Math. Ann.
DOI 10.1007/s00208-010-0601-8

Mathematische Annalen

The universality of ℓ_1 as a dual space

D. Freeman · E. Odell · Th. Schlumprecht

Received: 12 November 2009 / Revised: 2 June 2010

Abstract Let X be a Banach space with a separable dual. We prove that X embeds isomorphically into a $\mathcal{L}_\infty$ space Z whose dual is isomorphic to ℓ_1. If, moreover, U is a space with separable dual, so that U and X are totally incomparable, then we construct such a Z, so that Z and U are totally incomparable. If X is separable and reflexive, we show that Z can be made to be somewhat reflexive.

Mathematics Subject Classification (2000) 46B20

1 Introduction

In 1980 Bourgain and Delbaen [7] showed the surprising diversity of $\mathcal{L}_\infty$ Banach spaces whose duals are isomorphic to ℓ_1 by constructing such a space Z not containing an isomorph of c_0. Moreover, Z is *somewhat reflexive*, i.e., every infinite dimensional subspace of Z contains an infinite dimensional reflexive subspace. In fact, R. Haydon [15] proved the reflexive subspaces could be chosen to be isomorphic to ℓ_p spaces.

E. Odell and Th. Schlumprecht was supported by the National Science Foundation.

D. Freeman · Th. Schlumprecht (✉)
Department of Mathematics, Texas A&M University,
College Station, TX 77843-3368, USA
e-mail: schlump@math.tamu.edu

D. Freeman
e-mail: freeman@math.tamu.edu

E. Odell
Department of Mathematics, The University of Texas at Austin,
Austin, TX 78712-0257, USA
e-mail: odell@math.utexas.edu

Published online: 11 November 2010

The structure of Banach spaces X whose dual is isometric to ℓ_1 is more limited. Such a space X must contain c_0 [29] and in fact be an isometric quotient of $C(\Delta)$ [18]. Finally it was shown in [11] that such spaces must be c_0 saturated. Nevertheless, such a space need not be an isometric quotient of some $C(\alpha)$, for $\alpha < \omega_1$ [1].

The construction developed by Bourgain and Delbaen is quite general and allows for additional modifications. Very recently Argyros and Haydon [4] were able to adapt this construction to solve the famous *Scalar plus Compact Problem* by building an infinite dimensional Banach space, with dual isomorphic to ℓ_1, on which all operators are a compact perturbation of a multiple of the identity. In this paper we will prove three main theorems concerning isomorphic preduals of ℓ_1.

Theorem A *Let X be a Banach space with separable dual. Then X embeds into a $\mathcal{L}_\infty$ space Y with Y^* isomorphic to ℓ_1.*

Moreover, we have the following refinements of Theorem A.

Theorem B *Let X and U be totally incomparable infinite dimensional Banach spaces with separable duals. Then X embeds into a $\mathcal{L}_\infty$ space Z whose dual is isomorphic to ℓ_1, so that Z and U are totally incomparable.*

Theorem C *Let X be a separable reflexive Banach space. Then X embeds into a somewhat reflexive $\mathcal{L}_\infty$ space Z, whose dual is isomorphic to ℓ_1. Furthermore, if U is a Banach space with separable dual such that X and U are totally incomparable, then Z can be chosen to be totally incomparable with U.*

We recall that X and U are called totally incomparable if no infinite dimensional Banach space embeds into both X and U.

Since there are reflexive spaces of arbitrarily high countable Szlenk index [28] Theorem B (with $U = c_0$) as well as Theorem C solve a question of Alspach [2, Question 5.1] who asked whether or not there are $\mathcal{L}_\infty$ spaces with arbitrarily high Szlenk index not containing c_0. Moreover Alspach, in conference talks, asked whether Theorem A could be true. Furthermore, Theorem B with $U = c_0$ solves the longstanding open problem of showing that if X^* is separable and X does not contain an isomorph of c_0, then X embeds into a Banach space with a shrinking basis which does not contain an isomorph of c_0.

In Sect. 2 we review the skeletal aspects of the Bourgain–Delbaen construction of $\mathcal{L}_\infty$ spaces, following more or less, [4]. Theorem A will be proved in Sect. 4, while the proofs of Theorems B and C are presented in Sect. 5. The construction used to prove Theorem A will also be useful in the case where X^* is not separable. The construction proving Theorems B and C will be an *augmentation* of that used to prove Theorem A.

Section 3 contains background material necessary for our proof. We review some embedding theorems from [12,26] that play a role in the subsequent constructions. Terminology and definitions are given along with some propositions that facilitate their use. In particular, we define the notion of a c-decomposition and relate it to an FDD being shrinking (Proposition 3.11). This will be used to show that our $\mathcal{L}_\infty$ constructs have dual isomorphic to ℓ_1. We also show how Theorem 3.11 leads to an alternate and self contained proof of a less precise version of embedding Theorems 3.8 and 3.9, which is sufficient for their use in this paper.

We use standard Banach space terminology as may be found in [16] or [23]. We recall that X is $\mathcal{L}_\infty$ if there exist $\lambda < \infty$ and finite dimensional subspaces $E_1 \subseteq E_2 \subseteq \cdots$ of X so that $X = \overline{\bigcup_{n=1}^\infty E_n}$ and the Banach-Mazur distance satisfies

$$d\left(E_n, \ell_\infty^{\dim(E_n)}\right) \le \lambda, \quad \text{for all } n \in \mathbb{N}.$$

In this case we say X is $\mathcal{L}_{\infty,\lambda}$. S_X and B_X denote the unit sphere and unit ball of X, respectively. A sequence of finite dimensional subspaces of X, $(E_i)_{i=1}^\infty$ is an FDD (finite dimensional decomposition) if every $x \in X$ can be uniquely expressed as $x = \sum_{i=1}^\infty x_i$ where $x_i \in F_i$ for all $i \in \mathbb{N}$. It is usually required that $E_i \ne \{0\}$ for all $i \in \mathbb{N}$ for $(E_i)_{i=1}^\infty$ to be a finite dimensional decomposition, but it will be convenient for us to allow $E_i = \{0\}$ for some i's in Sect. 5.

We note that there are deep constructions of $\mathcal{L}_\infty$ spaces other then the ones in [7]. For example Bourgain and Pisier [8] prove that every separable Banach space X embeds into a $\mathcal{L}_\infty$ space Y so that Y/X is a Schur space with the Radon Nikodym Property. Dodos [10] used the Bourgain–Pisier construction to prove that for every $\lambda > 1$ there exists a class $(Y_\lambda^\xi)_{\xi<\omega_1}$ of separable $\mathcal{L}_{\infty,\lambda}$ spaces with the following properties. Each Y_λ^ξ is non-universal (i.e. $C[0,1]$ does not embed into Y_λ^ξ) and if X is separable with $\phi_{NU}(X) \le \xi$, then X embeds into Y_ξ^λ. Here ϕ_{NU} is Bourgain's ordinal index based on the Schauder basis for $C[0,1]$. Now $C[0,1]$ is a $\mathcal{L}_\infty$-space and is universal for the class of separable Banach spaces. Theorem A yields that the class of $\mathcal{L}_\infty$-spaces with separable dual is universal for the class of all Banach spaces with separable dual. We thank the second referee for promptly reviewing our paper.

2 Framework of the Bourgain–Delbaen construction

As promised, this section contains the general framework of the construction of *Bourgain–Delbaen spaces*. This framework is general enough to include the original space of Bourgain and Delbaen [7], the spaces constructed in [4], as well as the spaces constructed in this paper. We follow, with slight changes and some notational differences, the presentation in [4] and start by introducing *Bourgain–Delbaen sets*.

Definition 2.1 (Bourgain–Delbaen-sets) A sequence of finite sets $(\Delta_n : n \in \mathbb{N})$ is called a *Sequence of Bourgain–Delbaen Sets* if it satisfies the following recursive conditions:

Δ_1 is any finite set, and assuming that for some $n \in \mathbb{N}$ the sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ have been chosen, we let $\Gamma_n = \bigcup_{j=1}^n \Delta_j$. We denote the unit vector basis of $\ell_1(\Gamma_n)$ by $(e_\gamma^* : \gamma \in \Gamma_n)$, and consider the spaces $\ell_1(\Gamma_j)$ and $\ell_1(\Gamma_n \setminus \Gamma_j)$, $j < n$, to be, in the natural way, embedded into $\ell_1(\Gamma_n)$.

For $n \ge 1$, Δ_{n+1} will be the union of two sets $\Delta_{n+1}^{(0)}$ and $\Delta_{n+1}^{(1)}$, where $\Delta_{n+1}^{(0)}$ and $\Delta_{n+1}^{(1)}$ satisfy the following conditions.

The set $\Delta_{n+1}^{(0)}$ is finite and

$$\Delta_{n+1}^{(0)} \subset \big\{(n+1,\beta,b^*,f) : \beta \in [0,1], b^* \in B_{\ell_1(\Gamma_n)}, \\ \text{and} \quad f \in V_{(n+1,\beta,b^*)}\big\}, \tag{2.1}$$

where $V_{(n+1,\beta,b^*)}$ is a finite set for $\beta \in [0,1]$ and $b^* \in B_{\ell_1(\Gamma_n)}$.

$\Delta_{n+1}^{(1)}$ is finite and

$$\Delta_{n+1}^{(1)} \subset \left\{ (n+1,\alpha,k,\xi,\beta,b^*,f) : \begin{array}{l} \alpha,\beta \in [0,1], \\ k \in \{1,2,\dots n-1\}, \\ \xi \in \Delta_k,\, b^* \in B_{\ell_1(\Gamma_n \setminus \Gamma_k)} \\ \text{and} \quad f \in V_{(n+1,\alpha,k,\xi,\beta,b^*)} \end{array} \right\}, \tag{2.2}$$

where $V_{(n+1,\alpha,k,\xi,\beta,b^*)}$ is a finite set for $\alpha \in [0,1]$, $k \in \{1,2,\dots,n-1\}$, $\xi \in \Delta_k$, $\beta \in [0,1]$, and $b^* \in B_{\ell_1(\Gamma_n \setminus \Gamma_k)}$.

Moreover, we assume that $\Delta_{n+1}^{(0)}$ and $\Delta_{n+1}^{(1)}$ cannot both be empty.

If (Δ_n) is a sequence of Bourgain–Delbaen sets we put $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_n$. For $n \in \mathbb{N}$, and $\gamma \in \Delta_n$ we call n the *rank of* γ and denote it by $\mathrm{rk}(\gamma)$. If $n \geq 2$ and $\gamma = (n,\beta,b^*,f) \in \Delta_n^{(0)}$, we say that γ is *of type* 0, and if $\gamma = (n,\alpha,k,\xi,\beta,b^*,f) \in \Delta_n^{(1)}$, we say that γ is *of type* 1. In both cases we call β *the weight of* γ and denote it by $\mathrm{w}(\gamma)$ and call f the *free variable* and denote it by $\mathrm{f}(\gamma)$.

In case that $V_{(n+1,\beta,b^*)}$ or $V_{(n+1,\alpha,k,\xi,\beta,b^*)}$ is a singleton (which will be often he case) we sometimes suppress the dependency in the free variable and write $(n+1, \beta, b^*)$ instead of $(n+1,\beta,b^*,f)$ and $(n+1,\alpha,k,\xi,\beta,b^*)$ instead of $(n+1,\alpha,k,\xi, \beta,b^*,f)$.

Referring to a sequence of sets $(\Delta_n : n \in \mathbb{N})$ as Bourgain–Delbaen sets we will always mean that the sets $\Delta_n^{(0)}$, $\Delta_n^{(1)}$, Γ_n and Γ have been defined satisfying the conditions above. We consider the spaces $\ell_\infty(\bigcup_{j\in A} \Delta_j)$ and $\ell_1(\bigcup_{j\in A} \Delta_j)$, for $A \subset \mathbb{N}$, to be naturally embedded into $\ell_\infty(\Gamma)$ and $\ell_1(\Gamma)$, respectively.

We denote by $c_{00}(\Gamma)$ the real vector space of families $x = (x(\gamma) : \gamma \in \Gamma) \subset \mathbb{R}$ for which the *support*, $\mathrm{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$, is finite. The unit vector basis of $c_{00}(\Gamma)$ is denoted by $(e_\gamma : \gamma \in \Gamma)$, or, if we regard $c_{00}(\Gamma)$ to be a subspace of a dual space, such as $\ell_1(\Gamma)$, by $(e_\gamma^* : \gamma \in \Gamma)$. If $\Gamma = \mathbb{N}$ we write c_{00} instead of $c_{00}(\mathbb{N})$.

Definition 2.2 (Bourgain–Delbaen families of functionals) Assume that $(\Delta_n : n \in \mathbb{N})$ is a sequence of Bourgain–Delbaen sets. By induction on n we will define for all $\gamma \in \Delta_n$, elements $c_\gamma^* \in \ell_1(\Gamma_{n-1})$ and $d_\gamma^* \in \ell_1(\Gamma_n)$, with $d_\gamma^* = e_\gamma^* - c_\gamma^*$.

For $\gamma \in \Delta_1$ we define $c_\gamma^* = 0$, and thus $d_\gamma^* = e_\gamma^*$.

Assume that for some $n \in \mathbb{N}$ we have defined $(c_\gamma^* : \gamma \in \Gamma_n)$, with $c_\gamma^* \in \ell_1(\Gamma_{j-1})$, if $j \leq n$ and $\mathrm{rk}(\gamma) = j$. It follows therefore that $(d_\gamma^* : \gamma \in \Gamma_n) = (e_\gamma^* - c_\gamma^* : \gamma \in \Gamma_n)$ is a basis for $\ell_1(\Gamma_n)$ and thus for $k \leq n$ we have projections:

$$P_{(k,n]}^* : \ell_1(\Gamma_n) \to \ell_1(\Gamma_n), \quad \sum_{\gamma \in \Gamma_n} a_\gamma d_\gamma^* \to \sum_{\gamma \in \Gamma_n \setminus \Gamma_k} a_\gamma d_\gamma^*. \tag{2.3}$$

For $\gamma \in \Delta_{n+1}$ we define

$$c^*_\gamma = \begin{cases} \beta b^* & \text{if } \gamma = (n+1, \beta, b^*, f) \in \Delta^{(0)}_{n+1}, \\ \alpha e^*_\xi + \beta P^*_{(k,n]}(b^*) & \text{if } \gamma = (n+1, \alpha, k, \xi, \beta, b^*, f) \in \Delta^{(1)}_{n+1}. \end{cases} \tag{2.4}$$

We call $(c^*_\gamma : \gamma \in \Gamma)$, the *Bourgain–Delbaen family of functionals associated to* $(\Delta_n : n \in \mathbb{N})$. We will, in this case, consider the projections $P^*_{(k,n]}$ to be defined on all of $c_{00}(\Gamma)$, which is possible since $(d^*_\gamma : \gamma \in \Gamma)$ forms a vector basis of $c_{00}(\Gamma)$ and, (as we will observe later) under further assumptions, a Schauder basis of $\ell_1(\Gamma)$.

Remark 2.3 The reason for using $*$ in the notation for $P^*_{(k,m]}$ is that later we will show (with additional assumptions) that the $P^*_{(k,m]}$'s are the adjoints of coordinate projections $P_{(k,m]}$ on a space Y with an FDD $\mathbf{F} = (F_j)$ onto $\oplus_{j\in(k,m]} F_j$.

Of course we could, in the definition of $\Delta^{(0)}_{n+1}$ and $\Delta^{(1)}_{n+1}$, assume $\beta = 1$, rescale b^* accordingly, possibly increasing the number of free variables, then simply define $c^*_\gamma = b^*$, if γ is of type 0, or $c^*_\gamma = \alpha e^*_\xi + P^*_{(k,n]}(b^*)$, if γ is of type 1. Nevertheless, it will prove later more convenient to have this redundant representation which will allow us to change the weights of the elements of Γ and rescale the b^*'s, without changing the c^*_γ's. Moreover, it will be useful for recognizing that our framework is a generalization of the constructions in [4,7].

The next observation is a slight generalization of a result in [4], the main idea tracing back to [7].

Proposition 2.4 *Let* $(\Delta_n : n \in \mathbb{N})$ *be a sequence of Bourgain–Delbaen sets and let* $(c^*_\gamma : \gamma \in \Gamma)$ *be the corresponding family of associated functionals. Let* $(P^*_{(k,m]} : k < m)$ *and* $(d^*_\gamma : \gamma \in \Gamma)$ *be defined as in Definition 2.2. Thus*

$$P^*_{(k,n]} : c_{00}(\Gamma) \to c_{00}(\Gamma), \quad \sum_{\gamma\in\Gamma} a_\gamma d^*_\gamma \to \sum_{\gamma\in\Gamma_n\setminus\Gamma_k} a_\gamma d^*_\gamma.$$

For $n \in \mathbb{N}$, *let* $F^*_n = \operatorname{span}(d^*_\gamma : \gamma \in \Delta_n)$ *and for* $\theta \in [0, 1/2)$ *let* $C_1(\theta) = C_1 = 0$ *and if* $n \ge 2$,

$$C_n(\theta) = \sup\Big\{\beta \| P^*_{(k,m]}(b^*)\| : \gamma = (\tilde{n}, \alpha, k, \xi, \beta, b^*, f) \in \Delta^{(1)}_{\tilde{n}}, \\ k < m < \tilde{n} \le n, \beta > \theta\Big\},$$

with $\sup(\emptyset) = 0$, *and*

$$C_n = C_n(0) = \sup\Big\{\beta \| P^*_{(k,m]}(b^*)\| : \gamma = (\tilde{n}, \alpha, k, \xi, \beta, b^*, f) \in \Delta^{(1)}_{\tilde{n}}, \\ k < m < \tilde{n} \le n\Big\}.$$

Then

$$\oplus^n_{j=1} F^*_j = \operatorname{span}(e^*_\gamma : \gamma \in \Gamma_n) = \ell_1(\Gamma_n), \tag{2.5}$$

and if $C = \sup_n C_n < \infty$*, then* $\mathbf{F}^* = (F_n^*)$ *is an FDD for* $\ell_1(\Gamma)$ *whose decomposition constant* M *is not larger than* $1 + C$*. Moreover, for* $n \in \mathbb{N}$ *and* $\theta < 1/2$*,*

$$C_n \leq \max\,(2\theta/(1-2\theta), C_n(\theta)). \tag{2.6}$$

Proof As already noted, since $d_\gamma^* = e_\gamma^* - c_\gamma^*$, and $c_\gamma^* \in \ell_1(\Gamma_{n-1})$, for $n \in \mathbb{N}$ and $\gamma \in \Delta_n$, (2.5) holds. By induction on $n \in \mathbb{N}$ we will show that for all $0 \leq m < n$, $\|P_{[1,m]}^*|_{\ell_1(\Gamma_n)}\| \leq 1 + C_n$, and that (2.6) holds, whenever $\theta < 1/2$. For $n = 1$, and thus $m = 0$ and $C_1 = 0$, the claim follows trivially ($\|P_\emptyset^*\| \equiv 0$). Assume the claim is true for some $n \in \mathbb{N}$. Using the induction hypothesis and the fact that every element of $B_{\ell_1(\Gamma_{n+1})}$ is a convex combination of $\{\pm e_\gamma^* : \gamma \in \Gamma_{n+1}\}$ and $C_n(\theta) \leq C_{n+1}(\theta)$, it is enough to show that for all $\gamma \in \Delta_{n+1}$ and all $m \leq n$

$$\|P_{[1,m]}^*(e_\gamma^*)\| \leq 1 + C_{n+1} \quad \text{and} \tag{2.7}$$

$$\|\beta P_{(k,m]}^*(b^*)\| \leq \frac{2\theta}{1-2\theta} \vee C_n(\theta), \quad \text{if } \beta \leq \theta < 1/2 \quad \text{and} \tag{2.8}$$
$$\gamma = (n+1, \alpha, k, \xi, \beta, b^*, f) \in \Delta_{n+1}^{(1)}.$$

According to (2.4) we can write

$$e_\gamma^* = d_\gamma^* + c_\gamma^* = d_\gamma^* + \alpha e_\xi^* + \beta P_{(k,n]}^*(b^*),$$

with $\alpha, \beta \in [0, 1]$, $0 \leq k < n$, $\xi \in \Delta_k$ (put $k = 0$ and $\alpha = 0$ if γ is of type 0), and $b^* \in B_{\ell_1(\Gamma_n \setminus \Gamma_k)}$.

Thus

$$P_{[1,m]}^*(e_\gamma^*) = \alpha P_{[1,m]}^*(e_\xi^*) + \beta P_{(\min(m,k),m]}^*(b^*).$$

Now, if $k \geq m$, then $P_{[1,m]}^*(e_\gamma^*) = \alpha P_{[1,m]}^*(e_\xi^*)$ and thus our claim (2.7) follows from the induction hypothesis:

$$\|\alpha P_{[1,m]}^*(e_\xi^*)\| \leq 1 + C_k \leq 1 + C_{n+1}.$$

If $k < m$ it follows, again using the induction hypothesis in the type 0 case, that

$$\|P_{[1,m]}^*(e_\gamma^*)\| \leq \alpha\|e_\xi^*\| + \beta\|P_{(k,m]}^*(b^*)\| \leq 1 + C_{n+1}, \text{ which yields (2.7).}$$

In order to show (2.8), let $\gamma = (n+1, \alpha, k, \xi, \beta, b^*, f) \in \Delta_{n+1}^{(1)}$, with $\beta \leq \theta < 1/2$. We deduce from the induction hypothesis that

$$\begin{aligned}
&\|\beta P_{(k,m]}^*(b^*)\| \\
&\quad \leq \beta(\|P_{[1,m]}^*|_{\ell_1(\Gamma_n)}\| + \|P_{[1,k]}^*|_{\ell_1(\Gamma_n)}\|) \\
&\quad \leq 2\theta(C_n + 1)
\end{aligned}$$

$$\leq \begin{cases} 2\theta\,(C_n(\theta)+1)) \leq 2\theta C_n(\theta) + C_n(\theta)(1-2\theta) = C_n(\theta) & \text{if } C_n(\theta) > \frac{2\theta}{1-2\theta}, \\ 2\theta\left(\frac{2\theta}{1-2\theta}+1\right) = \frac{2\theta}{1-2\theta} & \text{otherwise,} \end{cases}$$

$$\leq \max\left(\frac{2\theta}{1-2\theta}, C_n(\theta)\right).$$

This finishes the induction step, and hence the proof. □

Remark 2.5 Let Γ be linearly ordered as $(\gamma_j : j \in \mathbb{N})$ in such a way that $\mathrm{rk}(\gamma_i) \leq \mathrm{rk}(\gamma_j)$, if $i \leq j$. Then the same arguments show that, under the assumption $C < \infty$ stated in Proposition 2.4, $(d^*_{\gamma_j})$ is actually a Schauder basis of ℓ_1 [4]. But, for our purpose, the FDD is the more useful coordinate system.

The spaces constructed in [4] satisfy the condition that for some $\theta < 1/2$ we have $\beta \leq \theta$, for all $\gamma = (n, \alpha, k, a^*, \beta, b^*, f) \in \Gamma$ of type 1. Thus in that case $C_n(\theta) = 0$, $n \in \mathbb{N}$, and the conclusion of Proposition 2.4 is true for $C \leq 2\theta/(1-2\theta)$ and, thus $M \leq 1/(1-2\theta)$.

The Bourgain–Delbaen sets we will consider in later sections will satisfy the following condition for some $0 < \theta < 1/2$:

$$\begin{aligned} &\text{For each } n \in \mathbb{N} \quad \text{and} \quad \gamma = (n, \alpha, k, \xi, \beta, b^*, f) \in \Delta_n^{(1)}, \\ &\text{either } \beta \leq \theta, \text{ or } b^* = e^*_\eta \text{ for some } \eta \in \Delta_m, k < m < n, \text{ such that } c^*_\eta = 0. \end{aligned} \tag{2.9}$$

Note that in the second case it follows that $e^*_\eta = d^*_\eta$ and so $P^*_{(k,m]}(e^*_\eta) = e^*_\eta$. Thus, $\beta\|P^*_{(k,m]}(b^*)\| = \beta\|e^*_\eta\| \leq 1$, and thus, we deduce that the assumptions of Proposition 2.4 are satisfied, namely that $\mathbf{F}^*$ is an FDD of ℓ_1 whose decomposition constant M is not larger than $\max(1/(1-2\theta), 2)$.

Assume we are given a sequence of Bourgain–Delbaen sets $(\Delta_n : n \in \mathbb{N})$, which satisfy the assumptions of Proposition 2.4 with $C < \infty$ and let M be the decomposition constant of the FDD (F^*_n) in $\ell_1(\Gamma)$. We now define the *Bourgain–Delbaen space associated to* $(\Delta_n : n \in \mathbb{N})$. For a finite or cofinite set $A \subset \mathbb{N}$, we let P^*_A be the projection of $\ell_1(\Gamma)$ onto the subspace $\oplus_{j\in A} F^*_j$ given by

$$P^*_A : \ell_1(\Gamma) \to \ell_1(\Gamma), \quad \sum_{\gamma\in\Gamma} a_\gamma d^*_\gamma \mapsto \sum_{\gamma\in A} a_\gamma d^*_\gamma.$$

If $A = \{m\}$, for some $m \in \mathbb{N}$, we write P^*_m instead of $P^*_{\{m\}}$. For $m \in \mathbb{N}$, we denote by R_m the restriction operator from $\ell_1(\Gamma)$ onto $\ell_1(\Gamma_m)$ (in terms of the basis (e^*_γ)) as well the usual restriction operator from $\ell_\infty(\Gamma)$ onto $\ell_\infty(\Gamma_m)$. Since $R_m \circ P^*_{[1,m]}$ is a projection from $\ell_1(\Gamma)$ onto $\ell_1(\Gamma_m)$, for $m \in \mathbb{N}$, it follows that the map

$$J_m : \ell_\infty(\Gamma_m) \to \ell_\infty(\Gamma), \quad x \mapsto P^{**}_{[1,m]} \circ R^*_m(x),$$

is an isomorphic embedding ($P^{**}_{[1,m]}$ is the adjoint of $P^*_{[1,m]}$ and, thus, defined on $\ell_\infty(\Gamma)$). Since R^*_m is the natural embedding of $\ell_\infty(\Gamma_m)$ into $\ell_\infty(\Gamma)$ it follows, for all $m \in \mathbb{N}$, that

$$R_m \circ J_m(x) = x, \quad \text{for } x \in \ell_\infty(\Gamma_m), \text{ thus} J_m \text{ is an extension operator,} \tag{2.10}$$

$$J_n \circ R_n \circ J_m(x) = J_m(x), \text{ whenever } m \le n \text{ and } x \in \ell_\infty(\Gamma_m), \tag{2.11}$$

and by Proposition 2.4,

$$\|J_m\| \le M. \tag{2.12}$$

Hence the spaces $Y_m = J_m(\ell_\infty(\Gamma_m))$, $m \in \mathbb{N}$, are finite-dimensional nested subspaces of $\ell_\infty(\Gamma)$ which (via J_m) are M-isomorphic images of $\ell_\infty(\Gamma_m)$. Therefore $Y = \overline{\bigcup_{m\in\mathbb{N}} Y_n}^{\ell_\infty}$ is a $\mathcal{L}_{\infty,M}$ space. We call Y the *Bourgain–Delbaen space associated to* (Δ_n). It follows from the definition of Y, and from 2.10, that for any $x \in \ell_\infty(\Gamma)$ we have

$$x \in Y \iff x = \lim_{m\to\infty} \|x - J_m \circ R_m(x)\| = 0. \tag{2.13}$$

Define for $m \in \mathbb{N}$

$$P_{[1,m]} : Y \to Y, \quad x \mapsto J_m \circ R_m(x).$$

We claim that $P_{[1,m]}$ coincides with the restriction of the adjoint $P^{**}_{[1,m]}$ of $P^*_{[1,m]}$ to the space Y. Indeed, if $n \in \mathbb{N}$, with $n \ge m$, and $x = J_n(\tilde{x}) \in Y_n$, and $b^* \in \ell_1(\Gamma)$ we have that

$$\begin{aligned}
&\langle P^{**}_{[1,m]}(x), b^*\rangle \\
&\quad = \langle x, P^*_{[1,m]}(b^*)\rangle \\
&\quad = \langle R_m(x), R_m \circ P^*_{[1,m]}(b^*)\rangle \text{ (since } P^*_{[1,m]}(b^*) \in \operatorname{span}(e^*_\gamma : \gamma \in \Gamma_m)) \\
&\quad = \langle P^{**}_{[1,m]} \circ R^*_m \circ R_m(x), b^*\rangle = \langle P_{[1,m]}(x), b^*\rangle.
\end{aligned}$$

Thus our claim follows since $\bigcup_n Y_n$ is dense in Y.

We therefore deduce that Y has an FDD (F_m), with $F_m = (P_{[1,m]} - P_{[1,m-1]})(Y)$, and as we observed in (2.12), $Y_m = \oplus_{j=1}^n F_j$ is, via J_m, M-isomorphic to $\ell_\infty(\Gamma_m)$ for $m \in \mathbb{N}$. Moreover, denoting by P_A the coordinate projections from Y onto $\oplus_{j\in A} F_j$, for all finite or cofinite sets $A \subset \mathbb{N}$, it follows that P_A is the adjoint of P^*_A restricted to Y, and P^*_A is the adjoint of P_A restricted to the subspace of Y^* generated by the F^*_n's.

As the next observation shows, $J_m|_{\ell_\infty(\Delta_m)}$ is actually an isometry for $m \in \mathbb{N}$.

Proposition 2.6 *For every $m \in \mathbb{N}$ the map $J_m|_{\ell_\infty(\Delta_m)}$ is an isometry between $\ell_\infty(\Delta_m)$ (which we consider naturally embedded into $\ell_\infty(\Gamma_m)$) and F_m.*

Proof Since $J_m(\ell_\infty(\Delta_m)) = (J_m - J_{m-1})(\Delta_m) = F_m$, for $m \in \mathbb{N}$, J_m is an isomorphism between $\ell_\infty(\Delta_m)$ and F_m. By 2.10, for $x \in \ell_\infty(\Delta_m)$, $\|J_m(x)\| \ge \|x\|$. In order to finish the proof we will show by induction on $n \in \mathbb{N}$ that $|e^*_\gamma(J_m(x))| \le 1$ for all $\gamma \in \Delta_n$ and $x \in \ell_\infty(\Delta_m)$, $\|x\| \le 1$.

If $n \le m$ this is clear since $R_m \circ J_m(x) = x$. Let $n > m$ and assume our claim is true for all $\gamma \in \Gamma_n$. Let $\gamma \in \Delta_{n+1}$ and write e^*_γ as $e^*_\gamma = \alpha e^*_\xi + \beta P^*_{(k,n]}(b^*) + d^*_\gamma$, with

$\alpha \in [0,1]$, $k < n$, $e^*_\xi \in \Delta_k$, and $b^* \in B_{\ell_1(\Gamma_n\setminus\Gamma_k)}$ ($\alpha = 0$, $k = 0$, and replace e^*_ξ by 0 if γ is of type 0). We have for $x \in \ell_\infty(\Delta_m)$, with $\|x\| \le 1$,

$$\begin{aligned}\langle e^*_\gamma, J_m(x)\rangle &= \langle P^*_{[1,m]}(e^*_\gamma), R^*_m(x)\rangle \\ &= \begin{cases} \beta\langle P^*_{(k,m]}(b^*), R^*_m(x)\rangle = \beta\langle P^*_{[1,m]}(b^*), R^*_m(x)\rangle = \beta\langle b^*, J_m(x)\rangle & \text{if } k<m \\ \alpha\langle e^*_\xi, R^*_m(x)\rangle = \alpha\langle P^*_{[1,m]}(e^*_\xi), R^*_m(x)\rangle = \alpha\langle e^*_\xi, J_m(x)\rangle & \text{if } k \ge m. \end{cases}\end{aligned}$$

Where the first equality in the first case holds since $\langle P^*_{[1,k]}(b^*), R^*_m(x)\rangle = 0$. Using our induction hypothesis, this implies our claim. □

Denote by $\|\cdot\|_*$ the dual norm of Y^*.

Proposition 2.7 *For all* $y^* \in \ell_1(\Gamma)$

$$\|y^*\|_* \le \|y^*\|_{\ell_1} \le M\|y^*\|_*. \tag{2.14}$$

and if $y^* \in \oplus_{j=m+1}^n F^*_j$, *with* $0 < m < n$, *then there is a family* $(a_\gamma)_{\gamma\in\Gamma_n\setminus\Gamma_m}$ *so that*

$$y^* = P^*_{(m,n]}\left(\sum_{\gamma\in\Gamma_n\setminus\Gamma_m} a_\gamma e^*_\gamma\right) \quad \textit{and} \quad \left\|\sum_{\gamma\in\Gamma_n\setminus\Gamma_m} a_\gamma e^*_\gamma\right\|_{\ell_1} \le M\|y^*\|_*. \tag{2.15}$$

Proof The first inequality in (2.14) is trivial. To show the second inequality we let $y^* \in \ell_1(\Gamma_n)$ for some $n \in \mathbb{N}$ and choose $x \in S_{\ell_\infty(\Gamma_n)}$ so that $\langle y^*, x\rangle = \|y^*\|_{\ell_1}$. Then, from (2.12) and (2.10),

$$\|y^*\|_* \ge \left\langle y^*, \frac{1}{M} J_n(x)\right\rangle = \frac{1}{M}\|y^*\|_{\ell_1}.$$

If $y^* \in \oplus_{j=m+1}^n F^*_j$, we can write y^* as

$$y^* = \sum_{\gamma\in\Gamma_n} \alpha_\gamma e^*_\gamma.$$

Since $P^*_{(m,n]}(e^*_\gamma) = 0$, for $\gamma \in \Gamma_m$, we obtain

$$y^* = P^*_{(m,n]}(y^*) = P^*_{(m,n]}\left(\sum_{\gamma\in\Gamma_n\setminus\Gamma_m} a_\gamma e^*_\gamma\right).$$

Moreover we obtain, from (2.14), that

$$\left\|\sum_{\gamma\in\Gamma_n\setminus\Gamma_m} a_\gamma e^*_\gamma\right\|_{\ell_1} \le \left\|\sum_{\gamma\in\Gamma_n} a_\gamma e^*_\gamma\right\|_{\ell_1} = \|y^*\|_{\ell_1} \le M\|y^*\|_*,$$

which yields (2.15). □

We now recall some more notation introduced in [4]. Assume that we are given a Bourgain–Delbaen sequence (Δ_n) and associated Bourgain–Delbaen family of functionals $(c^*_\gamma : \gamma \in \Gamma)$, corresponding to the Bourgain–Delbaen space Y, which admits a decomposition constant $M < \infty$. As above we denote its FDD by (F_n). For $n \in \mathbb{N}$ and $\gamma \in \Delta_n$, we have

$$e^*_\gamma = d^*_\gamma + c^*_\gamma = d^*_\gamma + \begin{cases} \beta b^* & \text{if } \gamma = (n, \beta, b^*, f) \in \Delta_n^{(0)}, \\ \alpha e^*_\xi + \beta P^*_{(k,n]}(b^*) & \text{if } \gamma = (n, \alpha, k, \xi, \beta, b^*, f) \in \Delta_n^{(1)}. \end{cases}$$

By iterating we eventually arrive (after finitely many steps) to a functional of type 0. By an easy induction argument we therefore obtain

Proposition 2.8 *For all $n \in \mathbb{N}$ and $\gamma \in \Delta_n$, there are $a \in \mathbb{N}$, $\beta_1, \beta_2, \ldots \beta_a \in [0, 1]$, $\alpha_1, \alpha_2, \ldots \alpha_a \in [0, 1]$ and numbers $0 = p_0 < p_1 < p_2 - 1 < p_2 < p_3 < p_3 - 1, \ldots < p_{a-1} < p_a - 1 < p_a = n$ in $\mathbb{N}_0$, vectors b^*_j, $j = 1, 2 \ldots a$, with $b^*_j \in B_{\ell_1(\Gamma_{p_j - 1} \setminus \Gamma_{p_{j-1}})}$, and $(\xi_j)_{j=1}^a \subset \Gamma_n$, with $\xi_j \in \Delta_{p_j}$, for $j = 1, 2 \ldots a$, and $\xi_a = \gamma$, so that*

$$e^*_\gamma = \sum_{j=1}^{a} \alpha_j d^*_{\xi_j} + \beta_j P^*_{(p_{j-1}, p_j)}(b^*_j). \tag{2.16}$$

Moreover for $1 \leq j_0 < a$

$$e^*_\gamma = \alpha_{j_0} e^*_{\gamma_{j_0}} + \sum_{j=j_0+1}^{a} \alpha_j d^*_{\xi_j} + \beta_j P^*_{(p_{j-1}, p_j)}(b^*_j). \tag{2.17}$$

We call the representations in (2.16) and (2.17) *the analysis of γ* and *partial analysis of γ*, respectively and let $\text{cuts}(\gamma) = \{p_1, p_2, \ldots p_a\}$, which we call the *set of cuts of γ*.

3 Embedding background and other preliminaries

Our constructions will depend heavily on some known embedding theorems. We review these in this section and add a bit more to facilitate their use. Zippin [30] proved that if X^* is separable, then X embeds into a space with a shrinking basis. So, in proving Theorem A, we could begin with such a space. However, to make our construction work, we need a quantified version of this theorem which appears in [12]. For Theorem C, we need a quantified reflexive version [26]. We begin with some notation and terminology.

Let $\mathbf{E} = (E_i)_{i=1}^\infty$ be an FDD for a Banach space Z. $c_{00}(\oplus_{i=1}^\infty E_i)$ denotes the linear span of the E_i's and if $B \subseteq \mathbb{N}$, $c_{00}(\oplus_{i \in B} E_i)$ is the linear span of the E_i's for $i \in B$. $P_n = P_n^{\mathbf{E}} : Z \to E_n$ *is the n^{th} coordinate projection for the FDD*, i.e., $P_n(z) = z_n$ if $z = \sum_{i=1}^\infty z_i \in Z$ with $z_i \in E_i$ for all i. For a finite set or interval $A \subseteq \mathbb{N}$,

$P_A = P_A^{\mathbf{E}} \equiv \sum_{n\in A} P_n^{\mathbf{E}}$. The *projection constant* of (E_n) in Z is

$$K = K(\mathbf{E}, Z) = \sup\left\{\left\|P_{[m,n]}^{\mathbf{E}}\right\| : m \le n\right\}.$$

$\mathbf{E}$ is *bimonotone* if $K(\mathbf{E}, Z) = 1$.

The vector space $c_{00}(\oplus_{i=1}^{\infty} E_i^*)$, where E_i^* is the dual space of E_i, is naturally identified as a ω^*-dense subspace of Z^*. Note that the embedding of E_i^* into Z^* is not, in general, an isometry unless $K(\mathbf{E}, Z) = 1$. Now we will often be dealing with a bimonotone FDD (via renorming) but when not we will consider E_i^* to have the norm it inherits as a subspace of Z^*. We write $Z^{(*)} = [c_{00}(\oplus_{i=1}^{\infty} E_i^*)]$. So $Z^{(*)} = Z^*$ if $(E_i)_{i=1}^{\infty}$ is shrinking, and then $\mathbf{E}^* = (E_i^*)_{i=1}^{\infty}$ is a boundedly complete FDD for Z^*.

For $z \in c_{00}(\oplus_{i=1}^{\infty} E_i)$ the *support of* z, $\mathrm{supp}_{\mathbf{E}}(z)$, is given by $\mathrm{supp}_{\mathbf{E}}(z) = \{n : P_n^{\mathbf{E}}(z) \neq 0\}$, and the *range of* z, $\mathrm{ran}_{\mathbf{E}}(z)$ is the smallest interval $[m, n]$ in $\mathbb{N}$ containing $\mathrm{supp}_{\mathbf{E}}(z)$.

A sequence $(z_i)_{i=1}^{\ell}$, where $\ell \in \mathbb{N}$ or $\ell = \infty$, in $c_{00}(\oplus_{i=1}^{\infty} E_i)$ is called a *block sequence* of (E_i) if $\max \mathrm{supp}_{\mathbf{E}}(z_n) < \min \mathrm{supp}_{\mathbf{E}}(z_{n+1})$ for all $n < \ell$. We write $z_n < m$ to denote $\max \mathrm{supp}_{\mathbf{E}}(z_n) < m$ and $z_n > m$ is defined by $\min \mathrm{supp}_{\mathbf{E}}(z_n) > m$.

Definition 3.1 [25] Let Z be a Banach space with an FDD $\mathbf{E} = (E_i)_{i=1}^{\infty}$. Let V be a Banach space with a normalized 1-unconditional basis $(v_i)_{i=1}^{\infty}$, and let $1 \le C < \infty$. We say that $(E_n)_{n=1}^{\infty}$ *satisfies subsequential C-V-upper estimates* if whenever $(z_i)_{i=1}^{\infty}$ is a normalized block sequence of $\mathbf{E}$ with $m_i = \min \mathrm{supp}_{\mathbf{E}}(z_i)$, $i \in \mathbb{N}$, then $(z_i)_{i=1}^{\infty}$ *is C-dominated by* $(v_{m_i})_{i=1}^{\infty}$. Precisely, for all $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$,

$$\left\|\sum_{i=1}^{\infty} a_i z_i\right\| \le C \left\|\sum_{i=1}^{\infty} a_i v_{m_i}\right\|.$$

Similarly, $(E_n)_{n=1}^{\infty}$ *satisfies subsequential C-V-lower estimates* if every such $(z_i)_{i=1}^{\infty}$ C-dominates $(v_{m_i})_{i=1}^{\infty}$.

We say that $(E_n)_{n=1}^{\infty}$ *satisfies subsequential V-upper estimates* or *subsequential V-lower estimates* if there exists a $C \ge 1$ so that $(E_n)_{n=1}^{\infty}$ satisfies subsequential C-V-upper estimates or subsequential C-V-lower estimates, respectively.

These are dual properties. If $(v_i^*)_{i=1}^{\infty}$ are the biorthogonal functionals of $(v_i)_{i=1}^{\infty}$ we define subsequential V^*-upper/lower estimates to mean as above with respect to $(v_i^*)_{i=1}^{\infty}$.

Proposition 3.2 [25, Proposition 2.14] *Let Z have a bimonotone* FDD $(E_i)_{i=1}^{\infty}$ *and let V be a Banach space with a normalized 1-unconditional basis $(v_i)_{i=1}^{\infty}$ with biorthogonal functionals $(v_n^*)_{n=1}^{\infty}$. Let $1 \le C < \infty$. The following are equivalent.*

a) *$(E_i)_{i=1}^{\infty}$ satisfies subsequential C-V-upper estimates in Z.*
b) *$(E_i^*)_{i=1}^{\infty}$ satisfies subsequential C-V^*-lower estimates in $Z^{(*)}$.*

Moreover, the equivalence holds if we interchange "upper" with "lower" in a) *and* b). *If the* FDD $(E_i)_{i=1}^{\infty}$ *is not bimonotone the proposition still holds but not with the same constants C. These changes depend upon $K(\mathbf{E}, Z)$.*

Recall that $A \subseteq B_{Z^*}$ is *d-norming for Z* $(0 < d \leq 1)$ if for all $z \in Z$,

$$d\|z\| \leq \sup\{|z^*(z)| : z^* \in A\}.$$

We will need a characterization of subsequential V-upper estimates obtained from norming sets.

Proposition 3.3 *Let Z have an* FDD $\mathbf{E} = (E_i)_{i=1}^\infty$ *and let V be a Banach space with a normalized 1-unconditional basis* $(v_i)_{i=1}^\infty$. *Let* $0 < d \leq 1$ *and let* $A \subseteq B_{Z^*}$ *be d-norming for Z. The following are equivalent.*

a) $(E_i)_{i=1}^\infty$ *satisfies subsequential V-upper estimates.*
b) *There exists* $C < \infty$ *so that for all* $z^* \in A$ *and any choice of k and* $1 \leq n_1 < \cdots < n_{k+1}$ *in* $\mathbb{N}$,

$$\left\| \sum_{i=1}^k \|z^* \circ P^{\mathbf{E}}_{[n_i, n_{i+1})}\| v^*_{n_i} \right\| \leq C.$$

Moreover, if $(E_i)_{i=1}^\infty$ *is bimonotone, then* a′) ⇒ b′) ⇒ b″) ⇒ a″) *where*

a′) $(E_i)_{i=1}^\infty$ *satisfies subsequential C-V-upper estimates.*
b′) *For every* $x^* \in S_{Z^*}$ *and any choice of k and* $1 \leq n_1 < n_2 < \cdots < n_{k+1}$ *in* $\mathbb{N}$,

$$\left\| \sum_{i=1}^k \|z^* \circ P^{\mathbf{E}}_{[n_i, n_{i+1})}\| v^*_{n_i} \right\| \leq C.$$

b″) *For every* $z^* \in A$ *and any choice of k and* $1 \leq n_1 < \cdots < n_{k+1}$ *in* $\mathbb{N}$,

$$\left\| \sum_{i=1}^k \|z^* \circ P^{\mathbf{E}}_{[n_i, n_{i+1})}\| v^*_{n_i} \right\| \leq C.$$

a″) $(E_i)_{i=1}^\infty$ *satisfies subsequential* Cd^{-1}*-V-upper estimates.*

Proof By renorming, we can assume that $(E_i)_{i=1}^\infty$ is bimonotone and thus we need only prove the "moreover" statement.

a′) ⇒ b′) follows from Proposition 3.2. Indeed, $(z^* \circ P^{\mathbf{E}}_{[n_i, n_{i+1})})_{i=1}^k$ is a block sequence of (E_i^*), whose sum has norm at most 1, and $\min \operatorname{supp}_{\mathbf{E}^*}(z^* \circ P^{\mathbf{E}}_{[n_i, n_{i+1})})$ can be assumed equal to n_i by standard perturbation arguments.

b′) ⇒ b″) is trivial.

b″) ⇒ a″). Let $(z_i)_{i=1}^n$ be a normalized block sequence of (E_i) with $m_i = \min \operatorname{supp}_{\mathbf{E}}(z_i)$ for $i \leq n$. Let $m_{n+1} = \max \operatorname{supp}_{\mathbf{E}}(z_n) + 1$. Let $(a_i)_1^n \subseteq \mathbb{R}$ and choose $z^* \in A$ with

$$\left| z^* \left(\sum_{i=1}^n a_i z_i \right) \right| \geq d \left\| \sum_{i=1}^n a_i z_i \right\|.$$

Thus,

$$\begin{aligned}\left\|\sum_{i=1}^n a_i z_i\right\| &\le d^{-1}\left|\sum_{i=1}^n a_i z^*(z_i)\right| \\ &= d^{-1}\left|\sum_{i=1}^n a_i z^* \circ P^{\mathbf{E}}_{[m_i,m_{i+1})}(z_i)\right| \\ &\le d^{-1}\sum_{i=1}^n |a_i|\,\left\|z^* \circ P^{\mathbf{E}}_{[m_i,m_{i+1})}\right\| \\ &= d^{-1}\left(\sum_{i=1}^n \left\|z^* \circ P^{\mathbf{E}}_{[m_i,m_{i+1})]}\right\| v^*_{m_i}\right)\left(\sum_{i=1}^n |a_i| v_{m_i}\right) \\ &\le C\, d^{-1}\left\|\sum_{i=1}^n a_i v_{m_i}\right\|, \text{ by b'')}.\end{aligned}$$

□

We recall some terminology concerning finite subsets of $\mathbb{N}$ which can be found for example in [27].

Definition 3.4 $[\mathbb{N}]^{<\omega}$ denotes the set of all finite subsets of $\mathbb{N}$ under the *pointwise topology*, i.e., the topology it inherits as a subset of $\{0, 1\}^{\mathbb{N}}$ with the product topology. Let $\mathcal{A} \subseteq [\mathbb{N}]^{<\omega}$. We say $\mathcal{A}$ is

i) *compact* if it is compact in the pointwise topology,
ii) *hereditary* if for all $A \in \mathcal{A}$, if $B \subseteq A$ then $B \in \mathcal{A}$,
iii) *spreading* if for all $A = (a_1, \dots, a_n) \in \mathcal{A}$ with $a_1 < a_2 < \cdots < a_n$ and all $B = (b_1, \dots, b_n) \in [\mathbb{N}]^{<\omega}$ with $b_1 < b_2 < \cdots < b_n$ and $a_i \le b_i$ for $i \le n$, $B \in \mathcal{A}$, such a B is called a *spread* of A,
iv) *regular* if $\{n\} \in \mathcal{A}$ for all $n \in \mathbb{N}$ and $\mathcal{A}$ is compact, hereditary and spreading.

We note that if $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$ is relatively compact, or equivalently if $\mathcal{A}$ does not contain an infinite strictly increasing chain, then there is a regular family, $\mathcal{B} \subset [\mathbb{N}]^{<\omega}$, containing $\mathcal{A}$.

Definition 3.5 Let $\mathcal{A} \subseteq [\mathbb{N}]^{<\omega}$ be a regular family. A sequence of sets in $[\mathbb{N}]^{<\omega}$, $A_1 < A_2 < \cdots < A_n$ (i.e., $\max A_i < \min A_{i+1}$ for $i < n$) is called $\mathcal{A}$-*admissible* if $(\min A_i)_{i=1}^n \in \mathcal{A}$.

Tsirelson spaces 3.6 Let $\mathcal{A} \subseteq [\mathbb{N}]^{<\omega}$ be a regular family of sets and let $0 < c < 1$. The Tsirelson space $T_{\mathcal{A},c}$ is the completion of c_{00} under the norm $\|\cdot\|_{\mathcal{A},c}$ which is given, implicitly, by the equation

$$\|x\|_{\mathcal{A},c} = \|x\|_\infty \vee \sup\left\{\sum_{i=1}^n c\|A_i x\|_{\mathcal{A},c} : n \in \mathbb{N},\ \text{and}\ A_1 < \cdots < A_n \text{ is } \mathcal{A}\text{-admissible}\right\}.$$

Here $A_i x = x|_{A_i}$. The unit vector basis (t_i) of c_{00} is always a shrinking and 1-unconditional basis for $T_{\mathcal{A},c}$. If the Cantor–Bendixson index of $\mathcal{A}$ (c.f. [27]) is at least ω then $T_{\mathcal{A},c}$ does not contain any isomorphic copy of ℓ_p or c_0, and hence $T_{\mathcal{A},c}$ must also be reflexive as every Banach space with an unconditional basis which does not contain an isomorphic copy of c_0 or ℓ_1 is reflexive.

If $\mathcal{A} = S_\alpha$ is the α^{th}-Schreier family of sets, where $\alpha < \omega_1$, we denote $T_{\mathcal{A},c}$ by $T_{c,\alpha}$. For more on these spaces (see e.g. [22,26] and the references therein). Let us recall that, for $n \in \mathbb{N}$, the spaces $T_{\alpha,c}$ and T_{α^n,c^n} are naturally isomorphic (via the identity).

Remark 3.7 We will later use the fact that if X has an FDD $(E_i)_{i=1}^\infty$ satisfying subsequential $T_{\mathcal{A},c}$-upper estimates for some regular family $\mathcal{A}$, then $(E_i)_{i=1}^\infty$ is shrinking. Indeed every normalized block sequence of $(E_i)_{i=1}^\infty$ must then be weakly null, since it is dominated by a weakly null sequence. This is equivalent to $(E_i)_{i=1}^\infty$ being shrinking.

Our embedding theorems, 3.8 and 3.9 below, refer to the Szlenk index, $S_z(X)$, [28]. If X is separable then $S_z(X)$ is an ordinal with $S_z(X) < \omega_1$ if and only if X^* is separable. Also $S_z(T_{c,\alpha}) = \omega^{\alpha\cdot\omega}$ [26, Proposition 7]. If $S_z(X) < \omega_1$ then $S_z(X) = \omega^\beta$ for some $\beta < \omega_1$. Much has been written on the Szlenk index (e.g., see [3,6,12–14,20,21,26]).

Theorem 3.8 [12, Theorem 1.3] *Let $\alpha < \omega_1$ and let X be a Banach space with separable dual. The following are equivalent.*

a) $S_z(X) \leq \omega^{\alpha\cdot\omega}$.
b) *X embeds into a Banach space Z having an* FDD *which satisfies subsequential $T_{c,\alpha}$-upper estimates, for some* $0 < c < 1$.

Theorem 3.9 [26, Theorem A] *Let $\alpha < \omega_1$ and let X be a separable reflexive Banach space. The following are equivalent.*

a) $S_z(X) \leq \omega^{\alpha\cdot\omega}$ *and* $S_z(X^*) \leq \omega^{\alpha\cdot\omega}$.
b) *X embeds into a Banach space Z having an* FDD *which satisfies both subsequential $T_{c,\alpha}$-upper estimates and subsequential $T^*_{c,\alpha}$-lower estimates, for some* $0 < c < 1$.

We note that the upper and lower estimates in both theorems are with respect to the unit vector basis (t_i) of $T_{c,\alpha}$ and its biorthogonal sequence (t_i^*), a basis for $T^*_{c,\alpha}$.

In order to use Theorem 3.8 in our proof of Theorem A, we need to reformulate what it means for an FDD for X to satisfy subsequential $T_{c,\alpha}$-upper estimates in terms of the functionals in X^*. We first need some more terminology.

Definition 3.10 Let $\mathbf{E} = (E_i)_{i=1}^\infty$ be an FDD for a space X and let $0 < c < 1$. Let $x \in c_{00}(\oplus_{i=1}^\infty E_i)$. A block sequence of $\mathbf{E}$, $(x_1, \ldots, x_\ell)$, is called a *c-decomposition of x* if

$$x = \sum_{i=1}^{\ell} x_i \text{ and, for every } i \leq \ell, \quad \begin{array}{ll} \text{either} & |\text{supp}_{\mathbf{E}}(x_i)| = 1 \\ \text{or} & \|x_i\| \leq c. \end{array} \tag{3.1}$$

Clearly every such x has a c-decomposition. The *optimal c-decomposition of x* is defined as follows. Set $n_1 = \min \operatorname{supp}_{\mathbf{E}}(x)$ and assume $n_1 < n_2 < \cdots < n_j$ have been defined. Let

$$n_{j+1} = \begin{cases} n_j + 1, & \text{if } \|P^{\mathbf{E}}_{n_j}(x)\| > c, \\ \min\{n : \|P^{\mathbf{E}}_{[n_j,n]}(x)\| > c\}, & \text{if } \|P^{\mathbf{E}}_{n_j}(x)\| \leq c \text{ and the "min" exists,} \\ 1 + \max \operatorname{supp}_{\mathbf{E}}(x), & \text{otherwise.} \end{cases}$$

There will be a smallest ℓ so that $n_{\ell+1} = 1 + \max \operatorname{supp}_{\mathbf{E}}(x)$. We then set for $i \leq \ell$, $x_i = P^{\mathbf{E}}_{[n_i,n_{i+1})}(x)$. Clearly $(x_i)_{i=1}^{\ell}$ is a c-decomposition of x. Moreover, and this will be important later, if (E_i) is bimonotone and $j \leq \lfloor \ell/2 \rfloor$, then $\|x_{2j-1} + x_{2j}\| > c$.

Let $\mathcal{A} \subseteq [\mathbb{N}]^{<\omega}$ be regular. We say that the *FDD $(E_i)_{i=1}^{\infty}$ for X is $(c, \mathcal{A})$-admissible in X* if every $x \in S_X \cap c_{00}(\oplus_{i=1}^{\infty} E_i)$ has an $\mathcal{A}$-admissible c-decomposition, $(x_i)_{i=1}^{k}$, where $(\operatorname{supp}_{\mathbf{E}}(x_i))_1^{\ell}$ is *$\mathcal{A}$-admissible*, i.e., $(\min \operatorname{supp}_{\mathbf{E}}(x_i))_{i=1}^{\ell} \in \mathcal{A}$.

Theorem 3.11 *Let $\mathbf{E} = (E_i)_{i=1}^{\infty}$ be a bimonotone* FDD *for a Banach space X. The following statements are equivalent.*

a) *(E_i) is shrinking.*
b) *For all $0 < c < 1$ there exists a regular family $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$ so that every $x^* \in B_{X^*} \cap c_{00}(\oplus_{i=1}^{\infty} E_i^*)$ has an optimal $\mathcal{A}$-admissible c-decomposition.*
c) *There exists $D \subset B_{X^*} \cap c_{00}(\oplus_{i=1}^{\infty} E_i^*)$, $0 < c < d \leq 1$ and a regular family $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$, so that D is d-norming for X, and every $x^* \in D$ admits an $\mathcal{A}$-admissible c-decomposition.*
d) *There exists $\alpha < \omega_1$, $0 < c < 1$, $1 \leq C$, and a subsequence $(t_{m_i})_{i=1}^{\infty}$ of the unit vector basis for $T_{c,\alpha}$, so that $(E_i)_{i=1}^{\infty}$ satisfies subsequential $C - (t_{m_i})_{i=1}^{\infty}$ upper estimates.*

Proof *a)* $\Rightarrow$ *b)*. Assume *b)* fails for some $0 < c < 1$. Then the set

$$\{(\min \operatorname{supp}_{E^*}(x_i^*))_{i=1}^{n} : (x_i^*)_{i=1}^{n} \text{ is the optimal } c\text{-decomposition of some } x^* \in B_{X^*} \cap c_{00}(\oplus_{i=1}^{\infty} E_i^*)\}$$

is not relatively compact in $[\mathbb{N}]^{<\omega}$. This yields a sequence $(n_i)_{i=1}^{\infty} \in [\mathbb{N}]^{\omega}$ so that for all $N \in \mathbb{N}$, there exists $x^*(N) \in B_{X^*} \cap c_{00}(\oplus_{i=1}^{\infty} E_i^*)$, with an optimal c-decomposition $(x_i^*(N))_{i=1}^{\ell(N)}$ so that $\min \operatorname{supp}_{E^*}(x_i^*(N)) = n_i$ for all $i \leq N$. After passing to a subsequence, we may assume that $\lim_{N\to\infty} x_i^*(N) = x_i^*$ for some $x_i^* \in B_{X^*} \cap c_{00}(\oplus_{i=1}^{\infty} E_i^*)$ with $\operatorname{supp}(x_i^*) \subset [n_i, n_{i+1})$ for all $i \in \mathbb{N}$. We have that $\|x_i^*(N) + x_{i+1}^*(N)\| \geq c$ for all $N \in \mathbb{N}$ and $1 \leq i < \ell(N)$, and hence $\|x_i^* + x_{i+1}^*\| \geq c$ for all $i \in \mathbb{N}$. Furthermore, $\|\sum_{i=1}^{N} x_i^*(N)\| \leq \|\sum_{i=1}^{\ell(N)} x_i^*(N)\| \leq 1$ for all $N \in \mathbb{N}$, and hence $\sup_{N\in\mathbb{N}} \|\sum_{i=1}^{N} x_i^*(N)\| \leq 1$. We conclude that (x_i^*) is not boundedly complete, and hence $(E_i)_{i=1}^{\infty}$ is not shrinking.

b) $\Rightarrow$ *c)* is trivial.

c) $\Rightarrow$ *d)*. Let D, $0 < c < d \leq 1$, and $\mathcal{A}$ be as in *c)*. We define

$$\mathcal{B} = \{n \cup B_1 \cup B_2 : n \in \mathbb{N}, B_1, B_2 \in A\} \cup \{\emptyset\}.$$

It is easily checked that $\mathcal{B} = \mathcal{B}_{\mathcal{A}}$ is regular. Let $(t_i)_{i=1}^{\infty}$ be the unit vector basis of $T_{c/d,\mathcal{B}}$. We will prove, by induction on $s \in \mathbb{N}$, that if $(x_i)_{i=1}^{k}$ is a normalized block sequence of $\mathbf{E}$ with finite length and $|\mathrm{supp}_{\mathbf{E}}(\sum_{i=1}^{k} x_i)| \le s$, then for all $(a_i)_1^k \subseteq \mathbb{R}$,

$$\left\| \sum_{i=1}^{k} a_i x_i \right\| \le c^{-1} \left\| \sum_{i=1}^{k} a_i t_{\min \mathrm{supp}_{\mathbf{E}}(x_i)} \right\|_{T_{c/d,\mathcal{B}}}. \tag{3.2}$$

This is trivial for $s = 1$ and also clear for $k = 1$, so we may assume $k > 1$. Assume it holds for all $s' \le s$. Let $(x_i)_{i=1}^{k}$ be a normalized block sequence of $\mathbf{E}$ with $|\mathrm{supp}_{\mathbf{E}}(\sum_{i=1}^{k} x_i)| = s + 1$. Let $m_i = \min \mathrm{supp}_{\mathbf{E}}(x_i)$ for $i \le k$ and set $m_{k+1} = 1 + \max \mathrm{supp}_{\mathbf{E}}(x_k)$. Let $(a_i)_{i=1}^{k} \subseteq \mathbb{R}$ and $c/d < \rho < 1$ be arbitrary. Since D is d-norming for X, there exists $x^* \in D$ with

$$\left| x^* \left(\sum_{i=1}^{k} a_i x_i \right) \right| \ge \rho d \left\| \sum_{i=1}^{k} a_i x_i \right\|.$$

Let $\tilde{x}^* = P^{\mathbf{E}^*}_{[m_1,m_{k+1})}(x^*)$ where $\mathbf{E}^* = (E_j^*)_{j=1}^{\infty}$ is the FDD for $X^{(*)}$. By the bimonotonicity of $\mathbf{E}$, $\|\tilde{x}^*\| \le 1$ and also $\|\tilde{x}^*(\sum_{i=1}^{k} a_i x_i)\| \ge \rho d \| \sum_{i=1}^{k} a_k x_i \|$. Furthermore, since x^* admits an $\mathcal{A}$-admissible c-decomposition, so does $\tilde{x}^*$. Let $(x_i^*)_{i=1}^{\ell}$ be an $\mathcal{A}$-admissible c-decomposition of $\tilde{x}^*$ and let $n_i = \min \mathrm{supp}_{\mathbf{E}^*}(x_i^*)$ for $i \le \ell$. Thus $(n_i)_{i=1}^{\ell} \in \mathcal{A}$.

If $\ell = 1$, then $\tilde{x}^* \in E_j^*$ for some j and so

$$\begin{aligned} \left\| \sum_{i=1}^{k} a_i x_i \right\| &\le (\rho d)^{-1} \left| \tilde{x}^* \left(\sum_{i=1}^{k} a_i x_i \right) \right| \le (\rho d)^{-1} |a_j| \\ &\le (\rho d)^{-1} \left\| \sum_{i=1}^{k} a_i t_{m_i} \right\| \le c^{-1} \left\| \sum_{i=1}^{k} a_i t_{m_i} \right\|, \text{ so (3.2) holds.} \end{aligned}$$

If $\ell > 1$, we proceed as follows. Define

$$\begin{aligned} B_1 &= \{m_i : i \le k \text{ and there exists } j \le \ell \text{ with } m_i \le n_j < m_{i+1}\}, \\ B_2 &= \{m_{i+1} : i \le k \text{ and } m_i \in B_1\}, \end{aligned}$$

and let $n = \min(B_1)$. Then $B \equiv B_1 \cup B_2 = \{n\} \cup (B_1 \backslash \{n\}) \cup B_2 \in \mathcal{B}_{\mathcal{A}}$. Indeed $B_2 \in \mathcal{A}$ since it is a spread of a subset of $(n_j)_{j=1}^{\ell} \in \mathcal{A}$, by the definition of B_1. Similarly $B_1 \backslash \{n\} \in \mathcal{A}$.

Write $B = \{m_{b_j} : j \le \ell'\}$ where $b_1 < b_2 < \cdots < b_{\ell'}$. Set $m_{b_{\ell'+1}} = m_{k+1}$. Since $k > 1$, $|\mathrm{supp}_{\mathbf{E}}(\sum_{i=b_j}^{b_{j+1}-1} x_i)| \le s$, for $j \le \ell'$, and our induction hypothesis applies to such blocks. Moreover, if $b_{j+1} \ne b_j + 1$ for some $j \le \ell'$, then there is at most one x_t^* whose support is not disjoint from $\oplus_{i=m_{b_j}}^{m_{b_{j+1}-1}} E_i^*$, since no n_i can satisfy

$m_{b_j} < n_i < m_{b_{j+1}}$. In addition, $|\mathrm{supp}_{\mathbf{E}^*}(x_t^*)| > 1$ in this case, and so $\|x_t^*\| \le c$ which yields

$$\left|\tilde{x}^*\left(\sum_{i=b_j}^{b_{j+1}-1} a_i x_i\right)\right| \le c\left\|\sum_{i=b_j}^{b_{j+1}-1} a_i x_i\right\|.$$

We obtain for $I = \{j \le \ell' : b_{j+1} \ne b_j + 1\}$ and $J = \{1, \ldots, \ell'\}\backslash I$,

$$\begin{aligned}
\rho\, d\left\|\sum_{i=1}^{k} a_i x_i\right\| &\le \left|\tilde{x}^*\left(\sum_{i=1}^{k} a_i x_i\right)\right| \\
&\le \left|\sum_{j\in I}\tilde{x}^*\left(\sum_{i=b_j}^{b_{j+1}-1} a_i x_i\right)\right| + \left|\sum_{j\in J}\tilde{x}^*(a_{b_j} x_{b_j})\right| \\
&\le \sum_{j\in I} c\left\|\sum_{i=b_j}^{b_{j+1}-1} a_i x_i\right\| + \sum_{j\in J}|a_{b_j}| \\
&\le \sum_{j\in I}\left\|\sum_{i=b_j}^{b_{j+1}-1} a_i t_{m_i}\right\| + \sum_{j\in J}\|a_{b_j} t_{m_{b_j}}\|, \\
&\qquad \text{by the induction hypothesis,} \\
&= \frac{d}{c}\sum_{j=1}^{\ell'}\frac{c}{d}\left\|\sum_{i=b_j}^{b_{j+1}-1} a_i t_{m_i}\right\| \le \frac{d}{c}\left\|\sum_{i=1}^{k} a_i t_{m_i}\right\|,
\end{aligned}$$

by definition of the norm for $T_{c/d,\mathcal{B}_\mathcal{A}}$. So

$$\rho\, c\left\|\sum_{i=1}^{k} a_i x_i\right\| \le \left\|\sum_{i=1}^{k} a_i t_{m_i}\right\|.$$

Since $\rho < 1$ was arbitrary this proves (3.2). Now the set $\mathcal{B}$ is regular, so its Cantor–Bendixson index $CB(\mathcal{B})$ is less than ω_1. By Proposition 3.10 in [27], if $\alpha < \omega_1$ is such that $CB(\mathcal{B}) \le \omega^\alpha$ then there exists $(m_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that $\{(m_i)_{i\in F} : F \in \mathcal{B}\} \subset S_\alpha$. It follows, from (3.2) that (E_i) satisfies subsequential $c^{-1} - (t_{m_i})_{i=1}^\infty$ upper estimates, where $(t_i)_{i=1}^\infty$ is the unit vector basis of $T_{c/d,\alpha}$.

d) ⇒ *a*) is immediate since (t_{m_i}) is weakly null. □

Remark 3.12 In Theorem 3.11, if the FDD (E_i) for X is not bimonotone, then the Proposition holds with slight modification. Let K be the projection constant of (E_i). The hypothesis "$0 < c < d$" in *c*) should be changed to "$0 < c < d/K$". This is seen by renorming X, in the standard way, so that (E_i) is bimonotone:

$$|||x||| = \sup_{m\le n}\left\|P^{\mathbf{E}}_{[m,n]}\right\|.$$

Then D becomes d/K-norming for $(X, |||\cdot|||)$. Furthermore, (3.2) becomes valid for $(X, ||\cdot||)$ with c^{-1} replaced by Kc^{-1}.

It is worth noting that Proposition 3.11 yields, as a corollary, the following less exact version of Theorem 3.8. A similar version of Theorem 3.9 would also follow.

Corollary 3.13 *Let X be a Banach space with X^* separable. Then there exists $\alpha < \omega_1$ and $0 < c < 1$ so that X embeds into a space Y, with an FDD (F_i) satisfying subsequential $T_{c,\alpha}$-upper estimates.*

Proof By Zippin's theorem [30], we may embed X into a space Z with a shrinking FDD (E_i). By Theorem 3.11 *d*), we obtain the result, except that the estimates are with respect to (t_{m_i}). We expand the FDD by inserting the basis vectors $(t_j)_{j\in(m_{i-1},m_i)}$ between E_{i-1} and E_i to obtain the desired FDD in a subspace of $Z \oplus T_{c,\alpha}$. □

Using Proposition 2.8 we can derive from Theorem 3.11 the following sufficient and necessary condition for the dual of a Bourgain–Delbaen space to be isomorphic to ℓ_1.

Corollary 3.14 *Let Y be the Bourgain–Delbaen space associated to a Bourgain–Delbaen sequence (Δ_n) satisfying condition (2.9) for some $\theta < 1/2$ (and thus the conclusion of Proposition 2.4 with $M \le \max(1/(1-2\theta), 2)$) and let $\mathbf{F} = (F_j)$ be the FDD of Y as introduced in Sect. 2 and $\mathbf{F}^* = (F_j^*)$. Define*

$$\mathcal{C} = \left\{ \mathrm{cuts}(\gamma) : \gamma \in \bigcup_{n=1}^{\infty} \Delta_n \right\}.$$

Then $\mathbf{F}$ is shrinking (and thus Y^* is isomorphic to ℓ_1) if $\mathcal{C}$ is compact, or equivalently, if $\mathcal{C}$ does not contain an infinite strictly increasing chain.

Proof Indeed, assuming (2.9), in the analysis of $\gamma \in \Gamma$

$$e_\gamma^* = \sum_{j=1}^{a} \alpha_j d_{\xi_j}^* + \beta_j P_{(p_{j-1},p_j)}^*(b_j^*).$$

all the β_j's are at most θ, except the ones for which the support of $P^{\mathbf{F}^*}_{(p_{j-1},p_j)}(b_j^*)$ (with respect to $\mathbf{F}^*$) is at most a singleton. Therefore the analysis of γ represents a c-decomposition of e_γ^* and, thus, Theorem 3.11 yields that $\mathbf{F}$ is shrinking. □

4 The proof of Theorem A

Let X be a separable Banach space. We will follow the generalized BD construction in Sect. 2 to embed X into a $\mathcal{L}_\infty$ space Y. Since X can be embedded into a space with basis (for example $C[0,1]$), we can assume that X has an FDD, which we denote by $\mathbf{E} = (E_i)$, and after a renorming, if necessary, we can assume that $\mathbf{E}$ is bimonotone. If X^* is separable then we can assume that $\mathbf{E}$ is shrinking by [30].

The Bourgain–Delbaen space Y, which we construct to contain X, will have Y^* isomorphic to ℓ_1, in the case that X^* is separable.

To begin we fix $0 < c \le 1/16$ and choose $0 < \varepsilon < c$, and $(\varepsilon_i)_{i=1}^\infty \subset (0, \varepsilon)$ with $\varepsilon_i \downarrow 0$ so that

$$\sum_{i=1}^\infty \varepsilon_i < \frac{\varepsilon}{8} \quad \text{and} \quad \sum_{i>n} \varepsilon_i < \frac{\varepsilon_n}{2} \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

Next, for $i \in \mathbb{N}$, we choose $R_i \subset (0, 1]$ and $\tilde{A}_i^* \subseteq S_{E_i^*}$ to be $\varepsilon_i/8$, dense in their respective supersets, with $1 \in R_i$ for all $i \in \mathbb{N}$. We then choose an appropriate countable subset, $D \subset B_{X^*} \cap c_{00}(\oplus E_i^*)$, which norms X.

Lemma 4.1 *There exists a set $D \subset \left(B_{X^*} \setminus \frac{1}{2} B_{X^*}\right) \cap c_{00}(\oplus E_i^*)$ with the following properties.*

a) $A_m^* := D \cap E_m^* = \frac{1}{1+\varepsilon/4} \tilde{A}_m^*$, *for* $m \in \mathbb{N}$.
b) $D \cap (\oplus_{j=m}^n E_j^*)$ *is finite, and* $(1-\varepsilon)$*-norms the elements of* $\oplus_{j=m}^n E_j$*, for all* $m < n$ *in* $\mathbb{N}$.
c) *Every* $x^* \in D$ *can be written as* $x^* = \sum_{i=1}^\ell r_i x_i^*$*, where* $(r_1 x_1^*, \ldots, r_\ell x_\ell^*)$*, is a* c*-decomposition of* x^* *and* $x_i^* \in D$*, and* $r_i \in R_{\max \operatorname{supp}(x_i^*)}$*, for* $i = 1, \ldots \ell$*. Moreover*

$$(\operatorname{supp}(x_i^*))_{i=1}^\ell \in \left\{ (\operatorname{supp}(z_i^*))_{i=1}^\ell : \begin{array}{c} (z_i^*)_{i=1}^\ell \text{ is the optimal } \frac{c}{1+\varepsilon/4}\text{-decomposition} \\ \text{of some } z^* \in B_{X^*} \cap c_{00}\left(\oplus_{j=1}^\infty E_j^*\right) \end{array} \right\}.$$

If (E_i) is 1-uncondtional in X then (a) and (b) can be replaced by

a') $A_m^* := D \cap E_m = \tilde{A}_m^*$, *for* $m \in \mathbb{N}$.
b') $D \cap \left(\oplus_{j \in B} E_j^*\right)$ *is finite, and* $(1-\varepsilon)$*-norms the elements of* $\oplus_{j \in B} E_j$*, for all finite* $B \subset \mathbb{N}$.

For D as in Lemma 4.1 and each $x^* \in D$ we pick such a c-decomposition $(r_1 x^*, r_2 x_2^*, \ldots r_\ell x^*)$ and call it the *special c-decomposition of* x^*. If $x^* \in A_j^* = D \cap E_j^*$, we let (x^*) be its own special c-decomposition.

Proof We abbreviate $\operatorname{supp}_{\mathbf{E}^*}(\cdot)$ by $\operatorname{supp}(\cdot)$, and we abbreviate $\operatorname{ran}_{\mathbf{E}^*}(\cdot)$ by $\operatorname{ran}(\cdot)$. Define

$$H = \frac{1}{1+\varepsilon/4} \left\{ \frac{\sum_{i=m}^n a_i x_i^*}{\|\sum_{i=m}^n a_i x_i^*\|} : m \le n, a_i \in R_i \text{ and } x_i^* \in \tilde{A}_i^* \text{ for } i \in [m, n] \right\}.$$

We note the following properties of H.

$$H \text{ is countable.} \tag{4.2}$$

$$H \cap \oplus_{i=1}^{n} E_i^* \text{ is finite for all } n \in \mathbb{N}. \tag{4.3}$$

$$H \cap \oplus_{i=m}^{n} E_i^* (1-\varepsilon)\text{-norms } \oplus_{i=m}^{n} E_i, \text{ for all } m \leq n \text{ in } \mathbb{N}. \tag{4.4}$$

If $x^* \in H$ and $\operatorname{supp}(x^*) \cap [m,n] \neq \phi, m \leq n,$ then

$$\frac{P_{[m,n]}^{\mathbf{E}^*}(x^*)}{\| P_{[m,n]}^{\mathbf{E}^*}(x^*)\|} \in (1+\varepsilon/4)H. \tag{4.5}$$

Set $H_n = \{h \in H : |\operatorname{ran}(h)| = n\}$ and thus $H = \bigcup_{n=1}^{\infty} H_n$. For each $n \in \mathbb{N}$ we will inductively define for $h \in H_n$, an element $\tilde{h} \in \left(B_{X^*} \setminus \frac{1}{2} B_{X^*}\right) \cap c_{00}(\oplus_{j=1}^{\infty} E_i^*)$. We then set $D_n = \{\tilde{h} : h \in H_n\}$ and $D = \cup_{n \in \mathbb{N}} D_n$.

If $h \in H_1$, let $\tilde{h} = h$. Let $n > 1$ and assume that D_m has been defined for all $m < n$. Let $h \in H_n$ and $(z_1^*, \ldots, z_\ell^*)$ be the optimal $c/(1+\varepsilon/4)$-decomposition of h. Note that $\ell \geq 2$ since $n > 1$ and $\|h\| = 1/(1+\varepsilon/4)$. We write the decomposition as

$$(s_i h_i)_{i=1}^{\ell} = \left(\|z_i^*\|(1+\varepsilon/4) \frac{z_i^*}{(1+\varepsilon/4)\|z_i^*\|} \right)_{i=1}^{\ell}.$$

By the definition of H, $\|z_i^*\| \leq 1/(1+\varepsilon/4)$ and so $0 < s_i = \|z_i^*\|(1+\varepsilon/4) \leq 1$ for $i \leq \ell$. If $h_i \notin H_1$, then $\|s_i h_i\| = \|z_i^*\| \leq c/(1+\varepsilon/4)$ and so $s_i \leq c$.

For $i \leq \ell$, choose $r_i \in R_{\max \operatorname{supp}(h_i)}$ with $|r_i - s_i| \leq \varepsilon_{\max \operatorname{supp}(h_i)}/4$ and $r_i \leq c$ if $h \notin H_1$. We define $\tilde{h} = \sum_{i=1}^{\ell} r_i \tilde{h}_i$. By induction, we will verify the following.

$$\operatorname{supp}(\tilde{h}) = \operatorname{supp}(h) \tag{4.6}$$

$$\|\tilde{h} - h\| \leq \sum_{j \in \operatorname{supp}(\tilde{h})} \varepsilon_j \tag{4.7}$$

$$\begin{aligned} &(r_1 \tilde{h}_1, .., r_\ell \tilde{h}_\ell) \text{ is a } c\text{-decomp of } \tilde{h}, \text{ with} \\ &r_i \in R_{\max \operatorname{supp}(\tilde{h}_i)} \text{ and } \tilde{h}_i \in \bigcup_{m<n} D_m, \text{ if } n > 1. \end{aligned} \tag{4.8}$$

The condition (4.6) is clear. To verify (4.7) we note that if $h_i \in H_1$, then

$$\|r_i \tilde{h}_i - s_i h_i\| \leq |r_i - s_i| < \varepsilon_{\max \operatorname{supp}(\tilde{h}_i)}/4.$$

If $h_i \notin H_1$, by the induction hypothesis,

$$\begin{aligned} \|r_i \tilde{h}_i - s_i h_i\| &\leq \|r_i(\tilde{h}_i - h_i)\| + \|(r_i - s_i)h_i\| \\ &\leq c \sum_{j \in \operatorname{supp}(\tilde{h}_i)} \varepsilon_j + \varepsilon_{\max \operatorname{supp}(h_i)}/4 \leq \sum_{j \in \operatorname{supp}(\tilde{h}_i)} \varepsilon_j. \end{aligned}$$

Thus $\|h - \tilde{h}\| \leq \sum_{i=1}^{\ell} \|r_i \tilde{h}_i - s_i h_i\| < \sum_{j \in \operatorname{supp}(\tilde{h})} \varepsilon_j$, which proves (4.7). (4.8) holds by construction. Equation (4.7) now yields,

$$1/2 \le 1/(1+\varepsilon/4) - \sum_{j\in \mathrm{supp}(\tilde h)} \varepsilon_j \le \|h\| - \|h-\tilde h\|$$
$$\le \|\tilde h\| \le \|h\| + \|h-\tilde h\| \le 1/(1+\varepsilon/4) + \sum_{j\in \mathrm{supp}(\tilde h)} \varepsilon_j \le 1.$$

Thus $D \subset B_{X^*}\setminus \frac{1}{2}B_{X^*}$. Properties $a)$, $b)$, and $c)$ of D follow from (4.6), (4.7), and (4.8).

If (E_i) is 1-unconditional, as defined, we instead begin with

$$H = \left\{ \frac{\sum_{i\in B} a_i x_i^*}{\|\sum_{i\in B}^n a_i x_i^*\|} : \emptyset \ne B \subset \mathbb{N}, |B| < \infty, a_i \in R_i \text{ and } x_i^* \in \tilde A_i^* \text{ for } i \in B \right\}.$$

We then follow the above construction, similarly without the $(1+\varepsilon/4)$-factors. These were necessary to ensure that the $\tilde h_j$'s were in B_{X^*}. □

Next we define Γ and a certain partial order on Γ and use that to define the Δ_n's.

$$\Gamma = \left\{ (r_1x_1^*, \dots, r_jx_j^*) : \begin{array}{l} j \ge 1 \text{ and there exists } y^* \in D \text{ so that} \\ (r_1x_1^*, \dots, r_jx_j^*) \text{ are the first } j \text{ elements} \\ \text{of the special } c- \text{ decomposition of } y^* \end{array} \right\}.$$

From Theorem 3.11 and Lemma 4.1 we deduce for $\mathcal{G} = \{\{\min \mathrm{supp}(x_j^*) : j \le \ell\} : (r_1x_1^*, \dots r_\ell x_\ell^*) \in \Gamma\}$

$$(E_i) \text{ is shrinking in } X \iff \mathcal{G} \text{ is compact.} \tag{4.9}$$

We first define an order on the bounded intervals in $\mathbb{N}$ by $[n_1, n_2] < [m_1, m_2]$ if $n_2 < m_2$ or $n_2 = m_2$ and $n_1 > m_1$. It is not hard to see that this is a well ordering. It is instructive to list the first few elements in increasing order (we let $[n, n] = n$):

$$(I_n)_{n=1}^\infty = (1, 2, [1, 2], 3, [2, 3], [1, 3], 4, [3, 4], [2, 4], [1, 4], 5 \dots)$$

If $\gamma = (x_1^*, \dots, x_\ell^*) \in \Gamma$ we let

$$\mathrm{ran}_{\mathbf{E}^*}\left(\sum_{i=1}^\ell x_i^*\right) \equiv \mathrm{ran}_{\mathbf{E}^*}(\gamma) \quad \text{and} \quad \mathrm{supp}_{\mathbf{E}^*}\left(\sum_{i=1}^\ell x_i^*\right) \equiv \mathrm{supp}_{\mathbf{E}^*}(\gamma).$$

For $\gamma \in \Gamma$ we define *the rank of* γ by $\mathrm{rk}(\gamma) = n$ if ran $\mathrm{supp}_{\mathbf{E}^*}(\gamma) = I_n$. We then define a partial order "$\le$" on Γ by $\gamma < \eta$ if $\mathrm{rk}(\gamma) < \mathrm{rk}_{\mathbf{E}^*}(\eta)$. If $\mathrm{rk}(\gamma) = \mathrm{rk}(\xi)$ and $\gamma \ne \eta$ we say that γ and η are incomparable. We next define an important subsequence $(m_j)_{j=1}^\infty$ of $\mathbb{N}$. For $j \in \mathbb{N}$ let $m_j = \mathrm{rk}(x^*)$ for $x^* \in A_j^*$. Thus $m_1 = 1, m_2 = 2$, $m_3 = 4$ and more generally $m_{j+1} = m_j + j$. Note that

$$\begin{array}{l} \text{for } \gamma \in \Gamma, i_0 = \max \mathrm{supp}_{\mathbf{E}^*}(\gamma) \\ \text{if and only if } m_{i_0} \le \mathrm{rk}(\gamma) < m_{i_0+1}. \end{array} \tag{4.10}$$

The following proposition is easily verified.

Proposition 4.2 *"$\leq$" is a partial order on Γ. Furthermore,*

a) *Every natural number is the rank of some element of Γ and the set of all such elements is finite.*
b) *If $j \in \mathbb{N}$ and $(z^*) \in \{\gamma : \mathrm{rk}(\gamma) = m_j\} = \{(rx^*) \in \Gamma : r \in R_j, x^* \in A_j^*\}$, then*

$$\{\gamma \in \Gamma : \gamma < z^*\} = \{\gamma \in \Gamma : \max \mathrm{supp}_{\mathbf{E}^*}(\gamma) < j\} \text{ and}$$
$$\{\gamma \in \Gamma : \gamma > (z^*)\} = \{\gamma \in \Gamma : \max \mathrm{supp}_{\mathbf{E}^*}(\gamma) \geq j \text{ and } \mathrm{supp}_{E^*}(\gamma) \neq \{j\}\}.$$

Proof Lemma 4.1 (b) implies that for any n there must be some $\gamma \in \Gamma$ of rank n, and if we let $s < t$, so that $I_n = (s, t]$, then

$$\#\{\gamma \in \Gamma : \mathrm{rk}(\gamma) = n\} \leq \sum_{\ell=1}^{t-s} \sum_{s=t_0<t_1<\ldots t_\ell=t} \prod_{j=1}^{\ell} \#R_{t_j} \cdot \#D \cap (\oplus_{j=t_{j-1}}^{t_j} E_j^*),$$

which yields (a). (b) follows easily from the definition of our partial order. □

For $n \in \mathbb{N}$, set $\Delta_n = \{\gamma \in \Gamma : \mathrm{rk}(\gamma) = n\}$. We will next define c_γ^* for $\gamma \in \Gamma$ (thus also defining $e_\gamma^* = c_\gamma^* + d_\gamma^*$). Following this we will show how the Δ_n's can be recoded to fit into the framework of Sect. 2. To begin,

i) we let $c_\gamma^* = 0$ if $\mathrm{rk}(\gamma) \in \{m_j : j \in \mathbb{N}\}$ (thus, in particular, $c_\gamma^* = 0$ if $\gamma \in \Delta_1$).

We proceed by induction and assume that c_γ^* has been defined for all $\gamma \in \Gamma_n = \bigcup_{j=1}^n \Delta_n$. Assume that $\gamma \in \Delta_{n+1}$ with $n + 1 \notin \{m_j : j \in \mathbb{N}\}$. Let $\gamma = (r_1 x_1^*, r_2 x_2^*, \ldots, r_\ell x_\ell^*)$. There are several cases.

ii) $\ell = 1$, so $\gamma = (r_1 x_1^*)$, where $|\mathrm{supp}_{\mathbf{E}^*}(x_1^*)| > 1$. Let $(s_1 y_1^*, s_2 y_2^*, \ldots, s_m y_m^*)$ be the special c-decomposition of x_1^* and note that $m \geq 2$, since $\|x_1^*\| \geq 1/2 > c$. Put $\xi = (s_1 y_1^*, s_2 y_2^*, \ldots, s_{m-1} y_{m-1}^*)$ and let η be the special c-decomposition of y_m^*. Define $c_\gamma^* = r_1 e_\xi^* + r_1 s_m e_\eta^*$.
iii) $\ell = 2$ and $|\mathrm{supp}_{\mathbf{E}^*}(x_1^*)| = 1$. Let $\xi = (x_1^*)$ and let η be the special c-decomposition of x_2^* and set $c_\gamma^* = r_1 e_\xi^* + r_2 e_\eta^*$.
iv) $\ell > 2$ or $\ell = 2$ and $|\mathrm{supp}_{\mathbf{E}^*}(x_1^*)| > 1$. Let $\xi = (r_1 x_1^*, r_2 x_2^*, \ldots r_{\ell-1} x_{\ell-1}^*)$ and let η be the special c-decomposition of x_ℓ^*. Define $c_\gamma^* = e_\xi^* + r_\ell e_\eta^*$.

Note that in the cases (ii), (iii) and (iv) $k := \mathrm{rk}(\xi) < \mathrm{rk}(\eta) \leq n$ and, furthermore, as can be shown inductively

$$\min \mathrm{supp}_{\mathbf{F}^*}(e_\gamma^*) \geq m_{\min \mathrm{ran}_{\mathbf{E}^*}(\gamma)} \quad \text{for all } \gamma \in \Delta_n. \tag{4.11}$$

For the recoding we proceed as follows. We will identify Δ_n with new sets $\tilde{\Delta}$ conforming to Definition 2.1. Set $\tilde{\Delta}_1 = \Delta_1 = \{(rx^*) : r \in R_1, x^* \in A_1^*\}$. For $n \geq 2$ we will identify Δ_n with $\tilde{\Delta}_n = \tilde{\Delta}_j^{(0)} \cup \tilde{\Delta}_j^{(1)}$. Assume this has be done for $j \leq n$. We let $\gamma \in \Delta_{n+1}$ and define $\tilde{\gamma}$ in the four cases above.

i) If $\gamma = (rx^*)$ with $r \in R_j$ and $x^* \in A_j^*$ for some $j \in \mathbb{N}$, and thus $\mathrm{rk}(\gamma) = m_j$, we let $\tilde{\gamma} = (m_j, 0, 0, rx^*)$, i.e. we choose $\beta = 0$, $b^* = 0$ and (rx^*) to be the free variable.

In the next three cases let ξ, η and $k = \mathrm{rk}(\xi)$, ℓ,m, r_j, $j \le \ell$, and s_j, $j \le m$, be as above in (ii), (iii) and (iv), and let $\tilde{\xi}$ and $\tilde{\eta}$ be the recodings of ξ and η.

ii) If $\gamma = (r_1 x_1^*)$, with $|\mathrm{supp}_{\mathbf{E}^*}| > 1$, we let $\tilde{\gamma} = (n+1, 2r_1, \frac{1}{2}(e^*_{\tilde{\xi}} + s_m e^*_{\tilde{\eta}}))$.

iii) If $\gamma = (r_1 x_1^*, r_2 x_2^*)$, with $|\mathrm{supp}_{\mathbf{E}^*}(x_1^*)| = 1$, let $\tilde{\gamma} = (n+1, r_1, k, \tilde{\xi}, r_2, e^*_{\tilde{\eta}})$.

iv) If $\gamma = (r_1 x_1^*, r_2 x_2^*, \ldots, r_\ell x_\ell^*)$, with $\ell > 2$ or $|\mathrm{supp}_{\mathbf{E}^*}(x_1^*)| > 1$, let $\tilde{\gamma} = (n+1, 1, k, \tilde{\xi}, r_\ell, e^*_{\tilde{\eta}})$.

In cases (i) and (ii), $\tilde{\gamma}$ is of type 0, while in the other cases it is of type 1. In cases (ii),(iii) and (iv) the set of free variables is a singleton and we have thus suppressed it. Definition 2.2 yields that the Bourgain–Delbaen space corresponding to the $\tilde{\Delta}_n$'s is exactly the same as the one obtained from the Δ_n's above. Indeed, in (ii), (iii) and (iv) the definition of $c^*_{\tilde{\gamma}}$ involves the projections $P^{\mathbf{F}^*}_{(k,n]}$. But $P^{\mathbf{F}^*}_{(k,n]}(e^*_\eta) = e^*_\eta$ by Propositions 4.2 and 4.11. Also, from our construction, we note that (2.9) is satisfied for the $\tilde{\Delta}_n$'s since the factors r involved are all at most $2c \le 1/8$, unless the relevant $b^* = e^*_\eta$ and $c^*_\eta = 0$, for some $\eta \in \Gamma$. It follows as in Remark 2.5, that $\mathbf{F}^* = (F_j^*)$ is an FDD for ℓ_1, whose decomposition constant M does not exceed 2.

Let $\gamma = (r_1 x_1^*, \ldots, r_\ell x_\ell^*) \in \Gamma$, $\ell \ge 2$. Then by iterating case (iv) we can compute the analysis of e^*_γ. Namely $e^*_\gamma = \sum_{j=3}^{\ell} (d^*_{\gamma_j} + r_j e^*_{\eta_j}) + e^*_{\gamma_2}$, where $\gamma_j = (r_1 x_1^*, \ldots, r_\ell x_\ell^*)$, for $2 \le j \le \ell$, and η_j is the special c-decomposition of x_j^*, for $3 \le j \le \ell$. By considering the different cases where $|\mathrm{supp}_{\mathbf{E}^*}(x_1^*)|$ has one or more elements we have

$$e^*_\gamma = \begin{cases} \sum_{j=1}^{\ell} d^*_{\gamma_j} + r_j e^*_{\eta_j} & \text{if } |\mathrm{supp}_{\mathbf{E}^*}(x_1^*)| = 1 \\ \sum_{j=2}^{\ell} (d^*_{\gamma_j} + r_j e^*_{\eta_j}) + d^*_{\gamma_1} + r_1 e^*_{\xi'} + r_1 s_m e^*_{\eta'} & \text{if } |\mathrm{supp}_{\mathbf{E}^*}(x_1^*)| > 1, \end{cases} \tag{4.12}$$

where in the bottom displayed formula, using case (ii), $\xi_1' = (s_1 y_1^*, \ldots, s_{m-1} y_{m-1}^*)$, where $(s_1 y_1^*, \ldots, s_{m-1} y_{m-1}^*, s_m y_m^*)$ is the special c-decomposi-tion of x_1^*) and η' is the special c-decomposition of y_m^*.

From 4.12, Corollary 3.14 and our construction using special c-decom-positions of elements of D, it follows that (F_i) is a shrinking FDD, if (E_i) is a shrinking FDD. Indeed, then the set $\{(\min \mathrm{supp}_{\mathbf{E}^*} x_i^*)_{i=1}^{\ell} : (r_1 x_1^*, \ldots, r_\ell x_\ell^*) \in \Gamma\}$ is compact. From the analysis (4.12) we see that $\mathcal{C} = \{\mathrm{cuts}(\gamma) : \gamma \in \Gamma\}$ is also compact.

To complete the proof of Theorem A it remains only to show that X embeds into Y, the Bourgain–Delbaen space associated to (Δ_n). As in Sect. 2 we let $J_m : \ell_\infty(\Gamma_m) \to Y \subset \ell_\infty(\Gamma)$ be the extension operator, for $m \in \mathbb{N}$.

Definition 4.3 For $i \in \mathbb{N}$, define $\phi_i : E_i \to \ell_\infty(\Delta_{m_i})$ by $\phi_i(x)(rx^*) = rx^*(x)$. Define $\phi : c_{00}(\bigoplus_{i=1}^\infty E_i) \to Y = \overline{\bigcup_m Y_m} \subseteq \ell_\infty(\Gamma)$ by $\phi(x) = \sum_i J_{m_i} \circ \phi_i(P_i^{\mathbf{E}} x) \in c_{00}(\bigoplus_{i=1}^\infty F_{m_i})$.

In proving that X embeds into Y we will use the following connection between the functionals e_γ^* and the elements $\gamma \in \Gamma$ deriving from the elements of D.

$$\text{If } n \notin \{m_j : j \in \mathbb{N}\} \quad \text{and} \quad \gamma = (r_1 x_1^*, \dots, r_\ell x_\ell^*) \in \Delta_n, \text{ then } c_\gamma^* = \alpha e_\xi^* + \beta e_\eta^*, \tag{4.13}$$

$$\text{where } \xi = (s_1 y_1^*, s_2 y_2^*, \dots, s_k y_k^*) \text{ and } \eta = (t_1 z_1^*, \dots, t_m z_m^*) \text{ are in } \Delta_{n-1},$$

$$\text{such that } \sum_{i=1}^{\ell} r_i x_i^* = \alpha \sum_{i=1}^{\ell} s_i y_i^* + \beta \sum_{i=1}^{\ell} t_i z_i^*.$$

This is easily verified using (ii), (iii) and (iv). Note that, since $A_i^* \subset B_{E_i^*}$ is $(1-\varepsilon/4)$-norming E_i, $(1-\varepsilon/4)\|x\| \le \|\phi_i(x)\| \le \|x\|$ for all $x \in E_i$.

Proposition 4.4 *The map ϕ extends to an isomorphism of X into Y, and*

$$(1-\varepsilon)\|x\| \le \|\phi(x)\| \le \|x\| \quad \textit{for all } x \in X.$$

Proof Using (4.13) and the definition of ϕ_j, $j \in \mathbb{N}$, we deduce, by induction on the rank of $\gamma \in \Gamma$, that for all $\gamma = (r_1 x_1^*, \dots, r_\ell x_\ell^*) \in \Gamma$ and all $x \in c_{00}(\oplus_{j=1}^\infty E_j)$,

$$e_\gamma^*(\phi(x)) = \sum_{j=1}^{\ell} r_j x_j^*(x).$$

Using the bimonotonicity of $\mathbf{E}$ in X, and the properties of the set $D \subset B_{X^*}$ as listed in Lemma 4.1 we obtain for $x \in c_{00}(\oplus_{j=1}^\infty E_j)$

$$\begin{aligned}(1-\varepsilon)\|x\| \le \sup_{x^* \in D} |x^*(x)| &= \sup_{\gamma = (r_1 x_1^*, \dots, r_\ell x_\ell^*) \in \Gamma} \left| \sum_{i=1}^{\ell} r_j x_j^*(x) \right| \\ &= \sup_{\gamma \in \Gamma} \left| e_\gamma^*(\phi(x)) \right| \le \|x\|,\end{aligned}$$

which implies our claim. □

We will be using the construction of Y and all the terminology and notation of that construction in the next two sections. In the proof of Theorems B and C we will also be using the construction for V replacing X where V has a normalized bimonotone basis $(v_i)_{i=1}^\infty$. In this case the v_i's play the role of the E_i's, more precisely E_i is replaced by $\text{span}(v_i)$. To help distinguish things we will write BD_X and BD_V for the respective $\mathcal{L}_\infty$ spaces containing isomorphs of X and V.

Finally, it is perhaps worth noting that, in the V case we could alter the proof slightly by allowing the scalars R_i to be negative and $\varepsilon_i/8$-dense in $[-1,1]\setminus\{0\}$ and take $A_j^* = \{\frac{1}{1+\varepsilon/4} v_j^*\}$. In the case that (v_i) is also 1-unconditional we can use $A_j^* = \{v_j^*\}$ (see the second part of Lemma 4.1). We would then obtain

Corollary 4.5 *Let V be a Banach space with a normalized bimonotone shrinking basis $(v_i)_{i=1}^\infty$. Then W embeds into a $\mathcal{L}_\infty$ space Z, with a shrinking basis $(z_i)_{i=1}^\infty$ so that $(v_i)_{i=1}^\infty$ is equivalent to some subsequence of $(z_i)_{i=1}^\infty$.*

In case that V is the Tsirelson space $T_{c,\alpha}$ the construction of a Bourgain–Delbaen space containing V becomes simpler.

Remark 4.6 Let X be the Tsirelson space $T_{c,\alpha}$, where $\alpha < \omega_1$ and $c \le 1/16$. In $T^*_{c,\alpha}$ there is a natural choice for the set D satisfying the conditions of Lemma 4.1 (1-unconditional case). Indeed, we let $D = \bigcup_{n=0}^\infty D_n$, where D_n, $n \ge 0$ is defined by induction

$$D_0 = \{\pm e_j^* : j \in \mathbb{N}\} \text{ and assuming } D_0, D_1 \dots D_n \text{ have been defined we let} \tag{4.14}$$
$$D_{n+1} = \left\{ c\sum_{i=1}^k x_i^* : \begin{array}{l} k \ge 2,\ x_i^* \in \bigcup_{j=0}^n D_j,\ \text{for } i \le k, \\ \{\min \operatorname{supp}(x_i^*) : i \le k\} \in S_\alpha,\ \text{and} \\ \max \operatorname{supp}(x_i^*) < \min \operatorname{supp}(x_{i+1}^*),\ \text{if } i < k. \end{array} \right\}.$$

In that case D 1-norms $T_{c,\alpha}$ and Γ also has a simple form in this case:

$$\Gamma_{\alpha,c} = \left\{ \left(cx_1^*, cx_2^*, \dots, cx_\ell^*\right) : \begin{array}{l} \ell \ge 2,\ x_i^* \in D,\ \text{for } i \le \ell, \\ \{\min \operatorname{supp}(x_i^*) : i \le \ell\} \in S_\alpha,\ \text{and} \\ \max \operatorname{supp}(x_i^*) < \min \operatorname{supp}(x_{i+1}^*),\ \text{if } i < \ell, \end{array} \right\} \cup D_0.$$

Our construction in Theorem A leads then to a Bourgain–Delbaen space containing isometrically $T_{c,\alpha}$ and it is very similar (but simpler) than the construction in [4] where a *mixed Tsirelson space* was used instead of $T_{c,\alpha}$.

In summary, our proof of Theorem A, then yields the following theorem.

Theorem 4.7 *Let X be a Banach space with a bimonotone FDD $\mathbf{E} = (E_j)$ and let $\varepsilon > 0$. Then X embeds into a Bourgain–Delbaen space Z having an FDD $\mathbf{F} = (F_j)$, such that*

a) *For $n \in \mathbb{N}$, there are embeddings $\phi_n : E_n \to F_{m_n}$, so that*

$$\phi : c_{00}\left(\oplus_{n=1}^\infty E_n\right) \to Z, \quad \sum x_n \mapsto \sum \phi_n(x_n)$$

extends to an isomorphism from X into Z with $(1-\varepsilon)\|x\| \le \|\phi(x)\| \le \|x\|$ for $x \in X$.

b) *$\mathbf{F}$ is shrinking (in Z) if $\mathbf{E}$ is shrinking (in X).*

From Theorem 4.7 and [12, Corollary 3.5] we obtain

Corollary 4.8 *There exists a collection* $\{Y_\alpha : \alpha < \omega_1\}$ *of* $\mathcal{L}_{\infty,2}$ *spaces such that* Y_α^* *is 2-isomorphic to* ℓ_1*, and* Y_α *is universal for the class* $\mathcal{D}_\alpha = \{X : X$ *separable and* $S_z(X) \le \alpha\}$*, for all* $\alpha < \omega_1$.

5 The proof of Theorems B and C

The constructions which will be used to prove Theorems B and C are *augmentations* of sequences of Bourgain–Delbaen sets as introduced in Sect. 2.

Definition 5.1 Assume that (Δ_n) is a sequence of Bourgain–Delbaen sets, and assume that (Δ_n) satisfies the assumptions of Proposition 2.4 with $C < \infty$, and hence $M < \infty$. We denote the Bourgain–Delbaen space associated with (Δ_n) by Y and its FDD by $\mathbf{F} = (F_n)$. Since we will deal with different Bourgain–Delbaen spaces we denote from now on the projections P_A of Y onto $\oplus_{j\in A} F_j$, $A \subset \mathbb{N}$ finite or cofinite, by $P_A^{\mathbf{F}}$.

An *augmentation of* (Δ_n), is then a sequence of finite, possibly empty, sets (Θ_n) having the property that $(\overline{\Delta}_n) := (\Delta_n \cup \Theta_n)$ is again a sequence of Bourgain–Delbaen sets. More concretely, this means the following. Θ_1 is a finite set and assuming that for some $n \in \mathbb{N}$, $(\Theta_j)_{j=1}^n$ have been chosen, we let $\overline{\Delta}_j = \Delta_j \cup \Theta_j$, $\Lambda_j = \bigcup_{i=1}^j \Theta_i$, and $\overline{\Gamma}_j = \bigcup_{i=1}^j \overline{\Delta}_i$, for $j \le n$, where Θ_{n+1} is the union of two sets, $\Theta_{n+1}^{(0)}$ and $\Theta_{n+1}^{(1)}$, which satisfy the following conditions.

$\Theta_{n+1}^{(0)}$ is finite and

$$\Theta_{n+1}^{(0)} \subset \left\{ (n+1, \beta, b^*, f) : \beta \in [0,1], b^* \in B_{\ell_1(\overline{\Gamma}_n)}, \text{ and } f \in W_{(n+1,\beta,b^*)} \right\}, \tag{5.1}$$

where $W_{(n+1,\beta,b^*)}$ is a finite set for $\beta \in [0,1]$ and $b^* \in B_{\ell_1(\overline{\Gamma}_n)}$.

$\Theta_{n+1}^{(1)}$ is finite and

$$\Theta_{n+1}^{(1)} \subset \left\{ (n+1, \alpha, k, \overline{\xi}, \beta, b^*, f) : \begin{array}{l} \alpha, \beta \in [0,1], \\ k \in \{1, 2, \ldots n-1\}, \\ \overline{\xi} \in \overline{\Delta}_k, \ b^* \in B_{\ell_1(\overline{\Gamma}_n \setminus \overline{\Gamma}_k)} \\ \text{and } f \in W_{(n+1,\alpha,k,\overline{\xi},\beta,b^*)} \end{array} \right\}, \tag{5.2}$$

where $W_{(n+1,\alpha,k,\overline{\xi},\beta,b^*)}$ is a finite set for $\alpha \in [0,1]$, $k \in \{1, 2, \ldots, n-1\}$, $\overline{\xi} \in \overline{\Delta}_k$, $\beta \in [0,1]$, and $b^* \in B_{\ell_1(\overline{\Gamma}_n \setminus \overline{\Gamma}_k)}$.

We denote the corresponding functionals (see Definition 2.2) by $c_{\overline{\gamma}}^*$ for $\overline{\gamma} \in \overline{\Gamma}$. We require also that $(\overline{\Delta}_n)$ satisfies the conditions of Proposition 2.4, so that $\overline{\mathbf{F}}^* = (\overline{F}_n^*)$, with $\overline{F}_n^* = \text{span}(e_{\overline{\gamma}}^* : \overline{\gamma} \in \overline{\Delta}_n)$ is an FDD of $\ell_1(\overline{\Gamma})$ whose decomposition constant $\overline{M}$ can be estimated as in Proposition 2.4. We denote then the associated Bourgain–Delbaen space by Z, and its FDD by $\overline{\mathbf{F}} = (\overline{F}_n)$. As in Sect. 2, we denote the projections from Z onto $\oplus_{i=k}^m \overline{F}_i$, by $P_{[k,m]}^{\overline{\mathbf{F}}}$, if $k < m$, or by $P_k^{\overline{\mathbf{F}}}$, if $k = m$. The restriction operator from $\ell_\infty(\overline{\Gamma})$ onto $\ell_\infty(\overline{\Gamma}_n)$ or $\ell_1(\overline{\Gamma})$ onto $\ell_1(\overline{\Gamma}_n)$ is denoted by $\overline{R}_n$ and the extension operator from $\ell_\infty(\overline{\Gamma}_n)$ to $\oplus_{j=1}^m \overline{F}_j \subset Z \subset \ell_\infty(\overline{\Gamma})$ is denoted by $\overline{J}_m$.

Note that by Corollary 3.14, under assumption (2.9), $\overline{\mathbf{F}}$ is shrinking in Z if $\{\text{cuts}(\gamma) : \gamma \in \Gamma\}$ is compact.

Remark 5.2 In general Y is not a subspace of Z. Nevertheless it follows from Proposition 2.6 that F_m is naturally isometrically embedded into $\overline{F}_m$ for $m \in \mathbb{N}$. Indeed, the map

$$\psi_m : F_m \to \overline{F}_m, \quad x \mapsto \overline{J}_m J_m^{-1}(x) = \overline{J}_m(x|_{\Delta_m}),$$

is an isometric embedding (where we consider $\ell_\infty(\Delta_m)$ to be naturally embedded into $\ell_\infty(\overline{\Delta}_m)$ and $\ell_\infty(\overline{\Delta}_m)$ naturally embedded into $\ell_\infty(\overline{\Gamma}_m)$). We put

$$\psi : c_{00}\left(\oplus_{j=1}^\infty F_j\right) \mapsto c_{00}\left(\oplus_{j=1}^\infty \overline{F}_j\right), \quad (x_j) \mapsto (\psi_j(x_j)). \tag{5.3}$$

We define ψ on $(\oplus_{j=1}^\infty F_j)_{\ell_\infty}$ by $\psi((x_j)_{j=1}^\infty) = (\psi_j(x_j))_{j=1}^\infty \in \prod_{j=1}^\infty \overline{F}_j$, a sequence in $(\overline{F}_j)_{j=1}^\infty$. Note that if $\overline{\gamma} \in \Lambda_n$ then we can regard, for $x = (x_j) \in (\oplus F_j)_{\ell_\infty}$, $c^*_{\overline{\gamma}}(\psi(x)) = c^*_\gamma(\sum_{j=1}^n \psi_j(x_j))$. It is worth noting that for $y \in c_{00}(\oplus_{j=1}^\infty F_j)$, $\psi(y)|_\Gamma = y$. Thus ψ extends such elements to elements of Z. However this extension is not necessarily bounded on Y. In any event, if we define $\pi(z) = z|_\Gamma$ for $z \in Z$ then $\pi : Z \to Y$.

The following provides a sufficient criterium for a subspace of Y to also embed into the augmented space Z.

Proposition 5.3 *Assume that X is a subspace of the Bourgain–Delbaen space Y with FDD $\mathbf{F} = (F_j)$ and which is associated to a Bourgain–Delbaen sequence (Δ_n). Assume moreover that $c_{00}(\oplus_{j=1}^\infty F_j) \cap X$ is dense in X.*

Let (Θ_n) be an augmentation of (Δ_n) with an associated space Z, and assume that $|c^*_{\overline{\gamma}}(\psi(x))| \le c_X\|x\|$ for all $\overline{\gamma} \in \Lambda = \bigcup_{j\in\mathbb{N}} \Lambda_j$ and all $x \in X$. Then ψ embeds X into Z and $\|x\| \le \|\psi(x)\| \le \max(1, c_X)\|x\|$. Furthermore, for $x \in X$, $\pi(\psi(x)) = x$. Thus $\pi : \psi(X) \to X$ is the inverse isomorphism of $\psi|_X$.

Remark 5.4 In [17;24, Lemma 3.1] it was shown that every separable Banach space X can be embedded into a Banach space W with FDD $\mathbf{E} = (E_j)$, so that $X \cap c_{00}(\oplus_{j=1}^\infty E_j)$ is dense in X. Moreover, (E_j) can be chosen to be shrinking if X^* is separable. Using the construction of Theorem A, we can therefore embed W into a Bourgain–Delbaen space Y which has an FDD $\mathbf{F} = (F_j)$ so that E_j embeds into F_{m_j} for some increasing sequence (m_j). It follows therefore that the image of X under the embedding into Y has the property needed in Proposition 5.3.

Proof of Proposition 5.3 For $x \in X$ and $\overline{\gamma} \in \overline{\Gamma}$ we first estimate $e^*_{\overline{\gamma}}(\psi(x))$. If $\gamma \in \Gamma$ then $e^*_\gamma(\psi(x)) = e^*_\gamma(x)$, and thus it follows that $\|\psi(x)\| \ge \|x\|_{\ell_\infty(\Gamma)} = \|x\|$ for all $x \in X$ and $\pi(\psi(x)) = x$. If $\overline{\gamma} \in \Lambda$ it follows that

$$\left|e^*_{\overline{\gamma}}(\psi(x))\right| = \left|c^*_{\overline{\gamma}}(\psi(x))\right| \le c_X\|x\|$$

and therefore the restriction of ψ to X is a bounded operator, still denoted by ψ, from X to $\ell_\infty(\overline{\Gamma})$, and $\|\psi\| \le \max(c_X, 1)$.

We still need to show that the image of X under ψ is contained in Z. However $\psi(X \cap c_{00}(\oplus_{j=1}^\infty F_j)) \subset Z$ since $\psi(X \cap F_j) \subset \psi(F_j) \subset \overline{F}_j \subset Z$ for all $j \in \mathbb{N}$. Thus the image of ψ on a dense subspace of X is contained in Z, and hence $\psi(X) \subset Z$.

Theorem 5.5 *Let Y be the Bourgain–Delbaen space associated to a sequence of sets (Δ_n) and let $\mathbf{F} = (F_n)$ be the FDD of Y. Let X be a subspace of Y and assume that $c_{00}(\oplus_{j=1}^\infty F_j) \cap X$ is dense in X and let V be a space with a 1-unconditional, and normalized basis (v_n).*

Then there is an augmentation (Θ_n) of (Δ_n) with an associated space Z and with FDD $\overline{\mathbf{F}} = (\overline{F}_n)$ so that the following hold.

a) *X embeds isometrically into Z via ψ.*
b) *If $\mathbf{F}$ and (v_i) are shrinking, then $\overline{\mathbf{F}}$ is also shrinking and, thus, Z^* is isomorphic to ℓ_1. Furthermore, if (z_n) is a normalized block basis in Z, with the property that*

$$\delta_0 = \inf_{n \in N} \operatorname{dist}(z_n, \psi(X)) > 0$$

then (z_n) has a subsequence (z'_n) which dominates (v_{k_n}) where $k_n = \max \operatorname{supp}_{\overline{\mathbf{F}}}(z'_n) + 1$, for $n \in \mathbb{N}$.
c) *If X has an FDD $\mathbf{E} = (E_n)$, with the property that $E_n \subset F_n$, for $n \in \mathbb{N}$, then in this case we can choose (Θ_n) so that*

$$c^*_{\overline{\gamma}}(\psi(x)) = 0, \quad \text{whenever, } \overline{\gamma} \in \Lambda = \bigcup_{j=1}^\infty \Theta_j \quad \text{and} \quad x \in X.$$

Moreover every normalized block sequence (z_n) satisfying

$$\begin{aligned} &\max \operatorname{supp}_{\overline{\mathbf{F}}}(z_n) + n + 2 < \min \operatorname{supp}_{\overline{\mathbf{F}}}(z_{n+1}) \\ &\text{and } \delta_0 = \inf_{n \in N} \operatorname{dist}(z_n, \psi(X)) > 0, \end{aligned} \tag{5.4}$$

dominates (v_{k_n}), where $k_n = \max \operatorname{supp}_{\overline{\mathbf{F}}}(z_n) + 1$.

Remark 5.6 In case (c) we allow some E_n to be the nullspace $\{0\}$. As noted in the introduction, this will be convenient. In the case of Theorem A, we actually had $E_j \subset F_{m_j}$, but we choose to simplify the notation in the arguments below.

Proof of Theorem 5.5 The construction of (Θ_n) will differ slightly depending on whether X has an FDD or not.

We use the construction of Sect. 4 for the space V with $c \le {}^1/16$ using as an FDD for V the basis $(v_i)_{i=1}^\infty$ and $A_j^* = \{\pm v_j^*\}$ for all $j \in \mathbb{N}$. We write $D^V, \Delta_n^V, \Gamma_n^V, \ldots$ to distinguish these sets from $\Delta_n, \Gamma_n, \ldots$ which came from the construction of Y. Thus we obtain a $\mathcal{L}_\infty$ space Y^V and a $\frac{1}{1-\varepsilon}$-embedding (see Proposition 4.4) $\phi^V : V \to Y^V$.The numbers $\varepsilon < c$ and $(\varepsilon_n) \subset (0, c)$ satisfy, as in Sect. 4, the condition (4.1).

Now $D^V = D$ is as defined in the unconditional case of Lemma 4.1 for the space V. We also note that in the case that V is the Tsirelson space, $T_{c,\alpha}$ with $\alpha < \omega_1$ and $c \le 1/16$ we could use D^V and $\Gamma^V = \Gamma_{c,\alpha}$ as defined in Remark 4.6.

We define by induction for all $n \in \mathbb{N}$ the sets Θ_n and the sets $\Theta_n^{(0)}$ and $\Theta_n^{(1)}$, if $n \ge 2$, satisfying (5.1) and (5.2). Moreover, we also define a map $\Theta_n \to \Gamma^V, \overline{\gamma} \mapsto \overline{\gamma}^V$ so that

$$\begin{aligned}
&\text{cuts}(\overline{\gamma}) \text{ is a spread of}\\
&\{\min \text{supp}_{V^*}(x_1^*), \min \text{supp}_{V^*}(x_2^*), \dots, \min \text{supp}_{V^*}(x_\ell^*)\},\\
&\text{where } \overline{\gamma}^V = (x_1^*, x_2^*, \dots, x_\ell^*) \in \Gamma^V,\\
&\text{for } \overline{\gamma} \in \Theta_n, \text{ and } \max \text{supp}_{V^*}(\overline{\gamma}^V) \le n.
\end{aligned} \tag{5.5}$$

The set of free variables will be a singleton, and α will always be chosen to be 1 in (5.2), so we suppress the free variable and α, in the definition of the elements of Θ_n.

To start the recursive construction we put $\Theta_1 = \emptyset$, and assuming $\Theta_j^{(0)}$ and $\Theta_j^{(1)}$ have been chosen for all $j \le n$, we proceed as follows. Λ_j, and $\overline{\Gamma}_j$, $j \le n$, $\overline{F}_j^*$ and $P^{\overline{\mathbf{F}}^*}_{(k,j]}$, $0 \le k < j \le n$, are given as in Definition 5.1. Since Y is a subspace of $\ell_\infty(\Gamma)$, and since $\Gamma_n \subset \overline{\Gamma}_n$, $e^*_{\overline{\gamma}}$, $\overline{\gamma} \in \overline{\Gamma}_n$, is a well defined functional on Y (and thus on X). The map $\psi : X \to \prod_{j=1}^\infty \overline{F}_j$ will be defined ultimately as in (5.3). At this point for $x \in X$, $\psi(x)|_{\overline{\Gamma}_n}$ is defined and so $e^*_{\overline{\gamma}}(\psi(x)) = c^*_{\overline{\gamma}}(\psi(x))$ is defined for $\overline{\gamma} \in \overline{\Gamma}_n$. Thus we can choose for $0 \le k < n$, finite sets

$$B_{(k,n]} \subset \begin{cases} \{b^* \in B_{\ell_1(\overline{\Gamma}_n \setminus \overline{\Gamma}_k)} : P^{\overline{\mathbf{F}}*}_{(k,n]}(b^*)|_{\psi(X)} \equiv 0\}, & \text{assuming } X \text{ has an FDD}\\ B_{\ell_1(\overline{\Gamma}_n \setminus \overline{\Gamma}_k)}, & \text{no assumptions on } X \end{cases}$$

which are symmetric and $\varepsilon_{n+1}/(2M+4)$ dense in their respective supersets. Then we put

$$\begin{aligned}
\Theta^{(0)}_{n+1} &= \Theta^{(0,1)}_{n+1} \cup \Theta^{(0,2)}_{n+1} \quad \text{with}\\
\Theta^{(0,1)}_{n+1} &= \{(n+1, rc, b^*) : (rv^*_{n+1}) \in \Gamma^V \text{ and } b^* \in B_{(0,n]}\}\\
\Theta^{(0,2)}_{n+1} &= \left\{ (n+1, r, e^*_{\overline{\eta}}) : \begin{array}{l} \overline{\eta} \in \Lambda_n, \exists x^* \in D^V \text{ so that}\\ (rx^*) \in \Gamma^V \text{ with } |\text{supp}_{V^*}(x^*)| > 1 \text{ and}\\ \overline{\eta}^V \text{ is the special c-decomposition of } x^* \end{array} \right\},
\end{aligned}$$

and

$$\begin{aligned}
\Theta^{(1)}_{n+1} &= \Theta^{(1,1)}_{n+1} \cup \Theta^{(1,2)}_{n+1} \quad \text{with}\\
\Theta^{(1,1)}_{n+1} &= \left\{ \overline{\gamma} = (n+1, k, \overline{\xi}, rc, b^*) : \begin{array}{l} k < n, \overline{\xi} \in \Theta_k, b^* \in B_{(k,n]},\\ (\overline{\xi}^V, rv^*_{n+1}) \in \Gamma^V_{n+1},\\ \text{with } |c^*_{\overline{\gamma}}(\psi(x))| \le \|x\|\\ \text{for all } x \in X \end{array} \right\}
\end{aligned}$$

$$\Theta_{n+1}^{(1,2)} = \left\{ \overline{\gamma} = (n+1, k, \overline{\xi}, r, e^*_{\overline{\eta}}) : \begin{array}{l} k < n, \overline{\xi} \in \Theta_k, \eta \in \Lambda_n, \exists x^* \in D^V \\ \text{with } |\text{supp}(x^*)| > 1, \text{ so that} \\ (\overline{\xi}^V, rx^*) \in \Gamma^V_{n+1}, \text{ and } \overline{\eta}^V \text{ is the} \\ \text{special } c\text{-decomposition of } x^* \\ \text{with } |c^*_{\overline{\gamma}}(\psi(x))| \leq \|x\| \\ \text{for all } x \in X \end{array} \right\}.$$

Note that for $(n+1, r, e^*_{\overline{\eta}}) \in \Theta^{(0,2)}_{n+1}$ or $(n+1, k, \overline{\xi}, r, e^*_{\overline{\eta}}) \in \Theta^{(1,2)}_{n+1}$ we have that $r \leq c$ since $|\text{supp}(x^*)| > 1$. We define for $\overline{\gamma} \in \Lambda_n$, $n \geq 2$,

$$\overline{\gamma}^V = \begin{cases} (rv^*_{n+1}) & \text{if } \overline{\gamma} = (n+1, rc, b^*) \in \Theta^{(0,1)}_{n+1}, \\ (rx^*) & \text{if } \overline{\gamma} = (n+1, r, e^*_{\overline{\eta}}) \in \Theta^{(0,2)}_{n+1}, \\ & \text{where } \overline{\eta}^V \text{ is the special c-decomposition of } x^*, \\ (\overline{\xi}^V, rv^*_{n+1}) & \text{if } \overline{\gamma} = (n+1, k, \overline{\xi}, rc, b^*) \in \Theta^{(1,1)}_{n+1}, \\ (\overline{\xi}^V, rx^*) & \text{if } \overline{\gamma} = (n+1, k, \overline{\xi}, r, e^*_{\eta}) \in \Theta^{(1,1)}_{n+1}, \\ & \text{where } \overline{\eta}^V \text{ is the special c-decomposition of } x^*. \end{cases}$$

Then condition (5.5) follows immediately for the elements of $\Theta^{(0)}_{n+1}$, while an easy induction argument proves it also for the elements of $\Theta^{(1)}_{n+1}$. It is worth pointing out that $\{\overline{\gamma}^V : \overline{\gamma} \in \Lambda\}$ is a proper subset of Γ^V, but nevertheless is sufficiently large for our purposes.

Proposition 2.4 yields that $(\overline{\Delta}_n)$ admits an associated Bourgain–Delbaen space Z with FDD $\overline{\mathbf{F}} = (\overline{F}_j)$ whose decomposition constant $\overline{M}$ is not larger than $\max(M, 1/(1-2c)) \leq \max(M, 2)$, where M is the decomposition constant of (F_j). If (F_j) and (v_n) are both shrinking in V, and thus, the optimal c-decompositions of elements of B_{V^*} are admissible with respect to some compact subset of $[\mathbb{N}]^{<\omega}$, our condition (5.5) together with Theorem 3.11 and Corollary 3.14 yield that the FDD $\overline{\mathbf{F}} = (\overline{\mathbf{F}})$ is shrinking in Z. The definition of $\Theta^{(1)}_n$ together with Proposition 5.3 imply that ψ isomorphically embeds X into Z.

To verify parts (b) and (c) of our Theorem and will need the following

Lemma 5.7 *Let (z^*_j) be a block basis in Z^* with respect to $\overline{\mathbf{F}}^*$ and $(\delta_j) \subset [0, 1]$ with $\sum_{j \in \mathbb{N}} \delta_j \leq 1$. Assume that $|z^*_j(\psi(x))| \leq \delta_j$ for all $j \in \mathbb{N}$ and $x \in B_X$. Define for $n \in \mathbb{N}$ $p_n = \min \text{supp}_{\overline{\mathbf{F}}^*}(z^*_n) - 1$ and $q_n = \max \text{supp}_{\overline{\mathbf{F}}^*}(z^*_n) + 1$ (thus $\text{supp}_{\overline{\mathbf{F}}^*}(z^*_n) \subset (p_n, q_n)$) and assume that*

$$z^*_n = P^{\overline{\mathbf{F}}*}_{(p_n,q_n)}(\tilde{z}^*_n) \text{ for some } \tilde{z}^*_n \in B_{(q_n,p_n)}, \quad \text{and} \quad q_n + n < p_{n+1}. \tag{5.6}$$

Then for any sequence $(\beta_j)_{j=1}^N$ *with* $w^* = \sum_{j=1}^N \beta_j v^*_{q_j} \in D^V$ *there exists* $\overline{\gamma} \in \Lambda_{N+q_N}$ *so that*

$$P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(e^*_{\overline{\gamma}}) = c\beta_n z^*_n, \quad \text{for all } n \le N, \quad \text{and}$$

$$P^{\overline{\mathbf{F}}^*}(e^*_{\overline{\gamma}})(\psi(x)) = \sum_{n=1}^N c\beta_n z^*_n(\psi(x)) \quad \text{if } x \in X. \tag{5.7}$$

Proof We prove our claim by induction on $N \in \mathbb{N}$. If $N = 1$ then $w^* = \pm v^*_{q_1}$, and we let $\overline{\gamma} = (q_n, c, \pm\tilde{z}^*_1) \in \Theta^{(0,1)}_{q_1}$. Then $e^*_{\overline{\gamma}} = d^*_{\overline{\gamma}} \pm c\tilde{z}^*_1$ and $P^{\overline{\mathbf{F}}^*}_{(p_1,q_1)}(e^*_{\overline{\gamma}}) = \pm cz^*_1$, depending on whether $\beta_1 = \pm 1$. Since $d^*_{\overline{\gamma}}(\psi(x)) = 0$ for $x \in X$ we also deduce the second part of (5.7).

Assume that our claim holds true for N and let $w^* = \sum_{j=1}^{N+1} \beta_j v^*_{q_j} \in D^V$. Then, by our choice of D^V (see Lemma 4.1), w^* has a special c-decomposition $(r_1 w^*_1, \ldots, r_\ell w^*_\ell)$, and we write w^*_j as $w^*_j = \sum_{i=N_{j-1}+1}^{N_j} \beta^{(j)}_i v^*_{q_i}$ with $\beta^{(j)}_i = \beta_i / r_j$, for $j \le \ell$ and $N_{j-1} + 1 \le i \le N_j$ and $N_0 = 0 < N_1 < \ldots N_\ell = N + 1$. Since $\ell \ge 2$, we can apply the induction hypothesis to each w^*_j and obtain $\overline{\eta}_j \in \Lambda_{q_{N_j} + N_j - N_{j-1}}$, $j = 1, 2 \ldots \ell$, so that $P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(e^*_{\overline{\eta}_j}) = c\beta^{(j)}_n z^*_n$ if $N_{j-1} < n \le N_j$. Now let

$$\overline{\gamma}_1 = \begin{cases} (q_1, cr_1, \operatorname{sign}(\beta_1)\tilde{z}^*_1)\} & \text{if } |\operatorname{supp}(w^*_1)| = 1 \\ (p_{N_1+1}, r_1, e^*_{\overline{\eta}_1}) & \text{if } |\operatorname{supp}(w^*_1)| > 1. \end{cases}$$

Note that, in the second case, by assumption (5.6) $q_{N_1} + N_1 < p_{N_1+1}$ and thus $\overline{\eta}_1 \in \Lambda_{p_{N_1+1}-1}$. Assuming we have chosen $\overline{\gamma}_{j-1}$, for $2 \le j \le \ell$ we let

$$\overline{\gamma}_j = \begin{cases} (q_{N_j}, \overline{\gamma}_{j-1}, cr_j, \operatorname{sign}(\beta_{N_j})\tilde{z}^*_{N_j}) & \text{if } |\operatorname{supp}(w^*_1)| = 1 \\ (q_{N_j} + N_j - N_{j-1} + 1, \overline{\gamma}_{j-1}, \operatorname{rk}(\gamma_{j-1}), r_j, e^*_{\overline{\eta}_j}) & \text{if } |\operatorname{supp}(w^*_1)| > 1. \end{cases}$$

Using the induction hypothesis on the $\overline{\eta}_j$'s, we deduce by induction on $j = 1, \ldots \ell$ that for $x \in X$

$$e^*_{\overline{\gamma}_j}(\psi(x)) = c^*_{\overline{\gamma}_j}(\psi(x)) \le \sum_{n=1}^{N_j} |c\beta_n z^*_n(\psi(x))| \le \sum_{n=1}^{N_j} \delta_n \|x\| \le \|x\|,$$

and thus $\overline{\gamma}_1 \in \Theta^{(0,1)}_{q_1}$, if $|\operatorname{supp}(w^*_1)| = 1$, and $\overline{\gamma}_1 \in \Theta^{(0,2)}_{p_{N_1+1}}$, if $|\operatorname{supp}(w^*_1)| > 1$, and $\overline{\gamma}_j \in \Theta^{(1,1)}_{q_{N_j}}$, if $|\operatorname{supp}(w^*_1)| = 1$, and $\overline{\gamma}_j \in \Theta^{(1,2)}_{q_{N_j}+N_j-N_{j-1}+1}$, if $|\operatorname{supp}(w^*_1)| > 1$, if $j = 2, 3 \ldots \ell$

Finally we choose $\overline{\gamma} = \overline{\gamma}_\ell$ which in both cases is an element of $\Lambda_{q_{N+1}+N+1}$. It follows for $n \le N$, and $1 \le j \le \ell$ such that $N_{j-1} < n \le N_j$ that

$$
\begin{aligned}
P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(e^*_{\overline{\gamma}}) &= P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(e^*_{\overline{\gamma}_j}) \\
&= \left\{ \begin{matrix} cr_j \operatorname{sign}(\beta_j) z^*_n & \text{if } |\operatorname{supp}(w^*_j)| = 1 \\ r_j P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(e^*_{\overline{\eta}_j}) & \text{if } |\operatorname{supp}(w^*_j)| > 1 \end{matrix} \right\} = \beta_n c z^*_n,
\end{aligned}
$$

which finishes the verification of the first part of (5.7), while the second part follows from the induction hypothesis applied to the $\overline{\eta}_j$'s. □

Continuation of the Proof of Theorem 5.5 To finish the proof we consider a normalized block basis (z_n) in Z, with $\delta_0 = \inf_n \operatorname{dist}(z_n, \psi(X)) > 0$ and the additional property (5.4) in the case where X has an FDD. Let $p_n = \min \operatorname{supp}_{\overline{\mathbf{F}}}(z_n) - 1$ and $q_n = \max \operatorname{supp}_{\overline{\mathbf{F}}}(z_n) + 1$. It follows that $q_n + n < p_{n+1}$, for $n \in \mathbb{N}$. In this case (X has an FDD) we choose $z^*_n \in \oplus_{j \in (p_n,q_n)} \overline{F}^*_j$, with $\|z^*_n\| \le 1$, $z^*_n(z_n) \ge \frac{\delta_0}{2M}$ and $z^*_n|_{\psi(X)} = 0$.

In the case (b) we proceed as follows. We choose $y^*_n \in Z^*$, $\|y^*_n\| \le 1$, so that $y^*_n(z_n) \ge \delta_0$ and $y^*_n|_{\psi(X)} \equiv 0$. After passing to subsequence and using the fact that (z_k) is weakly null, we can assume that y^*_n is w^*-converging, and after subtracting its w^* limit and possibly replacing δ_0 by a smaller number we can assume that (y^*_n) is w^* null.

After passing again to subsequences, we can assume that there exist p_n's and q_n's with

$$
\left\| P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(y^*_n) - y^*_n \right\| \le \varepsilon_n
$$

and $q_n + n < p_{n+1}$ for $n \in \mathbb{N}$. Then we let $z^*_n = P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(y^*_n)/(1+\varepsilon)$, and deduce that $\|z^*_n\| \le 1$ and $z^*_n(z_n) \ge \delta_0/(1+\varepsilon)) =: \delta'_0$.

In both cases we found $z^*_n \in \oplus_{p_n+1}^{q_n-1} F^*_j$, with $\|z^*_n\| \le 1$, $z^*_n(z_n) \ge \delta'_0$ and $z^*_n|_{\psi(X)} = 0$ in the first case and $\|z^*_n|_{\psi(X)}\| \le \varepsilon_n$ in the second.

By Proposition 2.7 we find $b^*_n \in \ell_1(\Gamma_{q_n-1} \setminus \Gamma_{p_n})$, for $n \in \mathbb{N}$ so that $\|b^*_n\|_{\ell_1} \le \overline{M}$ and $z^*_n = P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(b^*_n)$.

Using now the density assumption of $B_{(p_n,q_n)}$ we can choose $\tilde{b}^*_n \in B_{(p,q_n)}$ with $\|\tilde{b}^*_n - \frac{1}{\overline{M}} b^*_n\| \le \varepsilon_{q_n}/(2M+4) \le \varepsilon_{q_n}/2\overline{M}$, since $\overline{M} \le M \vee 2$. So if we let $\tilde{z}^*_n = P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(\tilde{b}^*_n)$, we deduce that $\|z^*_n/\overline{M} - \tilde{z}^*_n\| \le 2\overline{M}\varepsilon_{q_n}/2\overline{M} = \varepsilon_{q_n}$ and hence $\tilde{z}^*_n(z_n) \ge z^*_n(z_n)/\overline{M} - \|z^*_n/\overline{M} - \tilde{z}^*_n\| \ge \delta'_0/\overline{M} - \varepsilon_n$, for all $n \in \mathbb{N}$.

Let $n_0 \in \mathbb{N}$ be such that $\delta'_0 \ge 2\varepsilon_{n_0}\overline{M}$. It is enough to show that $(z_n)_{n \ge n_0}$ has lower $(v_{q_n})_{n \ge n_0}$ estimates. We can therefore assume without loss of generality that $n_0 = 1$. Let $(\alpha_j)_{j=1}^N \subset \mathbb{R}$ with $\| \sum_{j=1}^N \alpha_j v_{q_j} \| = 1$ and using Lemma 4.1 (in the unconditional case) we can choose $(\beta_j)_{j=1}^N \subset \mathbb{R}$ with $\sum_{j=1}^N \beta_j v^*_{q_j} \in D^V$ so that

$$
\sum_{j=1}^N \beta_j v^*_{q_j} \left(\sum_{j=1}^N \alpha_j v_{q_j} \right) = \sum_{j=1}^N \alpha_j \beta_j \ge (1-\varepsilon).
$$

Since (p_n) and (q_n) satisfy the assumptions of Lemma 5.7, we can choose $\overline{\gamma} \in \Lambda$ so that

$$e^*_\gamma\left(\sum_{j=1}^N \alpha_j z_j\right) = \sum_{j=1}^N \alpha_j\beta_j P^{\overline{\mathbf{F}}^*}_{(p_j,q_j)}(e^*_\gamma)(z_j)$$
$$= c\sum_{j=1}^N \alpha_j\beta_j z^*_j(z_j) \geq c(1-\varepsilon)\delta'_0/2\overline{M},$$

which finishes the proof of (b) and (c) and thus Theorem 5.5 in full.

We now prove Theorem B.

Proof of Theorem B Let X and U be totally incomparable spaces with separable duals.

By Theorem 3.8 U embeds into a space W with an FDD which satisfies subsequential $T_{c,\alpha}$-upper estimates for some $\alpha < \omega_1$ and some $0 < c < 1$. As noted before we can assume that, after possibly replacing α by one of its powers, we can assume that $c \leq {}^1/16$. We also noted that Proposition 7 in [26] calculates the Szlenk index of $T_{\alpha,c}$ to be $Sz(T_{\alpha,c}) = \omega^{\alpha\omega}$. We may thus choose $\beta > \alpha$ so that $Sz(T_{\beta,c}) > Sz(T_{\alpha,c})$. Furthermore, any infinite dimensional subspace of $T_{\alpha,c}$ has the same Szlenk index as $T_{\alpha,c}$. We immediately have that $T_{\alpha,c}$ and $T_{\beta,c}$ are totally incomparable, that is no infinite dimensional subspace of $T_{\alpha,c}$ is isomorphic to a subspace of $T_{\beta,c}$. This idea can be refined further to give that no normalized block sequence in $T_{\alpha,c}$ dominates a normalized block sequence in $T_{\beta,c}$.

Using Theorem A and Remark 5.4 we can embed X into a Bourgain–Delbaen space Y with shrinking FDD $\mathbf{F} = (F_j)$ so that $X \cap c_{00}(\oplus_{j=1}^\infty F_j)$ is dense in X. We apply now Theorem 5.5 to Y, with (v_j) being the unit vector basis of $T_{c,\beta}$, to obtain a Bourgain–Delbaen space Z, and an embedding ψ of X into Z, so that every normalized block sequence, which has a positive distance to $\psi(X)$, has a subsequence (z_i) which dominates some subsequence of (v_j). If (z_i) is equivalent to a basic sequence in U, then (z_i) is dominated by a subsequence of the unit vector basis for $T_{c,\alpha}$. Thus a subsequence of the unit vector basis for $T_{\alpha,c}$ must dominate a subsequence of (v_i) (the unit vector basis for $T_{\beta,c}$), which is a contradiction. Thus no normalized block sequence in Z, which has a positive distance to $\psi(X)$, is equivalent to a subsequence in U.

Now any normalized sequence in Z has a subsequence which is equivalent to a sequence in X or has a subsequence which has a positive distance to $\psi(X)$. In both cases it follows that the sequence is not equivalent to a sequence in U. Theorem B follows.

Proof of Theorem C Assume that X is reflexive. Using Theorem 3.9 we can assume that X has an FDD (E_i) which satisfies for some $\alpha < \omega_1$ both subsequential $T_{\alpha,c}$-upper and subsequential $T^*_{\alpha,c}$-lower estimates. As noted before we can assume that $c \leq {}^1/16$.

By Theorem 4.7 we can embed X into a Bourgain–Delbaen space Y with a shrinking FDD $\mathbf{F} = (F_j)$, associated to a sequence of Bourgain–Delbaen sets (Δ_n), via the mapping ψ given in (5.3).

Now we apply Theorem 5.5 (b) to the unit vector basis (v_j) of $T^*_{\alpha,c}$ and obtain an augmentation (Θ_n) of (Δ_n) generating a Bourgain–Delbaen space Z having an FDD

$\overline{\mathbf{F}} = (\overline{F}_j)$, so that every normalized block basis (z_n) in Z has a subsequence which is either equivalent to a block sequence in X, or which dominates a subsequence of (v_j). Moreover, the later case holds for all normalized block bases of (z_n). In both cases it follows that this subsequence is boundedly complete, and since it is shrinking it follows that it must span a reflexive space.

Similarly we can show the following result, whose proof we omit.

Theorem 5.8 *Let X be a Banach space with separable dual and let (u_j) be a shrinking basic sequence, none of whose subsequences is equivalent to a sequence in X. Then X embeds into a Bourgain–Delbaen space Z whose dual is isomorphic to ℓ_1, and which does not contain any sequence which is equivalent to any subsequence of (u_j).*

Using a construction similar to one in the proof of Theorem 5.5 we can show the following embedding result for spaces with an FDD satisfying subsequential lower estimates.

Theorem 5.9 *Let V be a Banach space with a normalized unconditional basis (v_i), having the following property.*

$$\begin{aligned}&\textit{There is a constant } C > 0 \textit{ so that for}\\&\textit{any two sequences } (p_n) \textit{ and } (q_n) \textit{ in } \mathbb{N},\\&\textit{with } p_1 < q_1 < p_2 < q_2 < \ldots, (v_{p_n})\\&C - \textit{dominates } (v_{q_n}).\end{aligned} \tag{5.8}$$

Let X be a Banach space with an FDD (E_i) which satisfies subsequential V-lower estimates. Then X embeds into a $\mathcal{L}_\infty$ space Z with an FDD $(\overline{F}_i)$ which satisfies skipped subsequential V'-lower estimates where V' is some subsequence of V. Furthermore, if (E_i) and (v_i) are both shrinking, then $(\overline{F}_i)$ can be chosen to be shrinking too.

Proof After renorming, we may assume that the FDD $\mathbf{E} = (E_i)$ is bimonotone and that the basis (v_i) is 1-unconditional. We use the construction of Sect. 4 to define a $\mathcal{L}_\infty$ space Y with an FDD $\mathbf{F} = (F_i)$ and an embedding $\phi : X \to Y$ such that $\phi(E_i) \subset F_{m_i}$ for some sequence $(m_i) \in [\mathbb{N}]^\omega$. For convenience, we will refer to the space $\phi(X)$ as X. As the FDD (E_i) satisfies subsequential V-lower estimates, there exists $K \geq 1$, so that

$$\begin{aligned}&\text{if } (x_i) \subset X \text{ is a normalized block sequence such}\\&\text{that } x_i \in \oplus_{j=m_{p_i}}^{m_{q_i}} F_j, \text{ with } 1 = p_1 < q_1 < p_2, \ldots,\\&\text{then } (x_i) K - \text{ dominates } (v_{q_i}).\end{aligned} \tag{5.9}$$

We now define the Banach space $\tilde{V} \cong V \oplus c_0$ with basis $(\tilde{v}_i)$ given by $\tilde{v}_{m_i} = v_i$ and $\tilde{v}_i = e_i$ if $i \notin \{m_j\}$, where (e_i) is the unit vector basis of c_0. It is clear that $(\tilde{v}_i)$ is a 1-unconditional normalized basic sequence, and that $(\tilde{v}_i)$ is shrinking if (v_i) is shrinking.

We denote the projection constant of (F_i) by M. The sets $(\overline{\Delta}_n)$, $\Theta^{(0,1)}$, $\Theta^{(0,2)}$, $\Theta^{(1,1)}$, and $\Theta^{(1,2)}$ are defined as in Theorem 5.5 for some constant $c < 1/K$, the basic sequence $(\tilde{v}_i)$, and some inductively chosen $\varepsilon_{n+1}/(2M+4)$-dense sets $B_{(k,n]} \subset B_{\ell_1(\overline{\Gamma}_n \setminus \overline{\Gamma}_k)}$ (i.e. we are using the case "no assumptions on X"). This construction yields that $(\overline{\Delta}_n)$ admits an associated Bourgain–Delbaen space Z with FDD $\overline{\mathbf{F}} = (\overline{F}_j)$ whose decomposition constant $\overline{M}$ is not larger than $\max(M, 1/(1-2c)) \leq \max(M, 2)$. If (F_j) and (v_n) are both shrinking in V, and thus, the optimal c-decompositions of elements of $B_{\tilde{V}^*}$ are admissible with respect to some compact subset of $[\mathbb{N}]^{<\omega}$, we have that the FDD $\overline{\mathbf{F}} = (\overline{\mathbf{F}})$ is shrinking in Z. Furthermore, we have an isometric embedding $\psi : X \to Z$.

Before continuing, we need the following lemma which is analogous to Lemma 5.7. $\square$

Lemma 5.10 *Let (z_j^*) be a block basis in Z^* with respect to $\overline{\mathbf{F}}^*$ such that there exist integers $p_1 < q_1 < p_2 < q_2 \ldots$ with $\text{supp}_{\overline{\mathbf{F}}^*}(z_n^*) \subset (m_{p_n}, m_{q_n})$ for all $n \in \mathbb{N}$. Assume that*

$$z_n^* = P^{\overline{\mathbf{F}}*}_{(m_{p_n}, m_{q_n})}(\tilde{z}_n^*) \quad \text{for some } \tilde{z}_n^* \in B_{(m_{p_n}, m_{q_n})}, \text{ for } n \in \mathbb{N}.$$

Then for any sequence $(\beta_j)_{j=1}^N$ with $w^* = \sum_{j=1}^N \beta_j v_{q_j}^* \in D^V$, there exists $\overline{\gamma} \in \Lambda_{N+k_N}$ so that

$$\begin{aligned} &P^{\overline{\mathbf{F}}^*}_{(m_{p_n}, m_{q_n})}(e^*_{\overline{\gamma}}) = c\beta_n z_n^*, \text{ if } n \leq N, \text{ and} \\ &P^{\overline{\mathbf{F}}^*}(e^*_{\overline{\gamma}})(\psi(x)) = \sum_{n=1}^N c\beta_n z_n^*(\psi(x)) \quad \text{if } x \in X. \end{aligned} \tag{5.10}$$

Since parts of the proof are essentially the same as the proof of Lemma 5.7 we will only sketch it and point out where both proofs differ.

Proof We will prove our claim by induction on N and the case $N = 1$ is exactly like in the proof of Lemma 5.7 (with p_j and q_j being replaced by m_{p_j} and m_{q_j}, respectively). To show the claim for $N + 1$, assuming the claim to be true for N, we let $w^* = \sum_{j=1}^{N+1} \beta_j \tilde{v}_{m_{q_j}} = \sum_{j=1}^{N+1} \beta_j v_{q_j} \in D^{\tilde{V}}$, and define $\ell \in \mathbb{N}$, $\ell \geq 2$ and $\overline{\gamma}_j$ and $\overline{\eta}_j$, $j = 1, 2 \ldots, \ell$, as in Lemma 5.7. We need only to show by induction on $j = 1, 2 \ldots \ell$, that $|e^*_{\overline{\gamma}_j}(\psi(x))| \leq \|x\|$ for $x \in X$ (without the assumption of Lemma 5.7 that $|z_j^*(\psi(x))| \leq \delta_j \|x\|$, for $j \leq \ell$). Using the induction hypothesis on the $\overline{\eta}_j$'s, we deduce by induction on $j = 1, \ldots \ell$ that for $x \in X$

$$\begin{aligned} |e^*_{\overline{\gamma}_j}(\psi(x))| &= |c^*_{\overline{\gamma}_j}(\psi(x))| \\ &\leq \sum_{n=1}^{N_j} |c\beta_n z_n^*(\psi(x))| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{N_j} c|\beta_n| \left\| P^{\overline{\mathbf{F}}}_{(m_{p_n}, m_{q_n})}(\psi(x)) \right\| \\
&= c\left(\sum_{n=1}^{N_j} \beta_n v^*_{q_n}\right)\left(\sum_{n=1}^{N_j} \| P^{\overline{\mathbf{F}}}_{(m_{p_n}, m_{q_n})}(\psi(x))\| v_{q_n}\right) \\
&\leq c\left\| \sum_{n=1}^{N_j} \| P^{\overline{\mathbf{F}}}_{(m_{p_n}, m_{q_n})}(\psi(x))\| \tilde{v}_{m_{q_n}} \right\| \\
&\leq c\left\| \sum_{n=1}^{N_j} \left(\left\| P^{\overline{\mathbf{F}}}_{(m_{p_n}, m_{q_n})}(\psi(x)) \right\| \tilde{v}_{m_{q_n}} \right.\right. \\
&\qquad \left.\left. + \left\| P^{\overline{\mathbf{F}}}_{[m_{q_n}, m_{p_{n+1}}]}(\psi(x)) \right\| \tilde{v}_{m_{p_{n+1}}} \right) \right\| \\
&\leq cK\|x\| \leq \|x\|
\end{aligned}
$$

[in the penultimate line we use the 1-unconditionality of $(\tilde{v}_j)$ and in the case of $j = \ell$ we put $p_{N_\ell+1} = m_{q_{N_\ell+1}}$, for the last line we use (5.9)] and thus $\overline{\gamma}_1 \in \Theta^{(0,1)}_{m_{q_1}}$, if $|\text{supp}(w^*_1)| = 1$, and $\overline{\gamma}_1 \in \Theta^{(0,2)}_{m_{p_{N_1+1}}}$, if $|\text{supp}(w^*_1)| > 1$, and $\overline{\gamma}_j \in \Theta^{(1,1)}_{m_{q_{N_j}}}$, if $|\text{supp}(w^*_1)| = 1$, and $\overline{\gamma}_j \in \Theta^{(1,2)}_{m_{q_{N_j}}+N_j-N_{j-1}+1}$, if $|\text{supp}(w^*_1)| > 1$, if $j = 2, 3 \ldots \ell$. We put then $\overline{\gamma} = \overline{\gamma}_\ell$, and the rest of the proof follows again like in Lemma 5.7. □

Continuation of the Proof of Theorem 5.8 To finish the proof we consider a normalized block basis (z_n) in Z such that there exists sequences $p_1 < q_1 < p_2 < q_2 \ldots$ with $\text{supp}_{\overline{\mathbf{F}}}(z_n) \subset (m_{p_n}, m_{q_n})$ for all $n \in \mathbb{N}$. We choose $z^*_n \in \oplus_{j\in(p_n,q_n)} \overline{F}^*_j$, with $\|z^*_n\| \leq 1$, $z^*_n(z_n) \geq \frac{1}{2M}$.

By Proposition 2.7 there exists $b^*_n \in \ell_1(\overline{\Gamma}_{q_n-1} \setminus \overline{\Gamma}_{p_n})$, for $n \in \mathbb{N}$ so that $\|b^*_n\|_{\ell_1} \leq \overline{M}$ and $z^*_n = P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(b^*_n)$. Using the density assumption of $B_{(p_n,q_n)}$, we choose $\tilde{b}^*_n \in B_{(p,q_n)}$ with $\|\tilde{b}^*_n - \frac{1}{M} b^*_n\| \leq \varepsilon_{q_n}/(2M+4) \leq \varepsilon_{q_n}/2\overline{M}$, since $\overline{M} \leq M \vee 2$. So if we let $\tilde{z}^*_n = P^{\overline{\mathbf{F}}^*}_{(p_n,q_n)}(\tilde{b}^*_n)$, we deduce that $\|z^*_n/\overline{M} - \tilde{z}^*_n\| \leq 2\overline{M}\varepsilon_{q_n}/2\overline{M} = \varepsilon_{q_n}$ and hence $\tilde{z}^*_n(z_n) \geq z^*_n(z_n)/\overline{M} - \|z^*_n/\overline{M} - \tilde{z}^*_n\| \geq 1/\overline{M} - \varepsilon_n$, for all $n \in \mathbb{N}$.

Let $(\alpha_j)_{j=1}^N \subset \mathbb{R}$ with $\|\sum_{j=1}^N \alpha_j v_{q_j}\| = 1$ and using Lemma 4.1 (in the unconditional case) we can choose $(\beta_j)_{j=1}^N \subset \mathbb{R}$ with $\sum_{j=1}^N \beta_j v^*_{q_j} \in D^V$ so that

$$
\sum_{j=1}^N \beta_j v^*_{q_j}\left(\sum_{j=1}^N \alpha_j v_{q_j}\right) = \sum_{j=1}^N \alpha_j \beta_j \geq (1-\varepsilon).
$$

Since (p_n) and (q_n) satisfy the assumptions of Lemma 5.7 (recall that $m_{j+1} = j + m_j$), we can choose $\overline{\gamma} \in \Lambda$ so that

$$e_{\gamma}^{*}\left(\sum_{j=1}^{N}\alpha_j z_j\right)=\sum_{j=1}^{N}\alpha_j\beta_j P_{(p_j,q_j)}^{\overline{\mathbf{F}}^{*}}(e_{\gamma}^{*})(z_j)$$
$$=c\sum_{j=1}^{N}\alpha_j\beta_j z_j^{*}(z_j)\geq c(1-\varepsilon)(1/\overline{M}-\varepsilon),$$

which gives that (z_n) dominates (v_{q_n}). Thus we may block the FDD $(\overline{F}_i)$ to achieve the theorem.

References

1. Alspach, D.: A l_1-predual which is not isometric to a quotient of $C(\alpha)$. In: Banach Spaces (Mérida, 1992). Contemp. Math., vol. 144. Am. Math. Soc., pp. 9–14 (1993)
2. Alspach, D.: The dual of the Bourgain–Delbaen space. Israel J. Math. **117**, 239–259 (2000)
3. Alspach, D., Judd, R., Odell, E.: The Szlenk index and local ℓ_1-indices. Positivity **9**(1), 1–44 (2005)
4. Argyros, S.A., Haydon, R.: A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact (preprint). arXiv:0903.3921
5. Bourgain, J.: On convergent sequences of continuous functions. Bull. Soc. Math. Belg. Sér. B **32**(2), 235–249 (1980)
6. Bourgain, J.: The Szlenk index and operators on $C(K)$-spaces. Bull. Soc. Math. Belg. Sér. B **31**(1), 87–117 (1979)
7. Bourgain, J., Delbaen, F.: A class of special $\mathcal{L}_\infty$ spaces. Acta Math. **145**(3-4), 155–176 (1980)
8. Bourgain, J., Pisier, G.: A construction of $\mathcal{L}_\infty$-spaces and related Banach spaces. Bol. Soc. Brasil. Mat. **14**(2), 109–123 (1983)
9. Casazza, P.G., Johnson, W.B., Tzafriri, L.: On Tsirelson space. Israel J. Math. **47**(2-3), 81–98 (1984)
10. Dodos, P.: On classes of Banach spaces admitting small universal spaces. Trans. A.M.S. **361**, 6407–6428 (2009)
11. Fonf, V.: A property of Lindenstrauss Phelps spaces. Funct. Anal. Appl. **13**(1), 66–67 (1979)
12. Freeman, D., Odell, E., Schlumprecht, Th., Zsák, A.: Banach spaces of bounded Szlenk index, II. Fund. Math. **205**(2), 161–177 (2009)
13. Godefroy, G.: The Szlenk index and its applications. General topology in Banach spaces, pp. 71–79. Nova Sci. Publ., Huntington, NY (2001)
14. Godefroy, G., Kalton, N.J., Lancien, G.: Szlenk indices and uniform homeomorphisms. Trans. Am. Math. Soc. **353**(10), 3895–3918 (2001)
15. Haydon, R.: Subspaces of the Bourgain–Delbaen space. Studia Math. **139**(3), 275–293 (2000)
16. Johnson, W.B., Lindenstrauss, J.: Basic concepts in the geometry of Banach spaces. In: Handbook of the Geometry of Banach Spaces, vol. I, pp. 1–84. North-Holland, Amsterdam (2001)
17. Johnson, W.B., Rosenthal, H.P., Zippin, M.: On bases, finite dimensional decompositions, and weaker structures in Banach spaces. Israel J. Math. **9**, 488–506 (1971)
18. Johnson, W.B., Zippin, M.: Separable L_1 preduals are quotients of $C(\Delta)$. Israel J. Math. **16**, 198–202 (1973)
19. Johnson, W.B., Zippin, M.: Subspaces and quotient spaces of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$. Israel J. Math. **17**, 50–55 (1974)
20. Judd, R., Odell, E.: Concerning Bourgain's ℓ_1 index of a Banach space. Israel J. Math. **108**, 145–171 (1998)
21. Lancien, G.: A survey on the Szlenk index and some of its applications. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **100**(1-2), 209–235 (2006)
22. Leung, D.H., Tang, W.-K.: The Bourgain ℓ_1-index of mixed Tsirelson space. J. Funct. Anal. **199**(2), 301–331 (2003)
23. Lindenstrauss, J., Tzafriri, L.: Classical Banach spaces, I. In: Sequence Spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92. Springer, Berlin/New York (1977)
24. Odell, E., Schlumprecht, Th.: Trees and branches in Banach spaces. Trans. Am. Math. Soc. **354**(10), 4085–4108 (2002)

25. Odell, E., Schlumprecht, Th., Zsák, A.: A new infinite game in Banach spaces with applications. In: Banach Spaces and Their Applications in Analysis, pp. 147–182. Walter de Gruyter, Berlin (2007)
26. Odell, E., Schlumprecht, Th., Zsák, A.: Banach spaces of bounded Szlenk index. Studia Math. **183**(1), 63–97 (2007)
27. Odell, E., Tomczak-Jaegermann, N., Wagner, R.: Proximity to ℓ_1 and distortion in asymptotic ℓ_1 spaces. J. Funct. Anal. **150**(1), 101–145 (1997)
28. Szlenk, W.: The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces. Studia Math. **30**, 53–61 (1968)
29. Zippin, M.: On some subspaces of Banach spaces whose duals are L_1 spaces. Proc. Am. Math. Soc. **23**, 378–385 (1969)
30. Zippin, M.: Banach spaces with separable duals. Trans. Am. Math. Soc. **310**(1), 371–379 (1988)