EQUILATERAL SETS IN UNIFORMLY SMOOTH BANACH SPACES.

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ABSTRACT. Let X be an infinite dimensional uniformly smooth Banach space. We prove that X contains an infinite equilateral set. That is, there exists a constant $\lambda > 0$ and an infinite sequence $(x_i)_{i=1}^\infty \subset X$ such that $\|x_i - x_j\| = \lambda$ for all $i \neq j$.

1. INTRODUCTION

A subset S of a Banach space X is called equilateral if there exists a constant $\lambda > 0$ such that $\|x - y\| = \lambda$ for all $x, y \in S$ with $x \neq y$. Much of the research on equilateral sets in Banach spaces is in estimating the maximal size of equilateral sets in finite dimensional Banach spaces, for some examples see [AP],[P],[S], and [SV]. Much less is known about equilateral sets in infinite dimensional Banach spaces. Instead of estimating the maximal size of equilateral sets in finite dimensional spaces, we consider the question of whether or not an infinite equilateral set exists in some given infinite dimensional Banach space. That is, given an infinite dimensional Banach space X, does there exist a sequence $(x_n)_{n=1}^\infty \subset X$ and a constant $\lambda > 0$ such that $\|x_n - x_m\| = \lambda$ for all $n \neq m$? For example, any subsymmetric basis is equilateral, such as the unit vector basis for ℓ_p for all $1 \leq p < \infty$ or the unit vector basis for Schlumprecht's space. On the other hand, the unit vector bases for Tsirelson's space and the hereditarily indecomposable Gowers-Maurey space are not subsymmetric, and yet they each have equilateral subsequences. Whether or not a given infinite dimensional Banach space contains an equilateral sequence is an isometric property. That is, it is possible for two infinite dimensional Banach spaces to be linearly isomorphic, and yet only one of them contain an equilateral sequence. Indeed, Terenzi constructed an equivalent norm $|\!|\!| \cdot |\!|\!|$ on ℓ_1 such that the Banach space $(\ell_1, |\!|\!| \cdot |\!|\!|)$ does not contain an equilateral sequence [T1],[T2]. Terenzi gave two distinct renormings of ℓ_1 which do not contain an equilateral sequence, and these are the only known infinite dimensional Banach spaces which do not contain an equilateral sequence. However, every renorming of c_0 does contain an equilateral sequence [MV]. Taken together, these two results are somewhat surprising as both ℓ_1 and c_0 are not distortable. We show that every uniformly smooth infinite dimensional Banach space contains an equilateral sequence.

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2. Asymptotic stability

Given a uniformly smooth Banach space X, before we can construct an equilateral sequence in X, we will need to first construct a sequence which is very close to being equilateral in certain ways. In this section we show how certain properties of weakly null sequences can be stabilized to make them "almost equilateral".

Let X be a uniformly smooth Banach space. For all $x \in X \setminus \{0\}$, there exists a unique functional $\phi_x \in S_{X^*}$ such that $\phi_x(x) = \|x\|$. Furthermore, the map $\Phi : X \setminus \{0\} \to S_{X^*}$ given by $\Phi(x) = \phi_x$ is uniformly continuous on subsets of X which are bounded away from 0.

Given a normalized weakly null sequence, the next lemma allows us to obtain a subsequence such that the difference between any two distinct elements is uniformly bounded away from 1. We recall that the *spreading model* generated by a semi normalized sequence $(x_i)_{i=1}^\infty$ in a Banach space X is a Banach space $(E, \|\cdot\|)$ with a basis $(e_i)_{i=1}^\infty$ satisfying

$$\Big\|\sum_{i=1}^n a_i e_i\Big\| = \lim_{k_1\to\infty}\lim_{k_2\to\infty}\dots\lim_{k_n\to\infty}\Big\|\sum_{i=1}^n a_i x_{k_i}\Big\|, \text{ for all } n\in\mathbb{N} \text{ and scalars } (a_i)_{i=1}^n.$$

See [O] for an expository reference on spreading models and the use of Ramsey theory in Banach spaces.

Lemma 2.1. *Let X be an infinite dimensional uniformly smooth Banach space and $(x_i)_{i=1}^\infty \subset X$ be a normalized weakly null sequence. Then $(x_i)_{i=1}^\infty$ has a subsequence with a spreading model $(e_i)_{i=1}^\infty$ such that $\|e_1 - e_2\| = \lambda > 1$.*

Proof. Every semi-normalized weakly null sequence in a Banach space has a subsequence with a 1-suppression unconditional spreading model (Proposition 2.3 (b) in [O]). Thus, after scaling and passing to a subsequence, we may assume that $(x_i)_{i=1}^\infty$ has a normalized 1-suppression unconditional spreading model $(e_i)_{i=1}^\infty$. Furthermore, as X is uniformly smooth, $[(e_i)]$ will be uniformly smooth as well. This can be seen as the property of being uniformly smooth is a uniform property of all two dimensional subspaces of a Banach space and for all $\varepsilon > 0$ every finite dimensional subspace of $[(e_i)]$ is $(1+\varepsilon)$-isomorphic to some subspace of X (Proposition 2.3 (c) in [O]). As $(e_i)_{i=}^\infty$ is 1-suppression unconditional, its sequence of biorthogonal functionals $(e_i^*)_{i=1}^\infty$ is normalized. We have that $e_1^*(e_1 - e_2) = 1$ and $-e_2^*(e_1 - e_2) = 1$. If $\|e_1 - e_2\| = 1$ then the normalizing unit functional of $e_1 - e_2$ would not be unique, and hence $\|e_1 - e_2\| > 1$. □

The following lemma allows us to choose a sequence which is asymptotically equilateral.

Lemma 2.2. *Let X be an infinite dimensional Banach space and $(x_i)_{i=1}^\infty \subset S_X$ be a normalized weakly null sequence with a spreading model $(e_i)_{i=1}^\infty$ such that $\|e_1 - e_2\| = \lambda > 1$. There exists a subsequence $(y_i)_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$ and a sequence of scalars $(a_i)_{i=1}^\infty \subset \mathbb{R}$ such that $a_i \to 1$, and $\lim_{i\to\infty}\|a_k y_k - a_i y_i\| = \lambda$ for all $k \in \mathbb{N}$.*

Proof. For all $x \in X$, we let $\phi_x \in S_X$ be a functional such that $\phi_x(x) = \|x\|$. We have that $\lim_{n\to\infty}\lim_{m\to\infty}\|x_n - x_m\| = \|e_1 - e_2\| = \lambda > 1$. Let $\varepsilon > 0$ be chosen so that $\lambda > 1 + \varepsilon$. By passing to a subsequence of $(x_i)_{i=1}^\infty$, we may assume for all $n \in \mathbb{N}$ that $\lambda_n := \lim_{m\to\infty}\|x_n - x_m\| > 1 + \varepsilon$. Moreover, we may assume that $\|x_n - x_m\| > 1 + \varepsilon$ for all $n, m \in \mathbb{N}$. If $\lambda_n = \lambda$ for all $n \in \mathbb{N}$ then setting $a_n = 1$ for all $n \in \mathbb{N}$ gives us our desired sequence. Thus, after passing to a subsequence again, we may assume that either $\lambda_n < \lambda$ for all $n \in \mathbb{N}$ or that $\lambda_n > \lambda$ for all $n \in \mathbb{N}$.

We first consider the case that $\lambda_n < \lambda$ for all $n \in \mathbb{N}$. We have for all $n, m \in \mathbb{N}$ that $\|x_n\| = \|x_m\| = 1$, $\|\phi_{x_n - x_m}\| = 1$, and $\phi_{x_n - x_m}(x_n - x_m) = \|x_n - x_m\| > 1 + \varepsilon$. Thus, $\phi_{x_n - x_m}(x_n) > \varepsilon$ for all $n, m \in \mathbb{N}$. Let $\overline{a}_n = 1 + (\lambda - \lambda_n)/\varepsilon$. Thus $\overline{a}_n \to 1$. By the definition of spreading model, we have that $\lim_{m\to\infty} \|a x_n - x_m\|$ exists for all $n \in \mathbb{N}$ and $0 \leq a \leq \overline{a}_n$. We have that,

$$\begin{aligned}\lim_{m\to\infty} \|\overline{a}_n x_n - x_m\| &\geq \lim_{m\to\infty} \phi_{x_n - x_m}(\overline{a}_n x_n - x_m)\\ &= (\overline{a}_n - 1)\lim_{m\to\infty} \phi_{x_n - x_m}(x_n) + \lim_{m\to\infty} \phi_{x_n - x_m}(x_n - x_m)\\ &\geq \lambda - \lambda_n + \lambda_n = \lambda.\end{aligned}$$

Thus, for all $n \in \mathbb{N}$, we have that $\lim_{m\to\infty} \|x_n - x_m\| = \lambda_n < \lambda \leq \lim_{m\to\infty} \|\overline{a}_n x_n - x_m\|$. Hence, we may choose by the Intermediate Value Theorem, applied to the function $a \mapsto \lim_{m\to\infty} \|a x_n - x_m\|$, a constant $1 < a_n \leq \overline{a}_n$ to yield $\lim_{m\to\infty} \|a_n x_n - x_m\| = \lambda$. As $\overline{a}_n \to 1$, we have that $a_n \to 1$, and hence $\lim_{m\to\infty} \|a_n x_n - a_m x_m\| = \lim_{m\to\infty} \|a_n x_n - x_m\| = \lambda$ for all $n \in \mathbb{N}$.

We now consider the case that $\lambda_n > \lambda$ for all $n \in \mathbb{N}$. By the definition of spreading models $\lim_{m\to\infty} \|a x_n - x_m\|$ exists for all $n \in \mathbb{N}$ and $0 \leq a \leq 1$. As $\lim_{m\to\infty} \|x_m\| = \lim_{m\to\infty} \|0 \cdot x_n - x_m\| = 1$ and $\lim_{m\to\infty} \|x_n - x_m\| > \lambda$, there exist by the Intermediate Value Theorem $0 < a_n < 1$ so that $\lim_{m\to\infty} \|a_n x_n - x_m\| = \lambda > 1 + \varepsilon$. After passing to a subsequence of $(x_i)_{i=1}^{\infty}$, we may assume that $\|a_n x_n - x_m\| > 1 + \varepsilon$ for all $m, n \in \mathbb{N}$.

Since for all $m, n \in \mathbb{N}$ we have $\|x_n\| = \|x_m\| = 1$ and $\|a_n x_n - x_m\| > 1 + \varepsilon$, it follows that $\phi_{a_n x_n - x_m}(x_n) > \varepsilon / a_n$, and, thus,

$$\begin{aligned}\lambda &= \lim_{m\to\infty} \|a_n x_n - x_m\|\\ &= \lim_{m\to\infty} \phi_{a_n x_n - x_m}(a_n x_n - x_m)\\ &= \lim_{m\to\infty} \phi_{a_n x_n - x_m}(x_n - x_m) - (1 - a_n)\phi_{a_n x_n - x_m}(x_n)\\ &\leq \lim_{m\to\infty} \|x_n - x_m\| - (1 - a_n)\varepsilon/a_n = \lambda_n - \varepsilon(1/a_n - 1).\end{aligned}$$

Since $\lambda = \lim_{n\to\infty} \lambda_n$ and $0 < a_n < 1$, for $n \in \mathbb{N}$, it follows that $a_n \to 1$. Hence, $\lim_{m\to\infty} \|a_n x_n - a_m x_m\| = \lim_{m\to\infty} \|a_n x_n - x_m\| = \lambda$ for all $n \in \mathbb{N}$.

□

By perturbing the asymptotically equilateral sequence given by Lemma 2.2 and passing to a subsequence, we obtain the following.

Lemma 2.3. *Let X be an infinite dimensional uniformly smooth Banach space and $(x_i)_{i=1}^{\infty} \subset X$ be a semi-normalized weakly null sequence. There exists a weakly null block sequence $(z_i)_{i=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ with $\lim_{i\to\infty} \|z_i\| = 1$ and a constant $\lambda > 1$ such that $\lim_{i\to\infty} \|z_k - z_i\| = \lambda$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \lim_{i\to\infty} \phi_{z_k - z_i}(z_\ell) = 0$ for all $\ell \in \mathbb{N}$.*

Proof. After passing to a subsequence and scaling, we assume by Lemmas 2.2 and 2.1 that there exists $\lambda > 1$ such that $\lim_{i\to\infty} \|x_i\| = 1$ and $\lim_{i\to\infty} \|x_k - x_i\| = \lambda$ for all $k \in \mathbb{N}$. For each $k > \ell$ we may pass to a subsequence of (x_i) such that $\lim_{i\to\infty} \phi_{x_k - x_i}(x_\ell)$ converges.

By taking the diagonal and passing to a further subsequence, we may assume that there exists $(b_\ell)_{\ell=1}^\infty \subset \mathbb{R}$ such that $\lim_{k\to\infty}\lim_{i\to\infty}\phi_{x_k-x_i}(x_\ell)=b_\ell$ for all $\ell\in\mathbb{N}$. Let x^* be a w^* accumulation point of $\{\phi_{x_k-x_i} : k,i\in\mathbb{N}\}$. As $(x_\ell)_{\ell=1}^\infty$ is weakly null, $\lim_{\ell\to\infty}x^*(x_\ell)=0$. Hence, $\lim_{\ell\to\infty}b_\ell=0$. If there exists a subsequence $(j_\ell)_{\ell=1}^\infty$ of $\mathbb{N}$ such that $b_{j_\ell}=0$ for all $\ell\in\mathbb{N}$ then setting $z_\ell=x_{j_\ell}$ gives our desired sequence. We thus may assume by passing to a subsequence that $b_\ell^2>|b_{\ell+1}|>0$ for all $\ell\in\mathbb{N}$. We set $v_\ell=x_{2\ell+1}-\frac{b_{2\ell+1}}{b_{2\ell}}x_{2\ell}$. Thus, $\lim_{k\to\infty}\lim_{i\to\infty}\phi_{x_k-x_i}(v_\ell)=0$ for all $\ell\in\mathbb{N}$. Furthermore, $\lim_{\ell\to\infty}\|v_\ell-x_{2\ell+1}\|=0$ as $b_\ell\to 0$ and $b_\ell^2>|b_{\ell+1}|>0$ for all $\ell\in\mathbb{N}$. As Φ is uniformly continuous on semi-normalized subsets of X, we have that $\lim_{k\to\infty}\lim_{i\to\infty}\phi_{v_k-v_i}(v_\ell)=0$ for all $\ell\in\mathbb{N}$. After passing to a subsequence of (v_i), we may assume by Lemma 2.2 that there exists a sequence of constants $c_\ell\to 1$ such that $\lim_{i\to\infty}\|c_kv_k-c_iv_i\|=\lambda$ for all $k\in\mathbb{N}$. As the map Φ is uniformly continuous on semi-normalized subsets of X and $c_k\to 1$, we have that $\lim_{k\to\infty}\lim_{i\to\infty}\phi_{c_kv_k-c_iv_i}(c_\ell v_\ell)=0$ for all $\ell\in\mathbb{N}$. Furthermore, we have that $(c_kv_k)_{k=1}^\infty$ is weakly null as $(x_{2k+1})_{k=1}^\infty$ and $(x_{2k})_{k=1}^\infty$ are weakly null. Thus letting $z_k=c_kv_k$ for all $k\in\mathbb{N}$ gives our desired sequence.

□

Given a Banach space X, recall that the *modulus of smoothness* of X is the function $\rho_X:[0,\infty)\to[0,\infty)$ defined by

$$\rho_X(\tau):=\sup\left\{\frac{1}{2}\|x+\tau y\|+\frac{1}{2}\|x-\tau y\|-1\ :\ x,y\in S_X\right\}\qquad\text{for all }\tau\in[0,\infty).$$

The modulus of smoothness quantifies the uniform smoothness of S_X, and a Banach space is uniformly smooth if and only if $\lim_{\tau\to 0^+}\frac{\rho_X(\tau)}{\tau}=0$.

Let X be a Banach space and let $(x_j)_{j=1}^\infty\subset X$ such that $\lim_{j\to\infty}\|x+x_j\|$ exists for all $x\in X$. The map $x\mapsto\lim_{j\to\infty}\|x+x_j\|$ is called a type. See [G] for a reference on types. The following Lemma gives a relationship between types and uniform smoothness.

Lemma 2.4. *Let X be a uniformly smooth Banach space and let $Y\subseteq X$ be a subspace. Let $(x_j)_{j=1}^\infty\subset X$ be a seminormalized weakly null sequence such that $\lim_{j\to\infty}\|y-ax_j\|$ exists for all $y\in Y$ and $a\in\mathbb{R}$. Define $\|\!|\cdot|\!\|$ on $Y\oplus\mathbb{R}$ by $\|\!|(y,a)|\!\|=\lim_{j\to\infty}\|y-ax_j\|$. Then $Y\oplus\mathbb{R}$ is a uniformly smooth Banach space under the norm $\|\!|\cdot|\!\|$ with modulus of smoothness at most the modulus of smoothness of X.*

Proof. Let $\rho_X:[0,\infty)\to[0,\infty)$ be the modulus of smoothness of X. Let $\tau>0$, and $(x,a),(y,b)\in S_{Y\oplus\mathbb{R}}$. Since $\lim_{j\to\infty}\|x-ax_j\|=1$ and $\lim_{j\to\infty}\|y-bx_j\|=1$, we have that,

$$\begin{aligned}
&\frac{1}{2}\|\!|(x,a)+\tau(y,b)|\!\|+\frac{1}{2}\|\!|(x,a)-\tau(y,b)|\!\|-1\\
&=\lim_{j\to\infty}\frac{1}{2}||x-ax_j+\tau(y-bx_j)||+\lim_{i\to\infty}\frac{1}{2}||x-ax_i-\tau(y-bx_j)||-1\\
&=\lim_{j\to\infty}\frac{1}{2}\left\|\frac{x-ax_j}{\|x-ax_j\|}+\tau\frac{y-bx_j}{\|y-bx_j\|}\right\|+\frac{1}{2}\left\|\frac{x-ax_j}{\|x-ax_j\|}-\tau\frac{y-bx_j}{\|y-bx_j\|}\right\|-1\\
&\le\rho_X(\tau).
\end{aligned}$$

Thus, $\rho_{Y\oplus\mathbb{R}}(\tau)\le\rho_X(\tau)$ and hence $Y\oplus\mathbb{R}$ is uniformly smooth under the norm $\|\!|\cdot|\!\|$. □

Lemma 2.5. *Let X be a uniformly smooth Banach space and let $Y \subseteq X$ be a subspace. Let $(x_j)_{j=1}^\infty \subset X$ be a seminormalized weakly null sequence such that $\lim_{j\to\infty} \|y - ax_j\|$ exists for all $y \in Y$ and $a \in \mathbb{R}$. Define $\||\cdot\||$ on $Y \oplus \mathbb{R}$ by $\||(y,a)\|| = \lim_{j\to\infty} \|y - ax_j\|$. Then for all $z, y \in Y$ and $a, b \in \mathbb{R}$,*

$$\phi_{(y,a)}((z,b)) = \lim_{j\to\infty} \phi_{y-ax_j}(z - bx_j).$$

Proof. Let $(y,a) \in S_{Y\oplus\mathbb{R}}$. We have that

$$\phi_{(y,a)}((y,a)) = \||(y,a)\|| = \lim_{j\to\infty} \|y - ax_j\| = \lim_{j\to\infty} \phi_{y-ax_j}(y - ax_j).$$

Let $(z,b) \in S_{Y\oplus\mathbb{R}}$ such that $\phi_{(y,a)}((z,b)) = 0$. Assume that $\lim_{j\to\infty} \phi_{y-ax_j}(z-bx_j) \neq 0$. Thus, there exists $c > 0$, $\sigma \in \{-1, 1\}$, and a subsequence $(k_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that $\sigma\phi_{y-ax_{k_j}}(z-bx_{k_j}) \geq c$ for all $j \in \mathbb{N}$. Let $\lambda > 0$.

$$\begin{aligned}
\||(y,a) + \lambda\sigma(z,b)\|| &= \lim_{j\to\infty} \|y - ax_j + \lambda\sigma(z - bx_j)\| \\
&\geq \liminf_{j\to\infty} \phi_{y-ax_{k_j}}(y - ax_{k_j} + \lambda\sigma(z - bx_{k_j})) \\
&= \lim_{j\to\infty} \phi_{y-ax_{k_j}}(y - ax_{k_j}) + \lambda \liminf_{j\to\infty} \sigma\phi_{y-ax_{k_j}}(z - bx_{k_j}) \\
&\geq 1 + \lambda c.
\end{aligned}$$

Hence, we have that

$$\phi_{(y,a)}(\sigma(z,b)) = \lim_{\lambda\to 0} \frac{\||(y,a) + \sigma\lambda(z,b)\|| - \||(y,a)\||}{\lambda} \geq \frac{(1+\lambda c) - 1}{\lambda} = c.$$

This is a contradiction as we have assumed that $\phi_{(y,a)}((z,b)) = 0$. Thus $\lim_{j\to\infty} \phi_{y-ax_j}(z - bx_j) = 0$ for all $(z,b) \in \phi_{(y,a)}^{-1}(0)$. We have as well that $\lim_{j\to\infty} \phi_{y-ax_j}(y-ax_j) = \phi_{(y,a)}((y,a))$. Thus, $\lim_{j\to\infty} \phi_{y-ax_j}(z - bx_j) = \phi_{(y,a)}((z,b))$ for all $(z,b) \in Y \oplus \mathbb{R}$. □

3. A uniform version of the Inverse Mapping Theorem

Let $d \in \mathbb{N}$ and $U \subset \mathbb{R}^d$ be a compact and convex subset whose interior contains the origin. We denote by $C_0^1(U, \mathbb{R}^d)$ the space of all continuously differentiable function $f : U \to \mathbb{R}^d$, with $f(0) = 0$. For $f \in C_0^1(U, \mathbb{R}^d)$, let f_i denote the i-th component of f, for $i \leq d$. The derivative function is denoted by Df, *i.e.*,

$$Df : U \to \mathbb{R}^{(d,d)} \quad \xi \mapsto \left[\frac{\partial f_i}{\partial x_j}(\xi)\right]_{1\leq i,j\leq d}.$$

$\mathbb{R}^{(d,d)}$ is the space of $d \times d$ matrices. Elements of $\mathbb{R}^{(d,d)}$ can be seen as operators on ℓ_2^d and we denote the operator norm on $\mathbb{R}^{(d,d)}$ by $\|\cdot\|_2$. We also denote the Euclidean norm on $\mathbb{R}^d$ by $\|\cdot\|_2$.

It follows for $f \in C_0^1(U, \mathbb{R}^{(d,d)})$ that the map $Df(\cdot)$ lies in $C(U, \mathbb{R}^{(d,d)})$, the space of all $\mathbb{R}^{(d,d)}$-valued continuous functions on U. For $M \in C(U, \mathbb{R}^{(d,d)})$ we let $\|M\|_\infty = \sup_\xi \|M(\xi)\|_2$ and for $f \in C_0^1(U, \mathbb{R}^d)$ we let $\|f\|_{(1,\infty)} = \|Df\|_\infty$. Then $\|\cdot\|_\infty$ and $\|\cdot\|_{(1,\infty)}$ are norms on $C(U, \mathbb{R}^{(d,d)})$ and $C_0^1(U, \mathbb{R}^d)$ respectively, which turn $C(U, \mathbb{R}^{(d,d)})$ and $C_0^1(U, \mathbb{R}^d)$ into Banach spaces, and the operator

$$D : C_0^1(U, \mathbb{R}^d) \to C(U, \mathbb{R}^{(d,d)}), \quad f \mapsto Df,$$

is an isometric embedding, onto the subspace of continuous functions

$$M = [M_{(i,j)}] : U \to \mathbb{R}^{(d,d)}, \quad \xi \mapsto [M_{(i,j)}(\xi)]_{1\le i,j\le d},$$

for which the ith row, $[M_{(i,j)}(\cdot)]_{1\le j\le d}$ is a conservative vector field, for all $i = 1, 2, \dots, d$.

From these considerations and the Theorem of Arzela-Ascoli we obtain the following compactness criterium.

Proposition 3.1. *A bounded subset $B \subset C^1_0(U, \mathbb{R}^d)$ is relatively norm compact if and only if $\{Df : f \in B\}$ is equicontinuous.*

For a decreasing function $\delta(\cdot) : (0, 1) \to (0, 1)$, with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, and a real number $R > 0$ we let $\mathcal{F}_{(\delta(\cdot),R)}$ be the set of all $f \in C^1_0(U, \mathbb{R}^d)$ for which $\|Df(0)\|_2 \le R$, $Df(0)$ is invertible, with $\|Df(0)^{-1}\|_2 \le R$, for which the modulus of continuity of Df is not larger than $\delta(\cdot)$, *i.e.* $\|Df(\xi) - Df(\eta)\|_2 \le \varepsilon$, for $\xi, \eta \in U$ with $\|\xi - \eta\|_2 \le \delta(\varepsilon)$. Note that $\mathcal{F}_{(\delta(\cdot),R)}$ is a closed and bounded set and $\{Df : f \in \mathcal{F}_{(\delta(\cdot),R)}\}$ is equicontinuous. Thus, $\mathcal{F}_{(\delta(\cdot),R)}$ is compact by Proposition 3.1.

We now state and prove a uniform version of the inverse mapping theorem. This will be used in proving our main result in Section 4.

Corollary 3.2. *Let $d \in \mathbb{N}$. For all $R > 0$ and decreasing functions $\delta(\cdot) : (0, 1) \to (0, 1)$, with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, there is an $\eta = \eta(\delta(\cdot), R)$, so that for all $f \in \mathcal{F}_{(\delta(\cdot),R)}$ we have $\eta B^d \subset f(U)$, where B^d denotes the Euclidean unit ball in $\mathbb{R}^d$.*

Proof. Assume our claim was not true. Then we could choose a sequence $f^{(n)} \subset \mathcal{F}_{(\delta(\cdot),R)}$, so that $\frac{1}{n}B^d \not\subset f^{(n)}(U)$, for all $n \in \mathbb{N}$.

As $\mathcal{F}_{(\delta(\cdot),R)}$ is compact, we may assume that $f^{(n)}$ converges in norm to some $f \in \mathcal{F}_{(\delta(\cdot),R)}$. By the Inverse Mapping Theorem f has a continuously differentiable inverse f^{-1} on some neighborhood $V \subset U$ of the origin. Since the sequence $(Df^{(n)})_{n=1}^\infty$ is bounded, the sequence $(f^{(n)})_{n=1}^\infty$ is equicontinuous and we can find $\rho > 0$ so that that for all $n \in \mathbb{N}$ $f^{(n)}(\rho B^d) \subset V$. For $n \in \mathbb{N}$ we consider the map

$$g^{(n)} : \rho B^d \to \mathbb{R}^d, \quad \xi \mapsto f^{-1} \circ f^{(n)}(\xi).$$

The sequence $(g^{(n)})_{n=1}^\infty$ converges in $C^1_0(\rho B^d, \mathbb{R}^d)$ to the identity. After possibly decreasing ρ and passing to a subsequence of the $(g^{(n)})$ we may assume that for all $n \in \mathbb{N}$

$$\|Dg^{(n)}(x) - \mathrm{Id}\,\|_2 \le \frac{1}{2} \text{ and } \|(Dg^{(n)}(x))^{-1} - \mathrm{Id}\,\|_2 \le \frac{1}{2}, \text{ for all } x \in \rho B^d, \tag{3.1}$$

$$\big\|g^{(n)}(z) - (g^{(n)}(x) + \langle Dg^{(n)}(x), z - x\rangle)\big\|_2 < \frac{1}{8}\|z - x\|_2, \text{ for all } x, z \in \rho B^d. \tag{3.2}$$

(3.1) can be achieved since $Dg^{(n)}(\cdot)$ uniformly converges to the identity matrix, and (3.2) can be achieved using the Taylor formula and the equicontinuity of the sequence $\big(Dg^{(n)}(\cdot)\big)$.

We claim that the image of ρB^d under $g = g^{(n)}$, $n \in \mathbb{N}$ contains $\frac{\rho}{4}B^d$.

Indeed, assume $y \in \frac{\rho}{4}B^d$. Choose $x_1 = y$ and note that

$$\begin{aligned}\|g^{(n)}(x_1) - y\|_2 &\le \|g^{(n)}(y) - Dg^{(n)}(0)(y)\|_2 + \|Dg^{(n)}(0)(y) - y\|_2 \\ &\le \frac{1}{8}\|y\|_2 + \frac{1}{2}\|y\|_2 \le \frac{\rho}{4}. \qquad \text{by (3.2) and (3.1).}\end{aligned}$$

Assume that we have chosen $x_1, x_2, \dots, x_m \in \rho B^d$ satisfying the following conditions for all $j = 1, 2, \dots, m$.

$$\|x_j - x_{j-1}\|_2 \le \frac{3}{2}\Big(\frac{1}{4}\Big)^{j-1}\rho \quad (\text{if } j > 1) \text{ and thus} \tag{3.3}$$

$$\|x_j\|_2 \le \frac{\rho}{4} + \rho\sum_{i=2}^{j}\frac{3}{2}\Big(\frac{1}{4}\Big)^{i-1} < \rho,$$

$$\|g(x_j) - y\|_2 \le \Big(\frac{1}{4}\Big)^{j}\rho. \tag{3.4}$$

Then we let

$$x_{m+1} = x_m + \big(Dg(x_m)\big)^{-1}\big(y - g(x_m)\big).$$

It follows from (3.1) and the induction hypothesis (3.4) that

$$\|x_{m+1} - x_m\|_2 \le \big\|(Dg(x_m))^{-1}\big\|_2 \cdot \|y - g(x_m)\|_2 \le \frac{3}{2}\Big(\frac{1}{4}\Big)^{m}\rho. \tag{3.5}$$

We now have that

$$\begin{aligned}\|g(x_{m+1}) - y\|_2 &= \big\|g(x_{m+1}) - (g(x_m) + Dg(x_m)(x_{m+1} - x_m)\big\|_2\\ &\le \frac{1}{8}\|x_{m+1} - x_m\|_2 \qquad \text{by (3.2)}\\ &\le \frac{1}{8}\frac{3}{2}\Big(\frac{1}{4}\Big)^{m}\rho < \Big(\frac{1}{4}\Big)^{(m+1)}\rho \qquad \text{by (3.5).}\end{aligned}$$

which finishes the induction step.

Letting $x = \lim_{m\to\infty} x_m = x_1 + \sum_{j=1}^{\infty}(x_{j+1} - x_j)$ it follows that

$$\|x\|_2 \le \frac{\rho}{4} + \frac{3\rho}{2}\sum_{j=1}^{\infty}\frac{1}{4^j} \le \frac{\rho}{4} + \frac{3}{2}\frac{1}{3}\rho < \rho,$$

and by (3.4) we have $g(x) = y$. Hence, the image of ρB^d under $g = g^{(n)}$, $n \in \mathbb{N}$ contains $\frac{\rho}{4}B^d$.

Finally we can find a positive $\rho' > 0$ so that $\rho' B^d \subset f(\frac{\rho}{4}B^d)$, and thus

$$\rho' B^d \subset f\Big(\frac{\rho}{4}B^d\Big) \subset f \circ g^{(n)}(\rho B^d) = f^{(n)}(\rho B^d) \subset f^{(n)}(U),$$

which contradicts $\frac{1}{n}B^d \not\subset f^{(n)}(U)$, for all $n \in \mathbb{N}$, and hence our proof is complete. □

4. Constructing an equilateral set

Given an infinite dimensional uniformly smooth Banach space X, our goal is to construct an equilateral sequence $(x_n)_{n=1}^{\infty} \subset X$. This will be done by first constructing a sequence $(z_n)_{n=1}^{\infty} \subset X$ which is "close" to being equilateral as in Section 2. We will then choose $\varepsilon_n \searrow 0$ and perturb $(z_n)_{n=1}^{\infty}$ by a triangular array of constants $(a_{i,n})_{1\le i\le n<\infty}$ (with $|a_{i,n}| < \varepsilon_n$ for all $1 \le i \le n$) such that if we set $x_n = (1+a_{n,n})z_n + \sum_{i=1}^{n-1} a_{i,n}z_i$ then $(x_n)_{n=1}^{\infty}$ is equilateral. The sequence $\varepsilon_n \searrow 0$ will be determined by the following lemma.

For $N \in \mathbb{N}$, $(\varepsilon_i)_{i=2}^{N} \subset [0, 1)$, and $1 > C > 0$ we define $A_{N\times N}(C, (\varepsilon_i)_{i=2}^{N})$ to be the set of $N \times N$ matrices $[a_{i,j}]_{1\le i,j\le N} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ which satisfy the following three properties.

(1) $|a_{i,j}| \leq 2$ for all $1 \leq i, j \leq N$,
(2) $|a_{i,i}| \geq C$ for all $1 \leq i \leq N$,
(3) $|a_{i,j}| \leq \varepsilon_j$ for all $1 \leq i < j \leq N$.

Lemma 4.1. *For all $C > 0$ there exists a sequence $(R_N)_{N=1}^{\infty} \subset (0, \infty)$ and a sequence $(\varepsilon_i)_{i=2}^{\infty} \subset (0, 1)$ such that A is invertible and $\|A^{-1}\| \leq R_N$ for all $A \in A_{N\times N}(C, (\varepsilon_i)_{i=2}^{N})$.*

Proof. We will prove the lemma by induction on $N \in \mathbb{N}$. For $N = 1$ the lemma holds for $R_1 = \frac{1}{C}$. We now let $N \in \mathbb{N}$ and assume that $(\varepsilon_i)_{i=2}^{N}$ has been chosen such that if $A \in A_{N\times N}(C, (\varepsilon_i)_{i=2}^{N})$ then A is invertible. We let $A' = A_{N+1\times N+1}(C, (\varepsilon_2, \varepsilon_3, ..., \varepsilon_N, 0)$.

If $[a_{i,j}]_{1\leq i,j\leq N+1} \in A'$ then $[a_{i,j}]_{1\leq i,j\leq N} \in A_{N\times N}(C, (\varepsilon_i)_{i=2}^{N})$ is invertible by the induction hypothesis, and hence $[a_{i,j}]_{1\leq i,j\leq N+1}$ is invertible because the last row of $[a_{i,j}]_{1\leq i,j\leq N+1}$ is linearly independent from the others. Thus, A' is a compact set of invertible matrices. As the set of invertible matrices on $\mathbb{R}^{N+1}$ is open, there exists ε_{N+1} such that $[a_{i,j}+\delta_{i,j}]_{1\leq,i,j\leq N+1}$ is invertible for all $[a_{i,j}]_{1\leq i,j\leq N+1} \in A'$ with $|\delta_{i,j}| \leq \varepsilon_{N+1}$ for all $1 \leq i, j \leq N + 1$. The map $A \mapsto A^{-1}$ is continuous on the set of invertible matrices, and hence there exists a constant $R_{N+1} > 0$ such that $\|[a_{i,j} + \delta_{i,j}]^{-1}_{1\leq,i,j\leq N+1}\| \leq R_{N+1}$ for all $[a_{i,j}]_{1\leq i,j\leq N+1} \in A'$ with $|\delta_{i,j}| \leq \varepsilon_{N+1}$ for all $1 \leq i, j \leq N + 1$ as this set is compact. Thus, $\|A^{-1}\| \leq R_{N+1}$ for all $A \in A_{(N+1)\times(N+1)}(C, (\varepsilon_i)_{i=2}^{N+1})$. □

We are now ready to prove our main result.

Theorem 4.2. *Let X be an infinite dimensional uniformly smooth Banach space. There exists a sequence $(x_i)_{i=1}^{\infty} \subset X$ and a constant $\lambda > 0$ such that $\|x_i - x_j\| = \lambda$ for all $i \neq j$.*

Proof. For all $x \in X \setminus \{0\}$, we let $\phi_x \in S_{X^*}$ be the unique functional such that $\phi_x(x) = \|x\|$. By Lemma 2.3, there exists a weakly null sequence $(z_i)_{i=1}^{\infty} \subset X$ such that $\lim_{i\to\infty} \|z_i\| = 1$ and a constant $2 > \lambda > 1$ such that $\lim_{i\to\infty} \|z_k-z_i\| = \lambda$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \lim_{i\to\infty} \phi_{z_k-z_i}(z_\ell) = 0$ for all $\ell \in \mathbb{N}$. After passing to a subsequence, we may assume that $\lim_{i\to\infty} \|x - z_i\|$ exists for all $x \in X$ and that $\|z_k - z_i\| > (1 + \lambda)/2$ for all $i \neq k$ and $\|z_i\| < (3 + \lambda)/4$ for all $i \in \mathbb{N}$. This gives us the following estimate for all $i \neq k$.

$$\phi_{z_k-z_i}(z_k) = \phi_{z_k-z_i}(z_k - z_i) + \phi_{z_k-z_i}(z_i) \geq \|z_k - z_i\| - \|z_i\| > \frac{1+\lambda}{2} - \frac{3+\lambda}{4} = \frac{\lambda-1}{4}. \tag{4.1}$$

We set $C = \frac{\lambda-1}{8}$, and thus we have that $\phi_{z_k-z_i}(z_k) > 2C > 0$ for all $i \neq k$. By Lemma 4.1 there exists $(R_N)_{N=1}^{\infty} \subset (0, \infty)$ and $(\varepsilon_i)_{i=2}^{\infty} \subset (0, 1)$ such that $\|A^{-1}\| \leq R_N$ for all $A \in A_{N\times N}(C, (\varepsilon_i)_{i=2}^{N})$. By induction on $N \in \mathbb{N}$, we shall produce a sequence $(x_i)_{i=1}^{\infty} \subset X$ and sequences of natural numbers $M_N = (m_i^N)_{i=1}^{\infty}$ with $M_0 = \mathbb{N}$ and M_N a subsequence of M_{N-1}, for all $N \in \mathbb{N}$, so that for all $N \in \mathbb{N}$, the following properties are satisfied.

(1) $\|x_i - x_j\| = \lambda$ for all $1 \leq i < j \leq N$,
(2) $\lim_{i\to\infty} \|x_k - z_{m_i^N}\| = \lambda$ for all $1 \leq k \leq N$,
(3) $\|x_i\| \leq 2$ for all $1 \leq i \leq N$,
(4) $\lim_{\ell\to\infty} \lim_{k\to\infty} \phi_{z_{m_\ell^N} - z_{m_k^N}}(x_i) = 0$ for all $1 \leq i \leq N$,
(5) $|\phi_{z_L - x_k}(x_i)| < \varepsilon_k$ for all $1 \leq i < k \leq N$ and $L \in M_N$.
(6) $|\phi_{z_L - x_k}(x_k)| > C$ for all $1 \leq k \leq N$ and $L \in M_N$.

Note that if we are able to construct such a sequence $(x_i)_{i=1}^{\infty}$ by induction, then $(x_i)_{i=1}^{\infty}$ would be equilateral by condition (1). Thus, all we need to do to complete the proof is to prove the induction argument. Let $N = 1$. We let $x_1 = z_1$ and $M_1 = (2, 3, 4, ...)$. Conditions (1) and (5) are trivially satisfied. Condition (2), (3), (4), and (6) are satisfied by our choice of $(z_i)_{i=1}^{\infty}$.

We now let $N \in \mathbb{N}$ and assume that we have constructed $(x_i)_{i=1}^{N}$ and $M_N = (m_i^N)_{i=1}^{\infty}$ to satisfy conditions (1) through (6). For each $K \in M_N$, we define a map $g^K : B_{\mathbb{R}^{N+1}} \to X$ by $g^K(a_1, ..., a_{N+1}) = (1+a_{N+1})z_K + \sum_{i=1}^{N} a_i x_i$. Our first goal is to show that there exists $\delta > 0$ and a subsequence M'_N of M_N such that if we set $x_{N+1} = g^K(a)$ for an arbitrary $a \in \delta B_{\mathbb{R}^{N+1}}$ and $K \in M'_N$, and if M_{N+1} is an arbitrary subsequence of $\{L \in M'_N | L > K\}$, then properties (3), (4), (5), and (6) would all hold.

As $\|z_K\| \leq (3 + \lambda)/4 < 2$ for all $K \in \mathbb{N}$, we may choose $\delta_1 > 0$ such that $\|g^K(a)\| \leq 2$ for all $a \in \delta_1 B_{\mathbb{R}^{N+1}}$ and $K \in M_1$. Thus, if $x_{N+1} = g^K(a)$ for an arbitrary $a \in \delta_1 B_{\mathbb{R}^{N+1}}$ and $K \in M_N$ then $\|x_{N+1}\| \leq 2$ and hence property (3) in the induction hypothesis would be satisfied.

For each $K \in \mathbb{N}$, we have that

$$\lim_{\ell\to\infty}\lim_{k\to\infty} \phi_{z_{m_\ell^N} - z_{m_k^N}}(g^K(a)) = \lim_{\ell\to\infty}\lim_{k\to\infty} (1 + a_{N+1})\phi_{z_{m_\ell^N} - z_{m_k^N}}(z_K) + \sum_{i=1}^{N} a_i \phi_{z_{m_\ell^N} - z_{m_k^N}}(x_i) = 0,$$

as $\lim_{k\to\infty} \lim_{i\to\infty} \phi_{z_k - z_i}(z_\ell) = 0$ for all $\ell \in \mathbb{N}$ and $\lim_{\ell\to\infty} \lim_{k\to\infty} \phi_{z_{m_\ell^N} - z_{m_k^N}}(x_i) = 0$ for all $1 \leq i \leq N$. Thus, if $x_{N+1} = g^K(a)$ for an arbitrary $a \in \delta_1 B_{\mathbb{R}^{N+1}}$ and $K \in M_N$ then property (4) in the induction hypothesis would be satisfied.

By (4), there exists a subsequence $M'_N = (m'^N_j)_{j=1}^{\infty}$ of M_N such that $|\phi_{z_L - z_K}(x_i)| \leq \varepsilon_{N+1}/2$ for all $1 \leq i \leq N$ and $L, K \in M'_N$ with $L > K$. The set $(g^K)_{K\in M'_N}$ is equicontinuous on $\delta_1 B_{\mathbb{R}^{N+1}}$, $g^K(0) = z_K$ for all $K \in M'_N$, and the map $x \mapsto \phi_x$ is uniformly continuous on $X \setminus \varepsilon B_X$ for all $\varepsilon > 0$. Thus, there exists $\delta_2 > 0$ with $\delta_2 < \delta_1$ such that $|\phi_{z_L - g^K(a)}(x_i)| < \varepsilon_{N+1}$ for all $1 \leq i \leq N$ and $K, L \in M'_N$ with $L > K$ and all $a \in \delta_2 B_{\mathbb{R}^{N+1}}$. Thus, if $x_{N+1} = g^K(a)$ for an arbitrary $a \in \delta_2 B_{\mathbb{R}^{N+1}}$ and $K \in M'_N$ then property (5) in the induction hypothesis would be satisfied for all $L \in M'_N$ with $L > K$. Similarly, (recalling that $\phi_{z_k - z_i}(z_k) > 2C$ for all $i \neq k$) after passing to a further subsequence of M'_N, we may assume that there exists $\delta > 0$ with $\delta < \delta_2$ such that if $x_{N+1} = g^K(a)$ for an arbitrary $a \in \delta B_{\mathbb{R}^{N+1}}$ and $K \in M'_N$ then property (6) in the induction hypothesis would be satisfied for all $L \in M'_N$ with $L > K$. Thus, if we set $x_{N+1} = g^K(a)$ for an arbitrary $a \in \delta B_{\mathbb{R}^{N+1}}$ and $K \in M'_N$, and if M_{N+1} is an arbitrary subsequence of $\{L \in M'_N | L > K\}$, then properties (3), (4), (5), and (6) would all hold.

Our next step is to show that we may choose $a \in \delta B_{\mathbb{R}^{N+1}}$, $K \in M'_N$, and a subsequence M_{N+1} of $\{L \in M'_N | L > K\}$ such that properties (1) and (2) hold for $x_{N+1} = g^K(a)$. For each $K \in M_N$ we define a map $f : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$f^K(a) = \left(\|g^K(a) - x_1\|, ..., \|g^K(a) - x_N\|, \lim_{j\to\infty} \|g^K(a) - z_{m'^N_j}\| \right) \quad \text{for all } a \in \mathbb{R}^{N+1}.$$

The derivative of f at 0, is given by $Df^K(0) = \left[\frac{\partial f_j^K}{\partial a_n}|_{a=0}\right]_{1\leq j,n\leq N+1}$. For $1 \leq j, n \leq N$,

$$\left.\frac{\partial f_j^K}{\partial a_n}\right|_{a=0} = \left.\frac{\partial}{\partial a_n}\right|_{a=0} \left\|(1 + a_{N+1})z_K + \sum_{i=1}^{N} a_i x_i - x_j\right\| = \phi_{z_K - x_j}(x_n). \tag{4.2}$$

For $1 \le n \le N$, we have by Lemma 2.5

$$(4.3)\quad \frac{\partial f^K_{N+1}}{\partial a_n}\bigg|_{a=0} = \frac{\partial}{\partial a_n}\bigg|_{a=0} \lim_{j\to\infty}\left\|(1+a_{N+1})z_K + \sum_{i=1}^N a_i x_i - z_{m'^N_j}\right\| = \lim_{j\to\infty}\phi_{z_K - z_{m'^N_j}}(x_n).$$

For $1 \le j \le N$, we have

$$(4.4)\quad \frac{\partial f^K_j}{\partial a_{N+1}}\bigg|_{a=0} = \frac{\partial}{\partial a_{N+1}}\bigg|_{a=0}\left\|(1+a_{N+1})z_K + \sum_{i=1}^N a_i x_i - x_j\right\| = \phi_{z_K - x_j}(z_K).$$

By Lemma 2.5,
(4.5)

$$\frac{\partial f^K_{N+1}}{\partial a_{N+1}}\bigg|_{a=0} = \frac{\partial}{\partial a_{N+1}}\bigg|_{a=0} \lim_{j\to\infty}\left\|(1+a_{N+1})z_K + \sum_{i=1}^N a_i x_i - z_{m'^N_j}\right\| = \lim_{j\to\infty}\phi_{z_K - z_{m'^N_j}}(z_K).$$

We first note that equations (4.2), (4.3), (4.4), and (4.5) imply that $\left|\frac{\partial f^K_j}{\partial a_n}|_{a=0}\right| \le 2$ for all $1 \le j, n \le N+1$ and $K \in M_N$ as $\|x_n\| \le 2$ for all $1 \le n \le N$ and $\|z_K\| \le 2$ for all $K \in M'_N$. By equation (4.2) and property (5), we have that $\left|\frac{\partial f^K_j}{\partial a_n}|_{a=0}\right| = \left|\phi_{z_K - x_j}(x_n)\right| < \varepsilon_j$ for all $K \in M_N$ and $1 \le n < j \le N$. By equation (4.3) we have that $\frac{\partial f^K_{N+1}}{\partial a_n}|_{a=0} = \lim_{j\to\infty}\phi_{z_K - z_{m'^N_j}}(x_n)$. Thus, by property (4), there exists $K_1 \in M'_N$ such that $\left|\frac{\partial f^K_{N+1}}{\partial a_n}|_{a=0}\right| < \varepsilon_{N+1}$ for all $1 \le n \le N$ and all $K \in M'_N$ with $K \ge K_1$. By equation (4.2) and property (6), we have that $\left|\frac{\partial f^K_j}{\partial a_j}|_{a=0}\right| = |\phi_{z_K - x_j}(x_j)| > C$ for all $K \in M'_N$ and all $1 \le j \le N$. By equation (4.5), we have that $\left|\frac{\partial f^K_{N+1}}{\partial a_{N=1}}|_{a=0}\right| = \left|\lim_{j\to\infty}\phi_{z_K - z_j}(z_K)\right| \ge \frac{\lambda-1}{4} > C$. Thus, we have that $Df^K(0) \in A_{(N+1)\times(N+1)}(C, (\varepsilon_i)_{i=2}^{N+1})$ and hence $\|(Df^K(0))^{-1}\| \le R_{N+1}$ for all $K \in M'_N$ with $K \ge K_1$.

Due to property (2) and $\lim_{i\to\infty}\|z_k - z_i\| = \lambda$ for all $k \in \mathbb{N}$, we have that $\lim_{j\to\infty} f^{m'^N_j}(0) = (\lambda, ..., \lambda)$. As $\|(Df^K(0))^{-1}\| \le R_{N+1}$ for all $K \in M'_N$ with $K \ge K_1$, we may apply Corollary 3.2 to obtain an integer $K \in M'_N$ with $K \ge K_1$ such that $(\lambda, ..., \lambda) \in f^K(\delta B_{N+1})$. Thus, there exists $a \in \delta B_{\mathbb{R}^{N+1}}$ such that $f^K(a) = (\lambda, ..., \lambda)$. We set $x_{N+1} = g^K(a)$ and $M_{N+1} = \{L \in M'_N \,|\, L > K\}$. As noted earlier, this choice of x_{N+1} and M_{N+1} satisfies properties (3), (4), (5), and (6) in the induction hypothesis. Furthermore, we have that

$$\|x_{N+1} - x_j\| = \left\|(1+a_{N+1})z_K + \sum_{i=1}^N a_i x_i - x_j\right\| = \lambda \quad \text{for all } 1 \le i \le N,$$

thus satisfying property (1). We have that

$$\lim_{j\to\infty}\|x_{N+1} - z_{m^{N+1}_j}\| = \lim_{j\to\infty}\left\|(1+a_{N+1})z_K + \sum_{i=1}^N a_i x_i - z_{m^{N+1}_j}\right\| = \lambda,$$

thus satisfying property (2) in the induction hypothesis. We have satisfied all properties in our induction hypothesis, and hence we obtain a sequence $(x_i)_{i=1}^\infty \subset X$ by induction which satisfies $\|x_i - x_j\| = \lambda > 0$ for all $i \ne j$. □

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