big) \\
&\quad \text{by (3.3),} \\
&\geq \sum_{m=N_0}^{\infty} \Big(|a_m| \int_{E_{2m}} \Big| f_t(y_{2m}) f(x_t) \Big| d\mu - 2^{-m} \sum_{n=m+1}^{\infty} |a_n| - 2^{-m}\Big(|a_m| + \sum_{n=N_0}^{m} 2|a_n| \Big) \Big) \\
&\quad \text{by (3.4),} \\
&\geq \sum_{m=N_0}^{\infty} \Big(|a_m| \int_{E_{2m}} \Big| f_t(y_{2m}) f(x_t) \Big| d\mu - 2^{-m+1} \Big) \qquad \text{as } \sum |a_n| = 1, \\
&\geq \sum_{m=N_0}^{\infty} |a_m| \int_{E_{2m}} \Big| f_t(y_{2m}) f(x_t) \Big| d\mu - \varepsilon/2 \qquad \text{as } 2^{-N_0+2} < \varepsilon/2, \\
&\geq \sum_{m=N_0}^{\infty} |a_m| (2\varepsilon - \varepsilon/2) - \varepsilon/2 \qquad \text{by (3.5),} \\
&= \varepsilon \qquad \text{as } \sum |a_n| = 1.
\end{aligned}$$

Hence, we have that $\| \sum_{n=N_0}^{\infty} a_n z_n \| \geq \varepsilon B_u^{-1} \sum_{n=N_0}^{\infty} |a_n|$. On the other hand,

$$\Big\| \sum_{n=N_0}^{\infty} a_n z_n \Big\| \leq \sum_{n=N_0}^{\infty} |a_n| \big(\|y_{2n}\| + \|y_{2n-1}\| \big) = 2 \sum_{n=N_0}^{\infty} |a_n|.$$

Thus, $(z_n)_{n=N_0}^{\infty}$ is equivalent to the unit vector basis of ℓ_1, which is a contradiction. □

Theorem 3.4 characterizes when $(x_t, f_t)_{t \in M}$ is a shrinking continuous Schauder frame for a separable Banach space X. Characterizing shrinking for separable Banach spaces is the most important case, but it is also of interest to determine when a continuous Schauder frame for a non-separable Banach space is shrinking. This is more challenging because we no longer have the theorem of Odell and Rosenthal. Note that the proofs of $(ii) \Rightarrow (i) \Rightarrow (iii)$ do not depend on X being separable, and are hence still valid when $(x_t, f_t)_{t \in M}$ is a continuous Schauder frame for any Banach space X. To prove the remaining direction $(iii) \Rightarrow (ii)$ for a non-separable Banach space X, we need to assume that either X has some additional structure or that the continuous Schauder frame $(x_t, f_t)_{t \in M}$ has some additional structure.

Theorem 3.5 *Let $(x_t, f_t)_{t \in M}$ be a continuous Schauder frame of a Banach space X with respect to a measure space (M, Σ, μ) such that $t \mapsto f_t$ is w-measurable. In the case that $(f_t)_{t \in M}$ is semi-discrete then the following are equivalent,*

(i) *The continuous Schauder frame $(x_t, f_t)_{t \in M}$ is shrinking.*
(ii) *The dual frame $(f_t, x_t)_{t \in M} \subseteq X^* \times X^{**}$ is a continuous Schauder Frame for X^*.*
(iii) *ℓ_1 does not embed isomorphically into X.*

In the case that every element of X^{**} is the w^* limit of a sequence in X then all of (i),(ii), and (iii) are true.

Proof Note that the proofs of the cases $(ii) \Rightarrow (i)$ and $(i) \Rightarrow (iii)$ in Theorem 3.4 hold here as well.

We now assume that $(f_t)_{t \in M}$ is semi-discrete and prove $(iii) \Rightarrow (ii)$. By Lemma 2.4, we may assume that $\|x_t\| = \|f_t\| = 1$ for all $t \in M$ and that $\mathcal{D}$ is the set of finite measure sets in M. Recall that $(f_t, x_t)_{t \in M}$ is a continuous* Schauder frame for X^*. Thus, for all $f \in X^*$ and all measurable $H \subseteq M$ there exists $f_H \in X^*$ with $f_M = f$ and $f_H(x) = \int_H f_t(x) f(x_t) d\mu(t)$ for all $x \in X$. To prove that $(f_t, x_t)_{t \in M}$ is a continuous Schauder frame for X^* we need to prove that $x^{**} f_H = \int_H x^{**}(f_t) f(x_t) d\mu(t)$ for all $x^{**} \in X^{**}$.

Let $f \in X^*$ and $x^{**} \in X^{**}$ with $\|f\| = 1 = \|x^{**}\|$. Let $H \in \Sigma$. Either there exists a σ-finite $H_0 \subseteq H$ such that the integral $\int_{H_0} x^{**}(f_t) f(x_t) d\mu(t)$ does not exist or the set $\{t \in H : x^{**}(f_t) f(x_t) \neq 0\}$ is σ-finite. Thus, we may assume without loss of generality that H is σ-finite. As H is σ-finite and $(f_t)_{t \in H}$ is semi-discrete, there exists a subset $H_1 \subseteq H$ with $\mu(H \setminus H_1) = 0$ such that the set $\{f_t\}_{t \in H_1}$ is separable. Hence, we may may choose a sequence $(y_n)_{n=1}^{\infty}$ in X with $\|y_n\| = 1$ for all $n \in \mathbb{N}$ so that $\lim_{n \to \infty} f_t(y_n) = x^{**}(f_t)$ for all $t \in H_1$ and $\lim_{n \to \infty} f_H(y_n) = x^{**}(f_H)$. The remaining proof of $(iii) \Rightarrow (ii)$ then follows the same as that of the separable case in Theorem 3.4.

We now assume that every element of X^{**} is the w^* limit of a sequence in X. Thus ℓ_1 cannot embed into X and hence (iii) is true. The proof of $(iii) \Rightarrow (ii)$ in Theorem 3.4 started off by assuming that ℓ_1 does not embed into X and then used that X was separable to obtain that every element of X^{**} is the w^* limit of a sequence in X. No other properties of X being separable were used in Theorem 3.4 and hence we may use the same proof to obtain (ii) here. Thus, we have that $(ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (ii)$ and that (iii) is true. Hence, all of (i),(ii), and (iii) are true. □

We now consider the generalization of boundedly complete to the continuous setting.

Definition 3.6 Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X. We say that $(x_t, f_t)_{t\in M}$ is *boundedly complete* if for all $x^{**} \in X^{**}$ we have that $P_E^{**}x^{**} = \int_E x^{**}(f_t)x_t d\mu$ for all $E \in \mathcal{D}$ and $\lim_{E\in\mathcal{D}} P_E^{**}x^{**} \in X$.

The duality between shrinking bases and boundedly complete bases is very useful in functional analysis. We will extend this to continuous Schauder frames, but we first prove the following lemma.

Lemma 3.7 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X over a measure space (M, Σ, μ). Suppose that the dual frame $(f_t, x_t)_{t\in M}$ is a continuous Schauder frame for X^*. Then for all $H \in \Sigma$,*

$$P_H^* f = \int_H f(x_t) f_t d\mu \qquad \text{for all } f \in X^*.$$

That is, P_H^* is the restriction operator for the Schauder frame $(f_t, x_t)_{t\in M}$.

Proof Let $f \in X^*$ and $H \in \Sigma$. As $(f_t, x_t)_{t\in M}$ is a continuous Schauder frame for X^*, there exists $f_H \in X^*$ such that $f_H = \int_H f(x_t) f_t d\mu$. As $(x_t, f_t)_{t\in M}$ is a continuous Schauder frame for X, we have that

$$(P_H^* f)(x) = f(P_H x) = \int_H f_t(x) f(x_t) d\mu(t) \qquad \text{for all } x \in X.$$

Thus, $f_H(x) = (P_H^* f)(x)$ for all $x \in X$. Hence, $f_H = P_H^* f$. □

Proposition 3.8 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X. Suppose that the dual frame $(f_t, x_t)_{t\in M}$ is a continuous Schauder frame for X^*. Then $(f_t, x_t)_{t\in M}$ is boundedly complete.*

Proof Let $x^{***} \in X^{***}$. As $X \subseteq X^{**}$, the restriction of x^{***} to X is some functional $x^* \in X^*$. Let $E \in \mathcal{D}$ and $y^{**} \in X^{**}$. Note that P_E^{**} is compact by Theorem 2.13, and hence $P_E^{**}(y^{**}) \in X$. We now have that,

$$\begin{aligned}
(P_E^{***}x^{***})(y^{**}) &= x^{***}(P_E^{**}y^{**}) \\
&= x^*(P_E^{**}y^{**}) \qquad \text{because } P_E^{**}y^{**} \in X, \\
&= y^{**}(P_E^* x^*)
\end{aligned}$$

$$= \int_E x^*(x_t)y^{**}(f_t)d\mu \qquad \text{by Lemma 3.7,}$$
$$= \int_E x^{***}(x_t)y^{**}(f_t)d\mu.$$

As P_E^{***} is compact, we have that $P_E^{***}x^{***} \in X^*$. Hence, $P_E^{***}x^{***} = \int_E x^{***}(x_t)f_t d\mu$.

We may now take the limit over the net $\mathcal{D}$ to achieve,

$$\lim_{E\in\mathcal{D}} \int_E x^{***}(x_t)f_t d\mu = \lim_{E\in\mathcal{D}} \int_E x^*(x_t)f_t d\mu = x^* \in X^*.$$

Thus, $(f_t, x_t)_{t\in M}$ is boundedly complete. □

Frames for Hilbert spaces are nicely characterized as projections of Riesz bases for larger Hilbert spaces [20]. Likewise, Schauder frames for Banach spaces are characterized as projections of Schauder bases for larger Banach spaces [10]. Except for the case where a Schauder frame is the union of a basis with finitely many other vectors, the construction in [10] will always result in a Banach space which contains c_0 [23]. However, a different construction can be used to prove that a Schauder frame is shrinking or boundedly complete if and only if it is the projection of a shrinking or boundedly complete Schauder basis [6]. This allows for constructing and studying frames by working directly with bases and then projecting onto a subspace. Essentially, a redundant frame may be dilated to a non-redundant basis. However, this concept of dilation is only possible for continuous frames over purely atomic measures. The following proposition shows that the reverse direction is still valid for continuous frames in that projecting continuous Schauder frames onto closed subspaces gives a continuous Schauder frame.

Proposition 3.9 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X. Let $Y \subseteq X$ be a complemented subspace and let $P : X \to Y$ be a bounded projection.*

(1) *$(Px_t, f_t|_Y)_{t\in M} \subseteq Y \times Y^*$ is a continuous Schauder frame for Y.*
(2) *If $(x_t, f_t)_{t\in M}$ is shrinking then $(Px_t, f_t|_Y)_{t\in M}$ is shrinking.*
(3) *If $(x_t, f_t)_{t\in M}$ is boundedly complete then $(Px_t, f_t|_Y)_{t\in M}$ is boundedly complete.*

Proof Let $y \in Y$ and $g \in Y^*$. Let $E \subseteq M$ be measurable and let P_E be the restriction operator for the Schauder frame $(x_t, f_t)_{t\in M}$ of X. Let $I_Y : Y \to X$ be the inclusion operator of Y into X. We will prove that PP_EI_Y is the restriction operator for the frame $(Px_t, f_t|_Y)_{t\in M}$. We have that

$$g(PP_EI_Yy) = P^*g(P_EI_Yy)$$
$$= \int_E f_t(I_Yy)P^*g(x_t)d\mu$$
$$= \int_E f_t|_Y(y)g(Px_t)d\mu.$$

Thus, $PP_E I_X y = \int_E f_t|_Y(y) P x_t d\mu$. Furthermore, $PP_M I_Y y = y$ as $(x_t, f_t)_{t\in M}$ is a continuous Schauder frame of X. Hence $(Px_t, f_t|_Y)_{t\in M}$ is a continuous Schauder frame of Y.

We now assume that $(x_t, f_t)_{t\in M}$ is shrinking. Let $E \in \mathcal{D}$ and $g \in Y^*$. We have that

$$\lim_{E\in\mathcal{D}} \|(PP_{E^c} I_Y)^* g\| = \lim_{E\in\mathcal{D}} \|I_Y^* P_{E^c}^*(P^* g)\| \leq \lim_{E\in\mathcal{D}} \|P_{E^c}^*(P^* g)\| = 0.$$

Thus, $(Px_t, f_t|_Y)_{t\in M}$ is shrinking.

We now assume that $(x_t, f_t)_{t\in M}$ is boundedly complete. Let $y^{**} \in Y^{**}$ and $E \in \mathcal{D}$. As, $PP_E I_Y : Y \to Y$ is a compact operator, $(PP_E I_Y)^{**} y^{**} \in Y \subseteq Y^{**}$. Let $g \in Y^*$. We have that

$$\begin{aligned}((PP_E I_Y)^{**} y^{**})(g) &= (P_E^{**}(I_Y^{**} y^{**}))(P^* g) \\ &= \int_E I_Y^{**} y^{**}(f_t) P^* g(x_t) d\mu \qquad \text{as } (x_t, f_t)_t \text{ is boundedly complete,} \\ &= \int_E y^{**}(f_t|_Y) g(Px_t) d\mu.\end{aligned}$$

Thus, $(PP_E I_Y)^{**} y^{**} = \int_E y^{**}(f_t|_Y) P x_t d\mu$. As $(x_t, f_t)_{t\in M}$ is boundedly complete there exists $x \in X$ such that $\lim_{E\in\mathcal{D}} P_E^{**}(I_Y^{**} y^{**}) = x$. Thus, we have that

$$\begin{aligned}\lim_{E\in\mathcal{D}} (PP_E I_Y)^{**} y^{**} &= P^{**} \lim_{E\in\mathcal{D}} P_E^{**}(I_Y^{**} y^{**}) \\ &= P^{**} x \\ &= Px \in Y \qquad \text{because } x \in X.\end{aligned}$$

Thus, $\lim_{E\in\mathcal{D}} (PP_E I_Y)^{**} y^{**} \in Y$ and $(Px_t, f_t|_Y)_t$ is boundedly complete. □

We now prove the analogue of Theorem 3.5 for boundedly complete Schauder frames.

Theorem 3.10 *Let $(x_t, f_t)_{t\in M}$ be a continuous Schauder frame for a Banach space X such that either $(f_t)_{t\in M}$ is semi-discrete or every x^{**} is the w^*-limit of a sequence in X. Then $(x_t, f_t)_{t\in M}$ is boundedly complete if and only if c_0 does not embed into X.*

Proof We first assume that $(x_t, f_t)_{t\in M}$ is not boundedly complete. By Lemma 2.16, $P_E^{**} x^{**} = \int_E x^{**}(f_t) x_t d\mu$ for all $E \in \mathcal{D}$ and $x^{**} \in X^{**}$. As $(x_t, f_t)_{t\in M}$ is not boundedly complete, there exists $x^{**} \in X^{**}$ such that $\lim_{E\in\mathcal{D}} \int_E x^{**}(f_t) x_t d\mu$ does not converge to an element of X. Hence, the net $\lim_{E\in\mathcal{D}} \int_E x^{**}(f_t) x_t d\mu$ is not norm Cauchy. This gives that there exists $\delta > 0$ and a sequence of sets $V_1 \subseteq W_1 \subseteq V_2 \subseteq W_2 \subseteq \ldots$ in $\mathcal{D}$ such that for $u_n := \int_{W_n \setminus V_n} x^{**}(f_t) x_t d\mu$, we have $\|u_n\| \geq \delta$ for every $n \in \mathbb{N}$.

Let $f \in X^*$. Let B_s be the suppression unconditionality constant of $(x_t, f_t)_{t\in M}$. We have by Lemma 2.16 that

$$\int_M |x^{**}(f_t)f(x_t)|d\mu \le 2 \sup_{E\in\mathcal{D}} \Big|\int_E x^{**}(f_t)f(x_t)d\mu\Big|$$
$$= 2 \sup_{E\in\mathcal{D}} |P_E^{**}x^{**}(f)| \le 2B_s\|x^{**}\|\|f\| \tag{3.6}$$

Thus, $|\int_M x^{**}(f_t)f(x_t)d\mu| < \infty$ and as $(W_n \setminus V_n)_{k=1}^\infty$ is pairwise disjoint, we have that

$$\lim_{n\to\infty} f(u_n) = \int_{W_n\setminus V_n} x^{**}(f_t)f(x_t)d\mu = 0.$$

Thus, $(u_n)_{n=1}^\infty$ is a weak null sequence and $\delta \le \|u_n\| \le B_s\|x^{**}\|$ for all $n \in \mathbb{N}$. By passing to a subsequence, we may assume without loss of generality that $(u_n)_{n=1}^\infty$ is a basic sequence. We will now prove that $(u_n)_{n=1}^\infty$ is equivalent to the unit vector basis of c_0. Let $(a_n)_{n=1}^\infty \in c_{00}$. We have that

$$\Big|f(\sum_{n=1}^\infty a_n u_n)\Big| = \Big|\sum_{n=1}^\infty \int_{W_n\setminus V_n} a_n x^{**}(f_t)f(x_t)d\mu\Big|$$
$$\le \sup_{n\in\mathbb{N}} |a_n| \int_M |x^{**}(f_t)f(x_t)|d\mu \quad \text{as } (W_n \setminus V_n)_{n=1}^\infty \text{ is pairwise disjoint,}$$
$$\le 2B_s\|x^{**}\|\|f\| \sup_{n\in\mathbb{N}} |a_n| \qquad \text{by (3.6).}$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of c_0.

We now assume that c_0 embeds into X. As c_0 is not distortable, there exists a sequence of unit vectors $(y_n)_{n=1}^\infty$ in X which is 2-equivalent to the unit vector basis of c_0. That is, $2^{-1}\sup|a_n| \le \|\sum_{n=1}^\infty a_n y_n\| \le 2\sup|a_n|$ for all $(a_n)_{n=1}^\infty \in c_0$. Let $\varepsilon > 0$. We will inductively choose sequences of sets $(E_n)_{n=1}^\infty, (F_n)_{n=1}^\infty \subseteq \mathcal{D}$ and a subsequence $(z_n)_{n=1}^\infty$ of $(y_n)_{n=1}^\infty$ such that for all $n \in \mathbb{N}$,

(1) $E_n \subseteq F_n$,
(2) $E_n \cap F_m = \emptyset$ for all $m < n$,
(3) $\|P_G z_n\| < \varepsilon 2^{-n}$ for all $G \in \mathcal{D}$ such that $G \cap F_n = \emptyset$,
(4) $\|P_{E_m} z_n\| < \varepsilon 2^{-n}$ for all $m < n$,
(5) $\|z_n - P_{E_n} z_n\| < \varepsilon 2^{-n}$.

We start with $z_1 = y_1$ and choose $E_1 \in \mathcal{D}$ such that $\|z_1 - P_{E_1}(z_1)\| < \varepsilon 2^{-1}$. By Lemma 2.10 there exists $F_1 \in \mathcal{D}$ such that $E_1 \subseteq F_1$ and $\|P_G z_1\| < \varepsilon 2^{-1}$ for all $G \in \mathcal{D}$ with $G \cap F_1 = \emptyset$. Thus, all the conditions are satisfied.

We now let $k \in \mathbb{N}$ and assume that $(E_n)_{n=1}^k$, $(F_n)_{n=1}^k$ and $(z_n)_{n=1}^k$ have been chosen. Choose $M \in \mathbb{N}$ such that $(z_n)_{n=1}^k$ is a subsequence of $(y_n)_{n=1}^M$. We have that P_{E_n} and P_{F_n} are compact operators for all $1 \le n \le k$. As $(y_n)_{n=M}^\infty$ converges weakly to 0, there exists $N > M$ such that $\max_{1\le n\le k}\|P_{E_n} y_N\| < \varepsilon 2^{-k-1}$ and $\sum_{n=1}^k \|P_{F_n} y_N\| < \varepsilon 2^{-k-2}$. Let $z_{k+1} = y_N$. As $z_{k+1} = \lim_{E\in\mathcal{D}} P_E z_{k+1}$, we may choose a set $E \succeq \cup_{n=1}^k F_n$ such that $\|z_{k+1} - P_E z_{k+1}\| < \varepsilon 2^{-k-2}$. Let $E_{k+1} = E \setminus \cup_{n=1}^k F_n$. Thus, $F_m \cap E_{k+1} = \emptyset$ for all $m \le k$ and

$$\|z_{k+1} - P_{E_{k+1}} z_{k+1}\| \leq \|z_{k+1} - P_E z_{k+1}\| + \sum_{n=1}^{k} \|P_{F_n} z_{k+1}\| < \varepsilon 2^{-k-2} + \varepsilon 2^{-k-2} = \varepsilon 2^{-k-1}.$$

Lastly, we choose $F_{k+1} \in \mathcal{D}$ such that $E_{k+1} \subseteq F_{k+1}$ and $\|P_G z_{k+1}\| < \varepsilon 2^{-k-1}$ for all $G \in \mathcal{D}$ with $G \cap F_{k+1} = \emptyset$. This finishes our induction.

Note that for all $j, N \in \mathbb{N}$ if $j < N$ then $E_N \cap F_j = \emptyset$ by (2) and hence $\|P_{E_N} z_j\| < \varepsilon 2^{-j}$ by (3). Likewise, if $N < j$ then $\|P_{E_N} z_j\| < \varepsilon 2^{-j}$ by (4). Hence we have that

$$\|P_{E_N} z_j\| < \varepsilon 2^{-j} \qquad \text{for all } j \neq N. \tag{3.7}$$

As $(z_n)_{n=1}^{\infty}$ is 2-equivalent to the unit vector basis of c_0, we have that $\| \sum_{j=1}^{n} z_j \| \leq 2$ for all $n \in \mathbb{N}$. Let $x^{**} \in X^{**}$ be a w^*-accumulation point of $(\sum_{j=1}^{n} z_j)_{n=1}^{\infty}$. For all $N \in \mathbb{N}$, let $f_N \in X^*$ such that $\|f_N\| = 1$ and $f_N(z_N) = 1$.

For all $n \geq N$,

$$\begin{aligned}
P_{E_N}^* f_N \left(\sum_{j=1}^{n} z_j \right) &= f_N \left(P_{E_N} \sum_{j=1}^{n} z_j \right) \\
&= f_N(z_N) - f_N(z_N - P_{E_N} z_N) + f_N \left(\sum_{1 \leq j \leq n;\, j \neq N} P_{E_N} z_j \right) \\
&\geq 1 - \|z_N - P_{E_N} z_N\| - \sum_{j \neq N} \|P_{E_N} z_j\| \\
&\qquad \text{as } f_N(z_N) = 1 \text{and} \|f_N\| = 1, \\
&> 1 - \varepsilon 2^{-N} - \sum_{j \neq N} \varepsilon 2^{-j} \qquad \text{by (5) and (3.7)} \\
&= 1 - \varepsilon
\end{aligned}$$

Thus, $P_{E_N}^* f_N(\sum_{j=1}^{n} z_j) \geq 1 - \varepsilon$ for all $n \geq N$. As x^{**} is a w^*-accumulation point of $(\sum_{j=1}^{n} z_j)_{n=1}^{\infty}$, we have that $x^{**}(P_{E_N}^* f_N) \geq 1 - \varepsilon$. This gives that $\|P_{E_N}^{**} x^{**}\| \geq 1 - \varepsilon$ as $\|f_N\| = 1$ for all $N \in \mathbb{N}$. As $(E_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{D}$ by (1) and (2), we have that $\lim_{E \in \mathcal{D}} P_E^{**} x^{**}$ cannot converges in norm. Thus, $(x_t, f_t)_{t \in M}$ is not boundedly complete. □

Theorem 3.11 *Let $(x_t, f_t)_{t \in M}$ be continuous Schauder frame for a Banach space X such that either: X is separable, $(f_t)_{t \in M}$ is semi-discrete, or every $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X. Then the following are equivalent:*

(1) *$(x_t, f_t)_{t \in M}$ is shrinking and boundedly complete,*
(2) *X does not contain an isomorphic copy of c_0 or ℓ_1,*
(3) *X is reflexive.*

Proof (3) ⇒ (2) is clear as c_0 and ℓ_1 are not reflexive. We now prove that (2) ⇒ (1). We assume that X does not contain an isomorphic copy of c_0 or ℓ_1. If either $(f_t)_{t \in M}$

is semi-discrete or every $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X then we may apply Theorems 3.5 and 3.10 to obtain that $(x_t, f_t)_{t\in M}$ is shrinking and boundedly complete. If X is separable, we have that every $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X [25] and hence $(x_t, f_t)_{t\in M}$ is shrinking and boundedly complete.

We now prove that (1) $\Rightarrow$ (3). We assume that $(x_t, f_t)_{t\in M}$ is shrinking and boundedly complete. Let $x^{**} \in X^{**}$. Since $(x_t, f_t)_{t\in M}$ is boundedly complete, there exists $x \in X$ such that $x = \lim_{E\in\mathcal{D}} \int_E x^{**}(f_t)x_t d\mu$. Let $f \in X^*$. As $(x_t, f_t)_{t\in M}$ is shrinking, Theorems 3.4 and 3.5 give that $f = \int_M x_t(f) f_t d\mu$. We now have that

$$\begin{aligned}
f(x) =& f\left(\lim_{E\in\mathcal{D}} \int_E x^{**}(f_t)x_t d\mu\right)\\
=& \lim_{E\in\mathcal{D}} f\left(\int_E x^{**}(f_t)x_t d\mu\right) \qquad \text{by continuity,}\\
=& \lim_{E\in\mathcal{D}} \int_E x^{**}(f_t) f(x_t) d\mu \qquad \text{by definition of the Pettis integral,}\\
=& \int_M x^{**}(f_t) f(x_t) d\mu \qquad \text{as } \lim_{E\in\mathcal{D}} \int_E x^{**}(f_t)x_t(f) d\mu \text{exists.}
\end{aligned}$$

On the other hand we have that

$$\begin{aligned}
x^{**}(f) =& x^{**}\left(\int_M f(x_t) f_t d\mu\right)\\
=& \int_M f(x_t)x^{**}(f_t) d\mu \qquad \text{by definition of the Pettis integral.}
\end{aligned}$$

Compiling the above, we have $f(x) = x^{**}(f)$ for all $f \in X^*$. Thus $x^{**} = x \in X$. As this was for arbitrary $x^{**} \in X^{**}$ we have that $X^{**} = X$ as desired. □

4 Sampling Continuous Schauder Frames

Many important frames for Hilbert spaces arise as samplings of continuous frames. In particular, wavelet frames, Gabor frames, and Fourier frames are all samplings of different continuous frames. Notably, all the frames introduced by Daubechies, Grossmann, and Meyer [11] in “Painless nonorthogonal expansions” are created by sampling different coherent states. Formally, if (M, Σ, μ) is a σ-finite measure space and $(x_t, f_t)_{t\in M}$ is a continuous frame of a Banach space X and $(t_j)_{j=1}^\infty$ is a sequence in M then $(x_{t_j}, f_{t_j})_{j=1}^\infty$ is called a sampling of $(x_t, f_t)_{t\in M}$. The discretization problem, posed by Ali et al. [3], asks when a continuous frame of a Hilbert space can be sampled to obtain a frame. A solution for certain types of continuous frames was obtained by Fornasier and Rauhut using the theory of co-orbit spaces [17] and a complete solution was recently given by Speegle and the second author [18] using the solution of the Kadison Singer Problem by Marcus et al. [24]. In particular, every bounded continuous frame on a Hilbert space may be sampled to obtain a discrete frame.

Problem 4.1 What are some Banach spaces where every bounded continuous Schauder frame may be sampled to obtain a discrete Schauder frame? What are some Banach spaces where there exists a bounded continuous Schauder frame which cannot be sampled to obtain a discrete Schauder frame?

Note that the discretization problem was solved for continuous Hilbert space frames, and Problem 4.1 is even open for reproducing pairs for Hilbert spaces. One reason for this is that the structure of positive operators factors heavily into the solution of the Kadison Singer Problem [24], but if $(x_t, f_t)_{t\in M}$ is a reproducing pair then $x_t \otimes f_t$ is only a positive operator when $f_t = \lambda_t x_t$ for some $\lambda_t \geq 0$.

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