

# ON SPREADING SEQUENCES AND ASYMPTOTIC STRUCTURES

D. FREEMAN, E. ODELL, B. SARI, AND B. ZHENG

ABSTRACT. In the first part of the paper we study the structure of Banach spaces with a conditional spreading basis. The geometry of such spaces exhibit a striking resemblance to the geometry of James' space. Further, we show that the averaging projections onto subspaces spanned by constant coefficient blocks with no gaps between supports are bounded. As a consequence, every Banach space with a spreading basis contains a complemented subspace with an unconditional basis. This gives an affirmative answer to a question of H. Rosenthal.

The second part contains two results on Banach spaces  $X$  whose asymptotic structures are closely related to  $c_0$  and do not contain a copy of  $\ell_1$ :

i) Suppose  $X$  has a normalized weakly null basis  $(x_i)$  and every spreading model  $(e_i)$  of a normalized weakly null block basis satisfies  $\|e_1 - e_2\| = 1$ . Then some subsequence of  $(x_i)$  is equivalent to the unit vector basis of  $c_0$ . This generalizes a similar theorem of Odell and Schlumprecht, and yields a new proof of the Elton-Odell theorem on the existence of infinite  $(1 + \varepsilon)$ -separated sequences in the unit sphere of an arbitrary infinite dimensional Banach space.

ii) Suppose that all asymptotic models of  $X$  generated by weakly null arrays are equivalent to the unit vector basis of  $c_0$ . Then  $X^*$  is separable and  $X$  is asymptotic- $c_0$  with respect to a shrinking basis  $(y_i)$  of  $Y \supseteq X$ .

## 1. INTRODUCTION

A basic sequence  $(x_i)$  in a Banach space is called *spreading* if it is equivalent to all of its subsequences. If, in addition, the sequence is unconditional then it is called *subsymmetric*. When  $(x_i)$  is spreading and weakly null it is automatically suppression unconditional. In Section 2 we will focus most of our attention on spreading sequences that are not unconditional. A famous example is the boundedly complete basis of the James space  $J$  and we shall see that much of the structure for  $J$  holds more generally for Banach spaces with a conditional spreading basis. We observe that if  $(e_i)$  is a normalized conditional spreading basis for  $X$  then the difference sequence  $(d_i) = (e_1, e_2 - e_1, e_3 - e_2, \dots)$  is a skipped unconditional basis for  $X$ . This means that if  $(x_j)$  is a normalized block basis of  $(d_i)$  with  $\text{supp}(x_j) < i_j < \text{supp}(x_{j+1})$  for some subsequence  $(i_j)$  of  $\mathbb{N}$ , then  $(x_j)$  is unconditional. Here  $\text{supp}(x_j)$  refers to the basis  $(d_i)$ , that is, if  $x_j = \sum_i b_i^j d_i$  then  $\text{supp}(x_j) = \{i : b_i^j \neq 0\}$ . It follows that, in the case  $(e_i)$  is spreading but not weakly null,  $\ell_1 \not\hookrightarrow X$  ( $\ell_1$  does not

---

Edward Odell (1947-2013).

Research of the first, second, and fourth author was supported by the National Science Foundation.

Research of the first author was supported by grant 353293 from the Simon's foundation and the third author was supported by grant 208290 from the Simon's Foundation.

embed isomorphically into  $X$ ) if and only if the difference basis  $(d_i)$  is shrinking. Also we show that  $c_0 \not\hookrightarrow X$  if and only if  $(e_i)$  is boundedly complete. Furthermore,  $c_0$  and  $\ell_1$  do not embed into  $X$  if and only if  $X$  is quasi-reflexive of order 1. It is interesting to note that these (except the skipped unconditionality result) were already observed in the 1970's by Brunel and Sucheston [BS] for ESA (equal sign additive) bases, which is a stronger property than spreading. However, our results are more general and the proofs are different. The crucial part of our approach is an unconditionality result, Theorem 2.3a, which is of independent interest. We also show that the well known averaging projection onto disjoint subsets of a subsymmetric basis remains bounded for the conditional spreading case as long as the subsets form a partition. One consequence is that  $X$  is isomorphic to  $D \oplus X$  where  $D$  is the subspace spanned by  $(d_{2n})_{n=1}^\infty$ . Moreover, every Banach space with a spreading basis contains a complemented subspace with an unconditional basis. This answers an open problem of H. Rosenthal.

In Section 3 we make a few remarks on Banach spaces that admit conditional spreading models. Our study of the conditional spreading sequences were motivated by the problems discussed in this section.

In section 4 we consider spaces whose asymptotic structure is closely related to  $c_0$ . In [OS] it was shown that if  $(x_i)$  is a basis for  $X$  and all spreading models of normalized block bases of  $(x_i)$  are 1-equivalent to the unit vector basis of  $c_0$ , then  $c_0$  embeds into  $X$ . Our first result of Section 4 generalizes this as follows. If  $(x_i)$  is weakly null and if every spreading model  $(e_i)$  generated by a *weakly null* block basis satisfies  $\|e_1 - e_2\| = 1$  and  $\ell_1 \not\hookrightarrow X$ , then  $c_0 \hookrightarrow X$ . This yields a quick proof of the Elton-Odell theorem [EO]. Namely, for every Banach space  $X$  there exists an infinite sequence  $(z_i)$  in the unit sphere  $S_X$  and  $\lambda > 1$  so that  $\|z_i - z_j\| \geq \lambda$  for all  $i \neq j$ . Indeed, if  $X$  contains  $\ell_1$  or  $c_0$  the result follows easily by the non-distortability of  $c_0$  and  $\ell_1$ . Otherwise, fix a weakly null normalized sequence  $(x_i)$ . By our theorem,  $(x_i)$  must have a normalized block basis with a spreading model  $(e_i)$  with  $\|e_1 - e_2\| > 1$  which yields an  $\varepsilon > 0$  and an infinite  $(1 + \varepsilon)$ -separated sequence.

One of the long standing open problems on asymptotic structures of Banach spaces is the following. Suppose that every spreading model of  $X$  is equivalent to the unit vector basis of  $c_0$  (or  $\ell_p$ ). Does  $X$  contain an asymptotic- $c_0$  (or asymptotic- $\ell_p$ ) subspace? We solve the  $c_0$  case with somewhat stronger assumption. If all normalized *asymptotic models*  $(e_i)$  of normalized weakly null arrays in  $X$  are equivalent to the unit vector basis of  $c_0$  and  $\ell_1 \not\hookrightarrow X$ , then  $X^*$  is separable and  $X$  is asymptotic- $c_0$  with respect to a shrinking basis  $(y_i)$  of  $Y \supseteq X$ . Recall that  $(e_i)$  is an asymptotic model of  $X$ , denoted by  $(e_i) \in AM_w(X)$ , if

there exists a normalized array  $(x_j^i)_{i,j \in \mathbb{N}}$  so that  $(x_j^i)_{j=1}^\infty$  is weakly null for all  $i \in \mathbb{N}$ , and for some  $\varepsilon_n \downarrow 0$ , all  $n$  and all  $(a_i)_1^n \subseteq [-1, 1]$  and  $n \leq k_1 < k_2 < \dots < k_n$

$$(1.1) \quad \left| \left\| \sum_{i=1}^n a_i x_{k_i}^i \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| \leq \varepsilon_n.$$

The notion of asymptotic models is a direct generalization of spreading models and it was introduced in [HO].  $X$  is asymptotic- $c_0$  if for some  $K < \infty$  for all  $n$  and all asymptotic spaces  $(e_i)_{i=1}^n$  are  $K$ -equivalent to the unit vector basis of  $\ell_\infty^n$  [MMT]. These notions are recalled in Section 4.

## 2. SPREADING BASES

We begin with a result solving a problem asked of us by S. A. Argyros.

**Theorem 2.1.** *Let  $(e_n)$  be a normalized basis for  $X$ . If every subspace spanned by a skipped block basis of  $(e_n)$  is reflexive then  $X$  is either reflexive or quasi-reflexive of order 1.*

*Proof.* The hypothesis yields that  $(e_n)$  is shrinking. If not, then for some normalized block basis  $(x_n)$  of  $(e_n)$  there exists  $f \in B_{X^*}$  and  $\varepsilon > 0$  with  $f(x_n) > \varepsilon$  for all  $n$ . But then  $(x_{2n})$  is a skipped block basis of  $(e_n)$  which cannot be shrinking, hence cannot span a reflexive space.

Let  $F \in X^{**}$ . Since the basis  $(e_i)$  is shrinking  $F$  is the  $w^*$ -limit of  $(\sum_{i=1}^n F(e_i^*)e_i)_{n=1}^\infty$  where  $(e_i^*)$  is the biorthogonal sequence to  $(e_i)$  (a basis for  $X^*$ ). We claim that if

$$\liminf_n |F(e_i^*)| = 0,$$

then  $F \in i(X)$ , where  $i(X)$  is the natural embedding of  $X$  into  $X^{**}$ .

Indeed, pick a subsequence  $(i_j)$  such that  $\sum_{j=1}^\infty |F(e_{i_j}^*)| < \infty$ . Let  $y = \sum_{j=1}^\infty F(e_{i_j}^*)e_{i_j}$ . Then  $y \in i(X)$ . Let  $G = F - y$ . Then  $G = w^* - \lim_n \sum_{j=1}^n \sum_{i_j < i < i_{j+1}} F(e_i^*)e_i$  and  $(\sum_{i_j < i < i_{j+1}} F(e_i^*)e_i)_{j=1}^\infty$  is a skipped block sequence which spans a reflexive subspace. Thus  $G \in i(X)$  and so is  $F$ .

Now suppose  $X$  is not reflexive and let  $G \in X^{**}$  and  $F \in X^{**} \setminus i(X)$ . Choose  $\lambda \in \mathbb{R}$  and a subsequence  $(i_n)$  of  $\mathbb{N}$  so that  $G(e_{i_n}^*) - \lambda F(e_{i_n}^*) \rightarrow 0$ . Then by the claim above we conclude that  $G - \lambda F \in i(X)$ . Therefore  $X^{**} = \mathbb{R}F \oplus i(X)$ .  $\square$

**Remark 2.2.** A generalization of the above from a basis to finite dimensional decompositions (FDD) is false. Indeed, the Argyros-Haydon space  $\mathfrak{X}_K$  has an FDD  $(M_n)$  with the property that every skipped blocking of  $(M_n)$  spans a reflexive subspace and yet its dual

is isomorphic to  $\ell_1$  (Theorem 9.1, [AH]). We thank Pavlos Motakis for pointing out the example.

We now turn to conditional spreading bases. Suppose that  $(e_i)$  is a normalized spreading basis for  $X$  which is not weakly null. Then the *summing functional*,

$$S\left(\sum_i a_i e_i\right) := \sum_i a_i$$

is bounded on  $X$ . Indeed for some  $\lambda \neq 0$ ,  $f \in X^*$ , and subsequence  $(i_n)$  of  $\mathbb{N}$  we have that  $f(e_{i_n}) - \lambda \rightarrow 0$  rapidly. So a perturbation of  $\lambda^{-1}f$  is constantly 1 on the  $e_{i_n}$ 's. Then it follows from the spreading property that  $S$  is bounded on  $X$ .

By renorming we can assume that  $(e_i)$  is normalized, 1-spreading and a bimonotone basis for  $X$ , and  $\|S\| = 1$ . This is easily achieved by replacing  $(e_i)$  by a spreading model of a subsequence, and then by the renorming  $\|x\| := \max(\|x\|, |S(x)|)$ . With this we also get that the functional  $S_I(\sum_i a_i e_i) := \sum_{i \in I} a_i$  is of norm one for any interval  $I$ . Note that the boundedness of  $S$  implies that the summing basis of  $c_0$  is dominated by every conditional spreading sequence.

**Theorem 2.3.** *Let  $(e_i)$  be a normalized 1-spreading, non weakly null, bimonotone basis for  $X$ .*

- a) *If  $(x_i)$  is a normalized block basis of  $(e_i)$  with  $S(x_i) = 0$  for all  $i$ , then  $(x_i)$  is suppression 1-unconditional.*
- b) *Let  $(d_i) = (e_1, e_2 - e_1, e_3 - e_2, \dots)$ . Then  $(d_i)$  is a skipped unconditional basis for  $X$ .*
- c)  *$(e_i)$  is boundedly complete if and only if  $c_0 \not\hookrightarrow X$ .*
- d)  *$(d_i)$  is shrinking if and only if  $\ell_1 \not\hookrightarrow X$ .*
- e)  *$\ell_1 \not\hookrightarrow X$  if and only if  $X^* = \mathbb{R}S \oplus [(e_i^*)]$ .*
- f)  *$c_0$  and  $\ell_1$  do not embed into  $X$  if and only if  $X$  is quasi-reflexive of order 1.*

*Proof.* For  $x, y \in X$  which are finitely supported with respect to the basis  $(e_i)$ , we write  $x \sim y$  if

$$x = \sum_{i=1}^k a_i e_{n_i} \quad \text{and} \quad y = \sum_{i=1}^k a_i e_{m_i} \quad \text{where} \quad n_1 < \dots < n_k, \quad m_1 < \dots < m_k.$$

- a) Let  $(x_i)$  be as in a). We now need the following lemma.

**Lemma 2.4.** *For all  $\varepsilon > 0$  and  $i_0 \in \mathbb{N}$  there exists  $m \in \mathbb{N}$  such that for all  $f \in S_{X^*}$  there exists  $\tilde{x} \in X$ ,  $\tilde{x} \sim x_{i_0}$  and  $\text{supp}(\tilde{x}) \subseteq [j, m]$ ,  $j = \min \text{supp}(x_{i_0})$ , so that  $|f(\tilde{x})| < \varepsilon$ .*

*Proof.* Let  $\varepsilon > 0$  and  $i_0 \in \mathbb{N}$ . Since  $|f(e_i)| \leq 1$  for any  $f \in S_{X^*}$ , by the pigeonhole principle there exists  $m$  with the following property:

Let  $j = \min \text{supp}(x_{i_0})$ . For all  $f \in S_{X^*}$  there exists  $\lambda \in [-1, 1]$  and  $F \subseteq [j, m]$  with  $|F| = k = |\text{supp}(x_{i_0})|$  so that for  $i \in F$ ,  $|f(e_i) - \lambda| < \varepsilon/k$ .

Place  $\tilde{x} \equiv \sum_{i \in F} a_i e_i$  on  $F$  so that  $\tilde{x} \sim x_{i_0}$ . Then  $S(\tilde{x}) = S(x_{i_0}) = 0$  and

$$|f(\tilde{x})| \leq |f(\tilde{x} - \lambda S(\tilde{x}))| + |\lambda S(\tilde{x})| = \left| \sum_{i \in F} a_i (f(e_i) - \lambda) \right| < \varepsilon.$$

□

Now let  $x = \sum_{i=1}^k a_i x_i$ ,  $\|x\| = 1$ ,  $\varepsilon > 0$ . Let  $F \subseteq \{1, 2, \dots, k\}$ . We will show that  $\|\sum_{i \in F} a_i x_i\| \leq 1 + \varepsilon$ . Let  $j_i = \min \text{supp}(x_i)$  for  $i \leq k$  and choose  $m_i$  by Lemma 2.4 for  $\varepsilon/k$  and  $j_i$ . Since  $(e_i)$  is 1-spreading we may assume that  $j_1 < m_1 < j_2 < m_2 < \dots$ . Let  $f \in S_{X^*}$  with  $f(\sum_{i \in F} a_i x_i) = \|\sum_{i \in F} a_i x_i\|$ . For  $i \notin F$ ,  $i \leq k$ , we choose  $\tilde{x}_i \sim x_i$  with  $\text{supp}(\tilde{x}_i) \subseteq [j_i, m_i]$  so that  $|f(\tilde{x}_i)| < \varepsilon/k$ . Then

$$\left\| \sum_{i \in F} a_i x_i \right\| \leq \left| f\left( \sum_{i \in F} a_i x_i + \sum_{i \notin F} a_i \tilde{x}_i \right) \right| + \left| f\left( \sum_{i \notin F} a_i \tilde{x}_i \right) \right| \leq \|x\| + \varepsilon = 1 + \varepsilon.$$

This proves a).

b) To see  $(d_i)$  is a basis for  $X$  we need only note that it is basic. This is an easy calculation that holds for any difference sequence  $(d_i)$  obtained from a normalized basic  $(e_i)$  that dominates the summing basis (i.e.,  $S$  is bounded). Indeed, for any  $n < m$

$$\begin{aligned} \left\| \sum_{i=1}^n a_i d_i \right\| &= \left\| \sum_{i=1}^{n-1} (a_i - a_{i+1}) e_i + a_n e_n \right\| \leq \left\| \sum_{i=1}^m a_i d_i \right\| + \|a_{n+1} e_n\| \\ &= \left\| \sum_{i=1}^m a_i d_i \right\| + \left| S\left( \sum_{i=n+1}^{m-1} (a_i - a_{i+1}) e_i + a_m e_m \right) \right| \leq 2 \left\| \sum_{i=1}^m a_i d_i \right\|. \end{aligned}$$

That  $(d_i)$  is skipped unconditional follows from a).

c) We need only show that if  $(e_i)$  is not boundedly complete, then  $c_0 \hookrightarrow X$ . Suppose that there exists  $(a_i) \subseteq \mathbb{R}$  so that  $\sup_n \|\sum_{i=1}^n a_i e_i\| = 1$  and  $\sum_{i=1}^{\infty} a_i e_i$  diverges. Choose  $\delta > 0$  and a subsequence  $(k_i)$  of  $\mathbb{N}$  so that  $\|x_i\| > \delta$  where  $x_i = \sum_{j=k_i}^{k_{i+1}-1} a_j e_j$  for  $i \in \mathbb{N}$ .

Choose a block sequence  $(y_i)$  of  $(e_i)$  so that  $y_{2i-1} \sim x_i$  and  $y_{2i} \sim x_i$  for all  $i$ . Then  $(y_{2i-1})$  and  $(y_{2i})$  are each equivalent to  $(x_i)$  and  $(y_{2i-1} - y_{2i})$  is unconditional by a). Furthermore

$$\sup_n \left\| \sum_{i=1}^n (y_{2i-1} - y_{2i}) \right\| \leq 2$$

and  $2 \geq \|y_{2i-1} - y_{2i}\| \geq \delta$  for all  $i$ . Thus  $(y_{2i-1} - y_{2i})$  is equivalent to the unit vector basis of  $c_0$ .

d) This follows easily since  $(d_i)$  is skipped unconditional.

e) Suppose  $\ell_1$  does not embed into  $X$ . By Rosenthal's  $\ell_1$  theorem [R] and the fact that  $(e_i)$  is spreading,  $(e_i)$  is weak Cauchy.

Let  $f \in X^*$ . Then  $f = w^* - \lim_n \sum_{i=1}^n f(e_i)e_i^*$ , and  $\lim_{i \rightarrow \infty} f(e_i) \equiv \lambda$  exists. Then  $f - \lambda S \in [(e_i^*)]$ . Indeed  $f - \lambda S = w^* - \lim_{n \rightarrow \infty} \sum_{i=1}^n b_i e_i^*$  where  $\lim_i b_i = 0$ . If the series is not norm convergent there exists  $\delta > 0$ ,  $(n_i) \in [\mathbb{N}]^\omega$ , and a normalized block basis  $(x_i)$  of  $(e_i)$  so that  $x_1 < e_{n_1} < x_2 < e_{n_2} < \dots$ , so that  $(f - \lambda S)x_i > \delta$  for all  $i$  and  $b_{n_i} \rightarrow 0$  rapidly. In particular,  $(x_i - S(x_i)e_{n_i})$  is unconditional and  $(f - \lambda S)(x_i - S(x_i)e_{n_i}) > \delta/2$  for all  $i$ . Thus  $(x_i - S(x_i)e_{n_i})$  is equivalent to the unit vector basis of  $\ell_1$ , a contradiction.

f) Let  $(u_n)$  be a skipped block basis of  $(d_i)$ , and assume  $c_0$  and  $\ell_1$  do not embed into  $X$ . Then  $(u_n)$  is unconditional and shrinking by b) and d) and is also boundedly complete since  $X$  does not contain  $c_0$ . Thus  $[(u_n)]$  is reflexive and Theorem 2.1 yields the result.  $\square$

If  $X$  has an unconditional basis and  $Y \subseteq X$  has non separable dual then  $\ell_1 \hookrightarrow Y$  [BP]. This also holds if  $X$  has a spreading basis. In fact, the result holds more generally.

**Proposition 2.5.** *Suppose  $X$  has a skipped unconditional basis and let  $Y \subseteq X$  with  $Y^*$  not separable. Then  $\ell_1$  embeds into  $Y$ .*

*Proof.* Assume that  $Y^*$  is not separable and  $\ell_1$  does not embed into  $Y$ . By Theorem 3.14 of [AJO] there exists an  $\ell_1^+$  weakly null tree  $(y_\alpha)_{\alpha \in T_\omega}$  in  $Y$ . Here  $T_\omega = \{(n_i)_1^k : n_1 < \dots < n_k, n_i \in \mathbb{N}, k \in \mathbb{N}\}$ .  $(y_{(\alpha, n)})_n$  is weakly null and normalized for all  $\alpha \in \{\emptyset\} \cup T_\omega$ . Furthermore, for some  $c > 0$ ,  $\|\sum_i a_i y_{\alpha_i}\| \geq c \sum_i a_i$  for all branches  $(\alpha_i)$  of  $T_\omega$  and  $a_i \geq 0$ . Using that the tree is weakly null and  $X$  has a skipped unconditional basis it is easy to find a branch  $(y_{\alpha_i})$  which is unconditional, hence is equivalent to the unit vector basis of  $\ell_1$ . This is a contradiction.  $\square$

**Remark 2.6.** The same proof also yields that if  $X$  is a subspace of a space with skipped unconditional finite dimensional decomposition and  $X^*$  is non-separable, then  $\ell_1$  embeds into  $X$ .

The next result answers a question asked of us by H. Rosenthal: If  $X$  has a spreading basis, does  $X$  contain a complemented subspace with an unconditional basis?

**Proposition 2.7.** *If  $(e_i)$  is a normalized spreading basis for  $X$  then the subspace  $Y$  spanned by the unconditional block basis  $[(e_{2n-1} - e_{2n})]$  is complemented in  $X$ .*

It suffices to prove that the complementary “projection”  $Q$  is bounded where

$$Q\left(\sum_i a_i e_i\right) = \sum_i \frac{a_{2i-1} + a_{2i}}{2} (e_{2i-1} + e_{2i}).$$

This is a consequence of the following more general result which is well known if the basis is subsymmetric.

**Theorem 2.8.** *Let  $(e_i)$  be a normalized bimonotone 1-spreading basis for  $X$ . Let  $(\sigma_j)_{j=1}^\infty$  be a partition of  $\mathbb{N}$  into successive intervals,  $\sigma_1 < \sigma_2 < \dots$ , with  $|\sigma_j| = n_j$  for  $j \in \mathbb{N}$ . Then the averaging operator*

$$Q\left(\sum_i a_i e_i\right) = \sum_{j=1}^\infty \left( \left( \sum_{i \in \sigma_j} a_i \right) / n_j \right) \left( \sum_{i \in \sigma_j} e_i \right)$$

is a bounded projection on  $X$  with  $\|Q\| \leq 3$ .

It is important to note that, unlike the subsymmetric case, there are no gaps allowed between blocks in this averaging operator.

*Proof.* It suffices to prove that for all  $k$ ,  $\|Qx\| \leq 3\|x\|$  if  $\text{supp}(x) \subseteq \bigcup_{i=1}^k \sigma_i$ . Let  $k \in \mathbb{N}$ ,  $x = \sum_{j=1}^{\max(\sigma_k)} a_j e_j$ . Let  $M$  be the least common multiple of  $(n_1, n_2, \dots, n_k)$  and set  $m_j = M/n_j$  for  $j \leq k$ .

We will construct vectors  $(y_i)_{i=1}^{2M}$  so that  $\frac{1}{2M} \sum_{j=1}^{2M} y_j = \bar{x} + \sum_{j=1}^M z_j$  where  $y_i \sim x$ ,  $2\bar{x} \sim Qx$  and  $z_j \sim \frac{1}{2M}x$  for  $j \leq M$ . It follows that

$$\|Qx\| = 2\|\bar{x}\| \leq 2\left(\|x\| + M \frac{1}{2M} \|x\|\right) = 3\|x\|.$$

To begin we spread  $x$  to obtain  $y_1$  so that the coordinates of  $y_1$  looks like this

$$y_1 = (a_1, a_2, \dots, a_{n_1}, 0, \dots, 0, a_{n_1+1}, \dots, a_{n_1+n_2}, 0, \dots, 0, a_{n_1+n_2+1}, \dots).$$

For each  $1 \leq j \leq k-1$ , we insert  $2n_j - 1$  zeros between the blocks of  $x$  corresponding to  $\sigma_j$  and  $\sigma_{j+1}$ , and let  $\gamma_j$  be the index set for the coordinates of the inserted block of zeros. The vectors  $y_2, \dots, y_{2M}$  will be spreads of  $y_1$ . The position of the first block  $(a_1, \dots, a_{n_1})$  is preserved for  $y_2, \dots, y_{m_1}$ . This block is then shifted one unit right for  $y_{m_1+1}, \dots, y_{2m_1}$ . Then another unit to the right for  $y_{2m_1+1}, \dots, y_{3m_1}$  and so on  $n_1$  times until reaching  $y_{2M} = y_{2n_1 m_1}$ . The same scheme is followed for the second block  $(a_{n_1+1}, \dots, a_{n_1+n_2})$  and the subsequent blocks. Thus the second block is preserved for  $y_2, \dots, y_{m_2}$  and then shifted once right for  $y_{m_2+1}, \dots, y_{2m_2}$ .

When we average the  $y_j$ 's,  $\bar{x}$  will be the average of the vectors  $y_1, \dots, y_{2M}$  restricted to the coordinates given by the union over  $1 \leq j \leq k$  of the first  $n_j$  coordinates of  $\gamma_j$ .

We give a simple example in the diagram below explaining this averaging procedure in the case  $k = 2$ ,  $n_1 = 2$ ,  $n_2 = 3$  and so  $M = 6$ ,  $m_1 = 3$ , and  $m_2 = 2$ .

$$\begin{array}{cccccccccccc}
a_1 & a_2 & 0 & 0 & 0 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & 0 & 0 & 0 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & 0 & 0 & 0 & 0 & a_3 & a_4 & \mathbf{a}_5 & 0 & 0 & 0 & 0 \\
0 & a_1 & \mathbf{a}_2 & 0 & 0 & 0 & a_3 & a_4 & \mathbf{a}_5 & 0 & 0 & 0 & 0 \\
0 & a_1 & \mathbf{a}_2 & 0 & 0 & 0 & 0 & a_3 & \mathbf{a}_4 & \mathbf{a}_5 & 0 & 0 & 0 \\
0 & a_1 & \mathbf{a}_2 & 0 & 0 & 0 & 0 & a_3 & \mathbf{a}_4 & \mathbf{a}_5 & 0 & 0 & 0 \\
0 & 0 & \mathbf{a}_1 & \mathbf{a}_2 & 0 & 0 & 0 & 0 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & 0 & 0 \\
0 & 0 & \mathbf{a}_1 & \mathbf{a}_2 & 0 & 0 & 0 & 0 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & 0 & 0 \\
0 & 0 & \mathbf{a}_1 & \mathbf{a}_2 & 0 & 0 & 0 & 0 & \mathbf{a}_3 & \mathbf{a}_4 & a_5 & 0 & 0 \\
0 & 0 & 0 & \mathbf{a}_1 & a_2 & 0 & 0 & 0 & 0 & \mathbf{a}_3 & \mathbf{a}_4 & a_5 & 0 \\
0 & 0 & 0 & \mathbf{a}_1 & a_2 & 0 & 0 & 0 & 0 & 0 & \mathbf{a}_3 & a_4 & a_5 \\
0 & 0 & 0 & \mathbf{a}_1 & a_2 & 0 & 0 & 0 & 0 & 0 & \mathbf{a}_3 & a_4 & a_5
\end{array}$$

The vector  $\bar{x}$  is the average of  $y_1, \dots, y_{2M}$  restricted to the coordinates given by bold type. The remaining coefficients are easily partitioned into  $M$  spreads of  $x$ .  $\square$

**Proposition 2.9.** *Let  $(e_i)$  be a normalized conditional spreading basis for  $X$ . Let  $D = [(d_{2n})]$ , where  $(d_n)$  is the difference basis. Then  $X \simeq D \oplus Y$  where  $Y = [(e_1 + e_2, e_3 + e_4, \dots)]$  is isomorphic to  $X$ .*

*Proof.* We may assume  $(e_i)$  is 1-spreading. By Proposition 2.7 and Theorem 2.8 it suffices to prove that  $(e_{2n-1} + e_{2n})_{n=1}^\infty$  dominates  $(e_n)$ . We will prove that if  $x = \sum_{i=1}^n a_i e_i$ ,  $\|x\| = 1$ , then  $\|\sum_{i=1}^n a_i (e_{2i-1} + e_{2i})\| \geq 2/3$ . Write  $x_1 = \sum_{i=1}^n a_i e_{3i-1}$ ,  $x_2 = \sum_{i=1}^n a_i e_{3i-2}$  and  $x_3 = \sum_{i=1}^n a_i e_{3i}$ . Assume  $\|x_1 + x_2\| = c$ . Let  $f \in S_{X^*}$ ,  $1 = f(x_1)$ . Then  $f(x_1 + x_2) \leq c$  so  $f(x_2) \leq c - 1$ . Also using  $\|x_1 + x_3\| = c$ ,  $f(x_3) \leq c - 1$ . Thus  $c \geq -f(x_2 + x_3) \geq 2 - 2c$  and so  $c \geq 2/3$ . Thus,  $\|\sum_{i=1}^n a_i (e_{2i-1} + e_{2i})\| = \|\sum_{i=1}^n a_i (e_{3i-2} + e_{3i-1})\| = c \geq 2/3$ . Note that the argument can easily be generalized for all  $\varepsilon > 0$  to get  $c \geq 1 - \varepsilon$ .  $\square$

It has been shown that spaces  $X$  whose dual are isomorphic to  $\ell_1$  are quite plentiful and need not contain  $c_0$  [BD]. Moreover, any  $Y$  with  $Y^*$  separable embeds into such a space [FOS]. But if  $X$  has a spreading basis,  $X^*$  is separable and  $\ell_1 \hookrightarrow X^*$ , then  $c_0 \hookrightarrow X$ . This holds more generally if  $X^*$  is separable and  $X^{**}$  is not separable, assuming a spreading basis, by Theorem 2.3. More can be said if  $X^*$  is isomorphic to  $\ell_1$ .

**Theorem 2.10.** *Let  $(e_i)$  be a normalized spreading basis for  $X$  and assume  $X^*$  is isomorphic to  $\ell_1$ . Then  $(e_i)$  is equivalent to either the unit vector basis of  $c_0$  or the summing basis.*



*Proof.* If  $(e_i)$  is weakly null then, it is unconditional. It follows that  $(e_i^*)$  is subsymmetric. Since  $X^* \simeq \ell_1$  some subsequence of  $(e_i^*)$  is equivalent to the unit vector basis of  $\ell_1$ , so  $(e_i^*)$  is such and so  $(e_i)$  is equivalent to the unit vector basis of  $c_0$ .

If  $(e_i)$  is not weakly null, then we consider the difference basis  $(d_i)$  of  $X$ . To show  $(e_i)$  is equivalent to the summing basis it suffices to show that  $(d_i)$  is equivalent to the unit vector basis of  $c_0$ . To do this, it suffices, by the triangle inequality, to show  $(d_{2i})$  is equivalent to the unit vector basis of  $c_0$  since  $(d_{2i})$  is equivalent to  $(d_{2i-1})$ . Now  $D = [(d_{2n})]$  is complemented in  $X$  and  $(d_{2n})$  is unconditional and shrinking. So  $(d_{2n}^*|_D)$  is an unconditional basis for  $D^*$  which is isomorphic to  $\ell_1$ , since it is complemented in  $X^* \simeq \ell_1$ . Thus  $(d_{2n}^*|_D)$  is equivalent to the unit vector basis of  $\ell_1$ . These are due to the facts that  $\ell_1$  is prime and has unique unconditional basis. Hence  $(d_{2n})$  is equivalent to the unit vector basis of  $c_0$ .  $\square$

### 3. REMARKS ON CONDITIONAL SPREADING MODELS

Recall that a normalized basic sequence  $(e_i)$  is a spreading model of a sequence  $(x_i)$  if for some  $\varepsilon_n \downarrow 0$ , for all  $n$ ,  $(a_i)_1^n \subseteq [-1, 1]$  positive integers  $n \leq k_1 < \dots < k_n$

$$(3.1) \quad \left| \left\| \sum_{j=1}^n a_j x_{k_j} \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| \leq \varepsilon_n.$$

In this case  $(e_i)$  is 1-spreading, and if  $(x_i)$  is weakly null, then  $(e_i)$  is suppression 1-unconditional. We denote by  $SP_w(X)$  the set of all spreading models of  $X$  generated by weakly null sequences. If  $(y_i)$  is normalized basic then, via Ramsey theory, some subsequence  $(x_i)$  of  $(y_i)$  generates a spreading model  $(e_i)$  as in (3.1) above. If  $(y_i)$  is normalized but does not have a basic subsequence then any basic spreading model admitted by  $(y_i)$  must be equivalent to the unit vector basis of  $\ell_1$ . Indeed, by Rosenthal's  $\ell_1$  theorem we may assume  $(y_i)$  is weak Cauchy. Every non-trivial weak Cauchy sequence has a basic subsequence (see the proof of Proposition 2.2, [Ro]). Thus a subsequence  $(x_i)$  of  $(y_i)$  weakly converges to a nonzero element  $x_0$ , and  $(x_i - x_0)$  generates an unconditional spreading model  $(u_i)$ . So  $(e_i)$  is equivalent to  $(x_0 + u_i)$  in  $\langle x_0 \rangle \oplus [(u_i)]$ . Since  $(e_i)$  is basic,  $(u_i)$  is not weakly null and therefore equivalent to the unit vector basis of  $\ell_1$ , and so is  $(e_i)$ .

One of the questions of interest about spreading models is whether there exists a “small” space that is universal for all (or a large class of) spreading models. Recall that the space  $C(\omega^\omega)$  is universal for all unconditional spreading models, that is, every subsymmetric basic sequence is a spreading model of  $C(\omega^\omega)$  [O]. In [AM] a remarkable example of a reflexive space is constructed so that every infinite dimensional subspace of it is universal

for all unconditional spreading models. For the case of conditional spreading models, S. A. Argyros raised the following which partly motivated our study of conditional spreading sequences above.

**Problem 3.1.** Let  $(e_i)$  be a conditional normalized spreading sequence. Does there exist a quasi-reflexive of order 1 space  $X$  with a normalized basis  $(x_i)$  which generates  $(e_i)$  as a spreading model?

We show that the answer is affirmative for the summing basis of  $c_0$ . For a given basis  $(e_i)$ , recall the space  $J(e_i)$ . For  $x \in J(e_i)$ , the norm is given by

$$\|x\| = \sup \left\{ \left\| \sum_{i=1}^k s_i(x) e_{p_i} \right\| : s_1 < s_2 < \dots < s_k \text{ are intervals in } \mathbb{N}, \min s_i = p_i \right\},$$

where  $s_i(x) = \sum_{j \in s_i} a_j$ ,  $s_i = [p_i, q_i)$ , and  $x = (a_j)$ .

**Proposition 3.2.** *Let  $(e_i)$  be the unit vector basis of the dual Tsirelson space  $T^*$ . Then the space  $J(e_i)$  is quasi-reflexive of order 1 and the spreading model generated by its natural basis is equivalent to the summing basis of  $c_0$ .*

*Proof.* In [BHO] it is shown that if  $(e_i)$  is a basis of a reflexive space, then  $J(e_i)$  is quasi-reflexive of order 1. Thus the first assertion follows since  $T^*$  is reflexive.

Also it is easy to see that any subsequence of the basis  $(u_i)$  of  $J(e_i)$  generates a spreading model equivalent to the summing basis  $(s_i)$ . Indeed, to estimate the norm of a vector  $x = \sum_{j=1}^k a_j u_{i_j}$  where  $k \leq i_1 < \dots < i_k$  note that for an arbitrary  $s_1 < \dots < s_k$  we have

$$\left\| \sum_{j=1}^k s_j(x) e_{i_j} \right\|_{T^*} \leq 2 \max_j | \sum_{i \in s_j} a_i |$$

and the latter expression is at most twice the summing norm of  $x$ . The reverse inequality is trivial (consider intervals  $s = [l, i_k]$ ,  $k \leq l \leq i_k$ ).  $\square$

In a follow-up work [AMS] the constructions similar to the above are studied in more detail and, in particular, Problem 3.1 is solved affirmatively.

#### 4. SPREADING AND ASYMPTOTIC MODELS

Our first result of this section is a strengthening of the  $c_0$ -part of the following theorem of Odell and Schlumprecht [OS]. *If  $X$  has a basis  $(x_i)$  so that every spreading model of a normalized block basis of  $(x_i)$  is 1-equivalent to the unit vector basis of  $c_0$  (respectively,  $\ell_1$ ), then  $X$  contains an isomorphic copy of  $c_0$  (respectively,  $\ell_1$ ).* Here we show that it is

sufficient to restrict the assumption to those spreading models generated by weakly null block bases.

**Theorem 4.1.** *Let  $(x_i)$  be a normalized weakly null basis for  $X$ . Assume that  $\ell_1$  does not embed into  $X$  and whenever  $(y_i)$  is a normalized weakly null block basis of  $(x_i)$  with spreading model  $(e_i)$ , then  $\|e_1 - e_2\| = 1$ . Then some subsequence of  $(x_i)$  is equivalent to the unit vector basis of  $c_0$ .*

**Remark.** The hypothesis yields that every spreading model  $(e_i)$  generated by a weakly null normalized sequence  $(y_i)$  is 1-equivalent to the unit vector basis of  $c_0$ . Indeed, we may assume  $(y_i)$  is a weakly null normalized block basis of  $(x_i)$ . Then  $\left(\frac{y_{2n-1} - y_{2n}}{\|y_{2n-1} - y_{2n}\|}\right)$  is a weakly null block basis generating the normalized spreading model  $(e_{2n-1} - e_{2n})$  and so  $\|e_1 - e_2 - e_3 + e_4\| = 1$ . By iteration of this argument, 1-spreading and the suppression 1-unconditionality of  $(e_i)$ ,

$$\left\| \sum_{i=1}^n \pm e_i \right\| = 1 \text{ for all } \pm 1 \text{ and all } n.$$

This implies  $(e_i)$  is 1-equivalent to the unit vector basis of  $c_0$ .

As it was pointed out in the introduction this immediately implies the following well known theorem of Elton and Odell [EO].

**Theorem 4.2** (Elton-Odell). *Let  $X$  be an infinite dimensional Banach space. Then there exists  $\lambda > 1$  and an infinite sequence  $(x_i) \subset S_X$  such that  $\|x_i - x_j\| \geq \lambda$  for all  $i \neq j$ .*

For the proof of Theorem 4.1 we need to recall some terminology. A collection  $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$  is called *thin* if there do not exist  $F, G \in \mathcal{F}$  with  $F$  being a proper initial segment of  $G$ .  $\mathcal{F}$  is *large* in  $M \in [\mathbb{N}]^\omega$  if for all  $N \in [M]^\omega$  there exists an initial segment  $F$  of  $N$  with  $F \in \mathcal{F}$ . For a sequence  $(x_i) \subseteq X$  and  $E \in [\mathbb{N}]^{<\omega}$  we set  $x_E = \sum_{i \in E} x_i$ . For a thin  $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$  we let

$$\mathcal{F}^I = \{G \in [\mathbb{N}]^{<\omega} : G \text{ is an initial segment of some } F \in \mathcal{F}\}.$$

**Lemma 4.3.** *Let  $X$  and  $(x_i)$  be as in the hypothesis of Theorem 4.1. Let  $\mathcal{F}$  be a collection of finite subsets of  $\mathbb{N}$  satisfying*

$$(4.1) \quad \sup\{\|x_E\| : E \in \mathcal{F}\} < \infty.$$

*Then there exists  $M \in [\mathbb{N}]^\omega$  so that for all  $E_1 < E_2 < \dots$  with  $E_i \in \mathcal{F} \cap [M]^{<\omega}$  for all  $i \in \mathbb{N}$ , the sequence  $(x_{E_i})$  is weakly null.*

*Proof.* By Elton's near unconditionality theorem [E], there exists  $M \subseteq \mathbb{N}$  such that for some  $C < \infty$  the subsequence  $(x_i)_{i \in M}$  satisfies for all  $E \subseteq F \in [M]^{<\omega}$ ,

$$(4.2) \quad \left\| \sum_{i \in E} \delta_i x_i \right\| \leq C \left\| \sum_{i \in F} \delta_i x_i \right\| \quad \text{for all choices of signs, } \delta_i = \pm 1.$$

Suppose that for some  $E_1 < E_2 < \dots$ ,  $E_i \in \mathcal{F}$  with  $E_i \subseteq M$  for all  $i$ , the sequence  $(x_{E_i})$  is not weakly null. Then after passing to a subsequence, there exists  $\varepsilon > 0$  and  $f \in B_{X^*}$  so that  $f(x_{E_j}) > \varepsilon$  for all  $j \in \mathbb{N}$ . Since  $X$  does not contain  $\ell_1$ , by Rosenthal's  $\ell_1$  theorem and passing to a further subsequence, we may assume that  $(x_{E_j})$  is weak Cauchy.

Let  $z_j = x_{E_{2j-1}} - x_{E_{2j}}$  for  $j \in \mathbb{N}$ . Then  $(z_j)$  is weakly null and moreover by (4.2)

$$n\varepsilon \leq \left\| \sum_{j \in G} x_{E_{2j-1}} \right\| \leq C \left\| \sum_{j \in G} z_j \right\|$$

for all  $|G| = n$ ,  $n \in \mathbb{N}$ . Thus  $(z_j/\|z_j\|)_j$  cannot have a  $c_0$  spreading model since  $\sup_j \|z_j\| < \infty$  by the assumption (4.1).  $\square$

**Lemma 4.4.** *Let  $X$  and  $(x_i)$  be as in the hypothesis of Theorem 4.1. Let  $\mathcal{F}$  be a thin collection of finite subsets of  $\mathbb{N}$  which is large in  $\mathbb{N}$ . Assume that  $(x_{E_i})$  is weakly null for all  $E_1 < E_2 < \dots$  in  $\mathcal{F}$  and*

$$(4.3) \quad \limsup_n \{ \|x_E\| : E \in \mathcal{F}, n \leq E \} = 1.$$

*Then there exists  $N = (n_i) \in [\mathbb{N}]^\omega$  so that  $\mathcal{G}$ , defined by,*

$$\mathcal{G} = \left\{ \bigcup_{i=1}^k E_i : k \in \mathbb{N}, n_k = \min(E_1), E_1 < \dots < E_k, E_i \in \mathcal{F} \cap [N]^{<\omega} \text{ for } i \leq k \right\}$$

*is thin and large in  $N$  and furthermore  $\mathcal{G}$  satisfies (4.3) (when  $\mathcal{G}$  replaces  $\mathcal{F}$ ).*

*Proof.* First we note that by passing to a subsequence, using  $(x_i)$  is normalized and weakly null, we may assume that

$$(4.4) \quad \liminf_{n \rightarrow \infty} \{ \|x_E\| : n \leq E \in [\mathbb{N}]^{<\omega} \} \geq 1.$$

Indeed, for each  $j \in \mathbb{N}$  we may choose  $f_j \in X^*$  with  $\|f_j\| = 1$  such that  $f_j(x_j) = \|x_j\| = 1$ . Fix  $\delta_n \downarrow 0$  and after passing to a subsequence we may assume that  $f_n(x_j) < \delta_n 2^{-j}$  for each  $n < j$ . Thus,  $1 - \delta_n < f_{\min E}(x_E) \leq \|x_E\|$ , for all  $n \leq E$  and (4.4) follows.

Let  $\varepsilon_k \downarrow 0$  and set

$$\mathcal{A}_k = \left\{ M \in [\mathbb{N}]^\omega : \text{if } E_1 < \dots < E_k, E_i \in \mathcal{F} \text{ for } i \leq k, \right. \\ \left. E = \bigcup_{i=1}^k E_i \text{ is an initial segment of } M, \text{ then } \|x_E\| \leq 1 + \varepsilon_k \right\}.$$

Note that as  $\mathcal{F}$  is thin and large in  $\mathbb{N}$ , for each  $M \in [\mathbb{N}]^\omega$  there exists unique  $E_1 < \dots < E_k$  with  $E_i \in \mathcal{F}$  for  $1 \leq i \leq k$  such that  $\cup_{i=1}^k E_i$  is an initial segment of  $M$ . Thus, whether or not a sequence  $M \in [\mathbb{N}]^\omega$  is contained in  $A_k$  depends entirely on a unique initial segment of  $M$ . This makes  $A_k \subset [\mathbb{N}]^\omega$  open in the product topology. Open sets are Ramsey, so we can find subsequences of  $\mathbb{N}$ ,  $M_1 \supset M_2 \supset \dots$ , so that either  $[M_k]^\omega \subseteq \mathcal{A}_k$  or  $[M_k]^\omega \cap \mathcal{A}_k = \emptyset$  for each  $k$ .

By the 1-equivalent to  $c_0$  spreading model hypothesis we must always have  $[M_k]^\omega \subseteq \mathcal{A}_k$ . Let  $N = (n_i)$  be a diagonal sequence,  $(n_i)_{i=k}^\infty \in M_k$  for all  $k$ . Define  $\mathcal{G}$  as in the statement of the lemma with respect to  $N$ .  $\square$

*Proof of Theorem 4.1.* We may assume, using [E] as in the proof of Lemma 4.3, that for some  $C < \infty$ ,

$$(4.5) \quad \|x_E\| \leq C\|x_F\| \text{ for all } E \subseteq F \in [\mathbb{N}]^{<\omega}.$$

We will show that for  $\alpha < \omega_1$  there exists  $N_\alpha = (n_i^\alpha)_i \in [\mathbb{N}]^\omega$  and  $\mathcal{G}_\alpha \subseteq [N_\alpha]^{<\omega}$  so that  $\mathcal{G}_\alpha$  is thin and large in  $N_\alpha$ . Moreover,  $\mathcal{G}_\alpha^I$  has Cantor-Bendixson index  $CB(\mathcal{G}_\alpha^I) \geq \omega^\alpha$  and

$$(4.6) \quad \sup\{\|x_E\| : E \in \mathcal{G}_\alpha, n_k^\alpha \leq E\} \leq 1 + \varepsilon_k$$

where  $\varepsilon_k \downarrow 0$  is fixed. By (4.5) we have that

$$(4.7) \quad \sup\{\|x_E\| : E \in \mathcal{G}_\alpha^I, n_k^\alpha \leq E\} \leq C(1 + \varepsilon_k).$$

Recall that if  $K$  is a countable set then its Cantor-Bendixson index will be a countable ordinal. Thus, the Cantor-Bendixson index of  $\cup_{\alpha < \omega_1} \mathcal{G}_\alpha^I$  is uncountable and it follows that for some  $N = (n_i) \in [\mathbb{N}]^\omega$ ,  $1_N$  is in the pointwise closure of

$$\{1_E : \|x_E\| \leq 2C, E \in [\mathbb{N}]^{<\omega}\} \text{ in } \{0, 1\}^\mathbb{N}.$$

Thus  $\sup_k \|\sum_{i=1}^k x_{n_i}\| < \infty$  and by (4.5) we obtain that  $(x_{n_i})$  is equivalent to the unit vector basis of  $c_0$ .

To begin we use Lemma 4.4 applied to  $\{\{j\} : j \in \mathbb{N}\}$  to obtain  $N_1 = (n_i^1)$  and  $\mathcal{G}_1 = \{E : n_k^1 = \min E, |E| = k, E \subseteq N_1\}$  satisfying (4.6), and note that  $CB(\mathcal{G}_1^I) = \omega$ . Assume  $N_\alpha$  and  $\mathcal{G}_\alpha$  are chosen to satisfy the given conditions. Choose  $\tilde{N}_{\alpha+1} \subseteq N_\alpha$  by Lemma 4.3. Then apply Lemma 4.4 to  $\tilde{N}_{\alpha+1}$  and  $\mathcal{G}_\alpha$  to obtain  $N_{\alpha+1}$  and  $\mathcal{G}_{\alpha+1}$ . By the definition of  $\mathcal{G}_{\alpha+1}$ ,  $CB(\mathcal{G}_{\alpha+1}^I) \geq \omega^{\alpha+1}$ .

If  $\alpha$  is a limit ordinal, choose  $\beta_n \uparrow \alpha$ , and let  $\tilde{N}_\alpha$  be a diagonal sequence of  $(N_{\beta_n})$  so that  $(\tilde{n}_i^\alpha)_{i=k}^\infty \subseteq N_{\beta_k}$  and (4.6) holds. Let  $\tilde{\mathcal{G}}_\alpha = \{E \subseteq \tilde{N}_\alpha : E \subseteq \mathcal{G}_{\beta_n} \text{ for some } n\}$ . Apply Lemmas 4.3 and 4.4 as above.  $\square$

Recall that the  $n$ -dimensional asymptotic structure of  $X$  (with respect to a fixed filter  $\text{cof}(X)$  of finite co-dimensional subspaces of  $X$ ) is the collection  $\{X\}_n$  of normalized basic sequences  $(e_i)_1^n$  satisfying the following. For all  $\varepsilon > 0$  and all  $X_1 \in \text{cof}(X)$  there exists  $x_1 \in S_{X_1}$  such that for all  $X_2 \in \text{cof}(X)$  there exists  $x_2 \in S_{X_2}$  so on so that for all  $X_n \in \text{cof}(X)$  there exists  $x_n \in S_{X_n}$  so that  $(x_i)_1^n$  is  $(1 + \varepsilon)$ -equivalent to  $(e_i)_1^n$  [MMT].  $X$  is asymptotic- $c_0$  if for some  $K < \infty$  and all  $n$ ,  $(e_i)_1^n \in \{X\}_n$  implies that  $(e_i)_1^n$  is  $K$ -equivalent to the unit vector basis of  $\ell_\infty^n$ . In this case  $X^*$  must be separable and the condition can be described in terms of weakly null trees. Namely,  $X$  is asymptotic- $c_0$  (assuming  $X^*$  is separable) if and only if for some  $K < \infty$  for all  $n \in \mathbb{N}$  and all normalized weakly null trees  $(x_\alpha)_{\alpha \in T_n}$  in  $X$ , some branch is  $K$ -equivalent to the unit vector basis of  $\ell_\infty^n$  where  $T_n = \{(k_1, k_2, \dots, k_n) : 1 \leq k_1 < \dots < k_n\}$ .

The following question is open.

**Problem 4.5.** Suppose that  $\ell_1$  does not embed into  $X$  and every spreading model generated by weakly null normalized sequences in  $X$  is equivalent to the unit vector basis of  $c_0$ . Does  $X$  contain an asymptotic- $c_0$  subspace? Does  $X$  contain a subspace  $Y$  with  $Y^*$  separable?

Note that the space  $JH$  constructed by Hagler [H] has non separable dual, does not contain  $\ell_1$  and every weakly null normalized sequence has a subsequence equivalent to the unit vector basis of  $c_0$ . So if the problem has affirmative answer it is necessary to pass to a subspace. We will prove a weaker theorem.

**Theorem 4.6.** *Suppose that a Banach space  $X$  does not contain an isomorphic copy of  $\ell_1$  and every asymptotic model  $(e_i)$  generated by weakly null arrays in  $X$  is equivalent to the unit vector basis of  $c_0$ . Then*

- i)  $X^*$  is separable and thus  $X$  embeds into a space  $Y$  with a shrinking basis  $(y_i)$ .*
- ii)  $X$  is asymptotic- $c_0$  (with respect to the basis  $(y_i)$ ).*

Recall that  $(e_i)$  is an asymptotic model of  $X$ , denoted by  $(e_i) \in AM_w(X)$ , generated by a normalized weakly null array  $(x_j^i)_{i,j \in \mathbb{N}}$  if  $(x_j^i)_{j=1}^\infty$  is weakly null for all  $i \in \mathbb{N}$ , and for some  $\varepsilon_n \downarrow 0$ , all  $n$  and all  $(a_i)_1^n \subseteq [-1, 1]$  and  $n \leq k_1 < k_2 < \dots < k_n$

$$(4.8) \quad \left| \left\| \sum_{i=1}^n a_i x_{k_i}^i \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| \leq \varepsilon_n.$$

Asymptotic models were introduced in [HO]. If every  $(e_i) \in AM_w(X)$  is equivalent to the unit vector basis of  $c_0$ , then there exists  $K < \infty$  so that every  $(e_i) \in AM_w(X)$  is  $K$ -equivalent to the unit vector basis of  $c_0$  [HO].

The hypothesis of the theorem can be contrasted with being asymptotic- $c_0$  as follows. The asymptotic model condition implies that for some  $K$ , and every  $n \in \mathbb{N}$  and normalized weakly null tree  $(x_\alpha)_{\alpha \in T_n}$  of a certain type, some branch is  $K$ -equivalent to the unit vector basis of  $\ell_\infty^n$ . The "certain type" condition is: there exist  $n$  normalized weakly null sequences  $(x_j^i)_{j=1}^\infty$ ,  $1 \leq i \leq n$  so that if  $\alpha = (\ell_1, \dots, \ell_k)$  then  $x_{\ell_k}^k = x_\alpha$ . In short, the successor sequences to each  $|\beta| = k - 1$  are all the same, depending only on  $k$ , for all  $1 \leq k \leq n$ . Theorem 4.6 states that if these specific normalized weakly null trees in  $X$  each have a branch  $K$ -equivalent to the unit vector basis of  $\ell_\infty^n$  then all normalized weakly null trees  $(x_\alpha)_{\alpha \in T_n}$  in  $X$  do as well.

*Proof of Theorem.* i) We first show that  $X^*$  is separable. Assume not. By a result of Stegall [S] for all  $\varepsilon > 0$  there exists  $\Delta \subseteq S_{X^*}$ ,  $\Delta$  is  $w^*$ -homeomorphic to the Cantor set, and a Haar like system  $(x_{n,i}) \subseteq X$ . More precisely, there exist a sequence  $(A_{n,i})$  of subsets of  $\Delta$  for  $n = 0, 1, 2, \dots$  and  $i = 0, 1, \dots, 2^n - 1$  such that  $A_{0,0} = \Delta$  and each  $A_{n,i}$  is the union of disjoint, non-empty, clopen subsets  $A_{n+1,2i}$  and  $A_{n+1,2i+1}$  with  $\lim_{n \rightarrow \infty} \sup_{0 \leq i < 2^n} \text{diam}(A_{n,i}) = 0$ , and Haar functions  $h_{n,i} \subseteq C(\Delta)$  (relative to  $(A_{n,i})$ ) so that

$$h_{2^{n+1}} := 1_{A_{n+1,2i}} - 1_{A_{n+1,2i+1}}, \quad n = 0, 1, \dots, \quad i = 0, 1, \dots, 2^n - 1.$$

Finally,  $(x_{n,i}) \subseteq X$  is a Haar like system (relative to  $(A_{n,i})$ ) if, indexing above Haar functions as  $h_{2^{n+1}} = h_{n,i}$ , we have  $\|x_{n,i}\| \leq 1 + \varepsilon$  for all  $(n, i)$  so that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|x_{n,i}|_{\Delta} - h_{n,i}\|_{C(\Delta)} < \varepsilon.$$

For simplicity in what follows we will assume  $x_{n,i}|_{\Delta} = h_{n,i}$  and ignore the tiny perturbations, and we will refer to the sets  $A_{n,i}$ 's as intervals. We will construct a Rademacher type system  $(r_n)$  from the  $x_{n,i}$ 's and conclude that  $\ell_1 \hookrightarrow X$  to get a contradiction.

Begin with  $r_1 \equiv x_{0,0}$  and suppose  $r_1, \dots, r_n \in \text{span}(x_{k,i})$  have been constructed so that for each choice of signs  $(\varepsilon_i)_1^n$  there is an interval  $I$  in  $\Delta$  on which for  $i \leq n$ ,  $r_i|_I = \varepsilon_i$ . Fix such an  $I$  and consider the subsequence  $(x_{k,l})$  that is 'supported' on  $I$ , that is,  $\text{supp} x_{k,l}|_{\Delta} \subseteq I$ . A further subsequence has pairwise disjoint support and a further subsequence of that is weak Cauchy. Thus the corresponding difference sequence is weakly null. The difference sequence has norm in  $[1, 2]$  and take values  $-1, 0, 1$  on  $I$ .

Now consider that this has been done for all  $2^n$  such  $I$ 's. Label the sequences as  $(d_j^i)_{j=1}^\infty$  for  $i \leq 2^n$ . By the asymptotic model hypothesis (applied to the weakly null array  $(d_j^i)_{j=1}^\infty$ ,  $i \leq 2^n$ ) we can form  $r_{n+1} = \sum_{i=1}^{2^n} d_{j_i}^i$  with  $1 \leq \|r_{n+1}\| \leq 2K$ .

If  $(a_n)_{n=1}^N \subseteq \mathbb{R}$  we choose an interval  $I \subseteq \Delta$  such that  $r_n|_I = \text{sign}(a_n)$  for all  $1 \leq n \leq N$ . Thus,  $\|\sum_n a_n r_n\| \geq \|\sum_n a_n r_n|_I\|_{C(I)} = \sum_n |a_n|$ . Thus  $(r_n)$  is a seminormalized sequence which dominates the unit vector basis of  $\ell_1$ . This contradicts that  $\ell_1$  does not embed into  $X$  and hence  $X^*$  must be separable. By Zippin's theorem  $X$  embeds into a space  $Y$  with a shrinking basis  $(y_i)$ .

ii) We proceed to show that  $X$  is an asymptotic- $c_0$  space with respect to the basis  $(y_i)$ . We need to prove that there exists a constant  $C$  such that for all  $n$  every asymptotic space  $(e_i)_1^n \in \{X\}_n$  is  $C$ -equivalent to the unit vector basis of  $\ell_\infty^n$ . If  $(e_i)_1^n \in \{X\}_n$  then also  $(\varepsilon_i e_i)_1^n \in \{X\}_n$  for all sequence of signs  $(\varepsilon_i)_1^n$ . Therefore, it is sufficient to show that there exists  $C$  such that for all  $n \in \mathbb{N}$  and for every asymptotic space  $(e_i)_1^n \in \{X\}_n$  we have  $\|\sum_{i=1}^n e_i\| \leq C$ .

Suppose this is not the case. Then for all  $C \geq 1$  there exists  $n$  and a normalized asymptotic tree (i.e., countably branching block tree)  $(x_\alpha)_{\alpha \in T_n}$  in  $X$  so that for every branch  $\beta = (x_i)_{i=1}^n$  of  $(x_\alpha)_{\alpha \in T_n}$  there exists  $f_\beta \in S_{X^*}$  with  $f_\beta(\sum_{i=1}^n x_i) > C$ .

We will construct weakly null seminormalized sequences  $(y_i^1)_{i \geq 1}, (y_i^2)_{i \geq 2}, \dots, (y_i^n)_{i \geq n}$  from the linear combinations of carefully chosen nodes of  $(x_\alpha)_{\alpha \in T_n}$  so that, after passing to subsequences in each and relabeling, the array  $\{y_i^k : 1 \leq k \leq n, i \geq 1\}$  satisfies  $\|y_i^k\| \leq K$  for all  $1 \leq k \leq n, i \geq 1$  and  $\|\sum_{k=1}^n y_{i_k}^k\| > C$  for all  $i_1 < \dots < i_n$ . This will contradict the assumption that all asymptotic models generated by weakly null arrays are  $K$ -equivalent to the unit vector basis of  $c_0$ .

We first describe a general procedure of extracting an array of weakly null sequences from a tree. The actual array will be obtained by applying this procedure to a carefully pruned tree (using our assumptions) that we describe later.

**Extracting arrays from trees.** The main idea of the construction is that each  $y_i^k$  is chosen to be a linear combination of nodes of  $(x_\alpha)_{\alpha \in T_n}$  from the  $k$ th level so that for every  $i_1 < \dots < i_n$  the union of the supports (with respect to the tree  $T_n$ ) of  $y_{i_1}^1, \dots, y_{i_n}^n$  contains a (unique) full branch of the tree  $T_n$ .

Let  $(x_\alpha)_{\alpha \in T_n}$  be the tree above. For  $(i_1, \dots, i_k) \in T_n$  we label the node  $x^k(i_1, \dots, i_k) := x_{(i_1, \dots, i_k)}$ . The superscript (which denotes the  $k$ th level in the tree) is redundant but we keep it for the sake of clarity.

We will construct the desired  $n$ -array so that all rows  $(y_i^k)_{i \geq k}$  and all diagonal sequences  $(y_{i_k}^k)_{k=1}^n, i_1 < \dots < i_n$  are blocks sequences. We will often prune the tree  $(x_\alpha)_{\alpha \in T_n}$  by deleting nodes and then relabel the remaining nodes. The pruned tree will always be a full (sub)-tree. Moreover, to ease the notation for later constructions we will relabel the



full subtree to match the indices so that the resulting array will have the property that for every diagonal sequence  $(y_{i_k}^k)_{k=1}^n$ ,  $i_1 < \dots < i_n$  the corresponding unique full branch is  $(x^1(i_1), x^2(i_1, i_2), \dots, x^n(i_1, i_2, \dots, i_n))$ .

The array is to be labelled as follows and constructed in diagonal order.

$$\begin{array}{cccccc} y_1^1 & y_2^1 & y_3^1 & y_4^1 & y_5^1 & \cdots \\ & y_2^2 & y_3^2 & y_4^2 & y_5^2 & \cdots \\ & & \ddots & \ddots & \ddots & \\ & & & y_n^n & y_{n+1}^n & \cdots \end{array}$$

Let  $y_i^1 = x^1(i)$  for all  $i \geq 1$ . So  $(y_i^1)_i$  is the sequence of initial nodes of  $(x_\alpha)_{\alpha \in T_n}$ . For the first diagonal sequence  $(y_1^1, \dots, y_n^n)$  take the leftmost branch of  $(x_\alpha)_{\alpha \in T_n}$ , that is,

$$(4.9) \quad y_1^1 = x^1(1), \quad y_2^2 = x^2(1, 2), \quad \dots, \quad y_n^n = x^n(1, \dots, n).$$

The node  $y_3^2$  will be a sum of two successors to the nodes  $x^1(1)$  and  $x^1(2)$  that comprise  $y_1^1$  and  $y_2^1$ , respectively. To do this we pick  $i_1 > 2$  and  $i_2 > 2$  large enough so that  $x^2(1, i_1)$  and  $x^2(2, i_2)$  are supported after  $x^1(2)$  (and hence after  $x^1(1)$ ). Delete the nodes  $x^2(1, j)$  for  $2 < j < i_1$  and the nodes  $x^2(2, j)$  for  $3 \leq j < i_2$  and relabel the remaining sequences so that the chosen nodes becomes  $x^2(1, i_1) = x^2(1, 3)$  and  $x^2(2, i_2) = x^2(2, 3)$ . Put

$$(4.10) \quad y_3^2 = x^2(1, 3) + x^2(2, 3).$$

We proceed in similar fashion so that each vector  $y_j^k$  of the  $k$ th row ( $j > k > 1$ ) is defined as a sum of nodes from the  $k$ th level of the tree  $(x_\alpha)_{\alpha \in T_n}$  and which are successors to the nodes that comprise the previously chosen vectors  $y_{k-1}^{k-1}, y_k^{k-1}, \dots, y_{j-1}^{k-1}$ . We pick the nodes so that the block conditions are satisfied and relabel the tree after deleting finitely many nodes. Thus  $y_4^3$  is a sum of nodes successor to the nodes of  $y_2^2$  and  $y_3^2$  and after relabeling the nodes it becomes

$$(4.11) \quad y_4^3 = x^3(1, 2, 4) + x^3(1, 3, 4) + x^3(2, 3, 4).$$

In general, suppose that  $y_j^{k-1}$  for  $k-1 \leq j < i$  and  $k \leq n$  are defined. Let

$$y_j^{k-1} = x^{k-1}(\bar{t}_1) + x^{k-1}(\bar{t}_2) + \dots = \sum_{m \in A_j^{k-1}} x^{k-1}(\bar{t}_m) \quad \text{for some } A_j^{k-1} \subset \mathbb{N}$$

be the enumeration of the (finitely many) nodes comprising  $y_j^{k-1}$ 's in the order they appear and where each  $\bar{t}_s$  is a  $k-1$ -tuple with maximal entry  $j$ .

We denote concatenation by  $(a_1, \dots, a_n) \frown a_{n+1} = (a_1, \dots, a_n, a_{n+1})$ . By passing to subsequences and relabeling the sequences of successor nodes

$$(x^k(\bar{t}_1 \frown l))_{l \geq j}, (x^k(\bar{t}_2 \frown l))_{l > \geq j}, (x^k(\bar{t}_3 \frown l))_{l \geq j}, \dots$$

we may assume each of these vectors are supported after the previously chosen ones. We define  $y_i^k$  as a sum of successors to the nodes comprising  $y_{k-1}^{k-1}, \dots, y_{i-1}^{k-1}$ . That is, we put

$$(4.12) \quad y_i^k = \sum_{j=k-1}^{i-1} \sum_{m \in A_j^{k-1}} x^k(\bar{t}_m \frown i).$$

Note that  $j$  is the maximal entry of  $\bar{t}_m \in A_j^{k-1}$  and hence  $x^k(\bar{t}_m \frown i)$  is a successor of  $x^k(\bar{t}_m)$  as  $j < i$ .

This completes the construction of the array. It follows that the support of any diagonal sequence  $(y_{i_k}^k)_{k=1}^n, i_1 < \dots < i_n$  contains the unique full branch  $(x^1(i_1), x^2(i_1, i_2), \dots, x^n(i_1, i_2, \dots, i_n))$  as desired.

**Pruning the tree.** For notational convenience we will denote branches

$$\beta = (x^1(i_1), x^2(i_1, i_2), \dots, x^n(i_1, i_2, \dots, i_n))$$

of the tree by  $\beta = (i_1, i_2, \dots, i_n)$ . From the construction the support of (the sum of) each sequence  $y_{i_1}^1, \dots, y_{i_n}^n$  consists of the unique full branch  $\beta = (i_1, i_2, \dots, i_n)$  and other off-branch nodes whose numbers add up quickly as  $i_n$  gets large. By our assumption there is a branch functional  $f_\beta$  so that

$$(4.13) \quad f_\beta \left( \sum_{k=1}^n x^k(i_1, \dots, i_k) \right) > C.$$

Our goal here is to show that for all  $\varepsilon > 0$  we can prune the tree so that the array (with respect to the pruned tree) satisfies

$$(4.14) \quad \|y_i^k\| \leq K + \varepsilon, \text{ for all } 1 \leq k \leq n, i \geq k,$$

and

$$(4.15) \quad f_\beta \left( \sum_{k=1}^n y_{i_k}^k \right) \geq C - \varepsilon.$$

Let  $\varepsilon > 0$ . Fix  $(\varepsilon_k)_{k=1}^n$  so that  $\sum_{k=1}^n \varepsilon_k < \varepsilon$ . Let  $(x_\alpha)_{\alpha \in T_n}$  be a full subtree satisfying block conditions described in the above construction. That is, every sequence of successor nodes of  $(x_\alpha)_{\alpha \in T_n}$  is a block basis and whenever  $y_i^k$  is defined as in (4.25) the sequences  $(y_{i_1}^1, \dots, y_{i_n}^n)$  are blocks as well.

As before we will proceed in diagonal order (of the array). Let  $y_i^1 = x^1(i)$  for all  $i \geq 1$ . For the first diagonal sequence  $(y_1^1, \dots, y_n^n)$  again we take the leftmost branch of  $(x_\alpha)_{\alpha \in T_n}$ , that is,

$$(4.16) \quad y_1^1 = x^1(1), y_2^2 = x^2(1, 2), \dots, y_n^n = x^n(1, \dots, n).$$

The condition (4.14) is clearly satisfied since the tree is normalized and the condition (4.15) follows from the assumption (4.13).

We wish to define  $y_3^2$  as in (4.10). This will require two steps. First consider the sequences of level 2 successor nodes

$$(x^2(1, l))_{l \geq 3}, (x^2(2, l))_{l \geq 3}.$$

By our main assumption, the array formed by these sequences can be refined to generate an asymptotic model  $K$ -equivalent to the unit vector basis of  $\ell_\infty^2$ . Thus by passing to subsequences, relabeling and ignoring tiny perturbations we can assume that for all  $3 \leq l_1 < l_2$ ,

$$(4.17) \quad \|x^2(1, l_1) + x^2(2, l_2)\| \leq K.$$

This will ensure that whenever  $y_3^2$  is defined as in (4.10) the condition (4.14) is satisfied. The second refinement towards ensuring (4.15) is somewhat more complicated.

Consider again the sequences of successor nodes  $(x^2(1, l))_{l \geq 3}$  and  $(x^2(2, l))_{l \geq 3}$ . By the main assumption each of these sequences generate spreading models which are  $K$ -equivalent to the unit vector basis of  $c_0$ . Fix  $N \geq 1 + K^2/\varepsilon_1^2 + 2K/\varepsilon_1$ . By passing to subsequences and relabeling we can assume that both  $(x^2(1, l))_{l=3}^{N+3}$  and  $(x^2(2, l))_{l=3}^{N+3}$  are  $K$ -equivalent to the unit vector basis of  $\ell_\infty^N$ . For every branch  $\beta = (i_1, \dots, i_n)$  of  $T_n$  we let  $f_{(i_1, i_2, \dots, i_n)}$  denote the corresponding branch functional satisfying (4.13). For each  $3 \leq l \leq N + 3$ ,  $f_{(1, l) \wedge \bar{j}}$  and  $f_{(2, l) \wedge \bar{j}}$  are the branch functionals for branches extending  $(1, l)$  and  $(2, l)$  respectively, where  $\bar{j}$  is an  $(n - 2)$ -tuple. We stabilize the values of these functionals on the chosen nodes. That is, by passing to subsequences and ignoring tiny perturbations we can assume that for all  $\bar{j}, \bar{j}'$  we have

$$f_{(1, l) \wedge \bar{j}}(x^2(2, t)) = f_{(1, l) \wedge \bar{j}'}(x^2(2, t)) \text{ and } f_{(2, l) \wedge \bar{j}}(x^2(1, t)) = f_{(2, l) \wedge \bar{j}'}(x^2(1, t)),$$

for all  $3 \leq l, t \leq N + 3$ .

**Claim.** There exist  $3 \leq l_1, l_2 \leq N + 3$  so that for all  $\bar{j}$

$$(4.18) \quad |f_{(1, l_1) \wedge \bar{j}}(x^2(2, l_2))| < \varepsilon_1 \text{ and } |f_{(2, l_2) \wedge \bar{j}}(x^2(1, l_1))| < \varepsilon_1.$$

For any functional  $f$  of norm at most 1 and sequence  $(x_t)_{t=1}^n$  which is  $K$ -equivalent to the unit vector basis of  $\ell_\infty^n$  there is a sequence of signs  $\delta_t = \pm 1$  so that

$$(4.19) \quad \sum_{t=1}^n |f(x_t)| = \left| f\left(\sum_{t=1}^n \delta_t x_t\right) \right| \leq K.$$

It follows that the cardinality  $|\{t : |f(x_t)| \geq \varepsilon_1\}| \leq K/\varepsilon_1$ . Thus for each  $l$  and  $\bar{j}$ ,

$$|A_l| := \left| \{t : |f_{(1,l)\bar{j}}(x^2(2,t))| < \varepsilon_1\} \right| \geq N - K/\varepsilon_1.$$

Then for any  $B \subset \{3, \dots, N+3\}$  with  $K/\varepsilon_1 + 1 \leq |B| < K/\varepsilon_1 + 2$  we have

$$\left| \bigcap_{l \in B} A_l \right| \geq 1.$$

Indeed,  $N - |B|K/\varepsilon_1 \geq N - K^2/\varepsilon_1^2 - 2K/\varepsilon_1 \geq 1$ . Fix such a subset  $B$  and let  $l_2 \in \bigcap_{l \in B} A_l$ . Then  $|f_{(1,l)\bar{j}}(x^2(2,l_2))| < \varepsilon_1$  for all  $l \in B$ . Now consider the functionals  $f_{(2,l_2)\bar{j}}$ . Since  $(x^2(1,l))_{l \in B}$  is  $K$ -equivalent to the unit vector basis of  $\ell_\infty^{|B|}$  and  $|B| \geq K/\varepsilon_1 + 1$ , by a similar argument as above, there is  $l_1 \in B$  such that  $|f_{(2,l_2)\bar{j}}(x^2(1,l_1))| < \varepsilon_1$ , proving the claim.

Now we relabel the nodes as  $x^2(1,l_1) = x^2(1,3)$  and  $x^2(2,l_2) = x^2(2,3)$  (by deleting finitely many nodes) and put

$$(4.20) \quad y_3^2 = x^2(1,3) + x^2(2,3).$$

At this stage the pruned tree has the following *gap property* of the branch functionals  $f_{(1,3)\bar{j}}$  and  $f_{(2,3)\bar{j}}$ .

$$\begin{aligned} f_{(1,3)\bar{j}}\left((y_1^1 + y_3^2) - (x^1(1) + x^2(1,3))\right) &= f_{(1,3)\bar{j}}\left(x^2(2,3)\right) < \varepsilon_1, \\ f_{(2,3)\bar{j}}\left((y_2^1 + y_3^2) - (x^1(2) + x^2(2,3))\right) &= f_{(2,3)\bar{j}}\left(x^2(1,3)\right) < \varepsilon_1 \end{aligned}$$

We have that  $x^1(1)$  and  $x^2(1,3)$  are the nodes on the branch of  $(1,3) \bar{j}$ . Thus the first above inequality states that the branch functional  $f_{(1,3)\bar{j}}$  is small on the off branch part of  $y_1^1 + y_3^2$ , and the second above inequality states that the branch functional  $f_{(2,3)\bar{j}}$  is small on the off branch part of  $y_2^1 + y_3^2$ . This will be important for us as the branch functionals  $f_\beta$  are defined to be large on their branch. We will eventually be able to obtain (4.15) by showing that  $f_\beta$  is greater than  $C$  on the branch part of  $\sum_{k=1}^n y_{i_k}^k$  and  $f_\beta$  is smaller than  $\varepsilon$  on the off branch part of  $\sum_{k=1}^n y_{i_k}^k$  where  $\beta = (i_1, \dots, i_n)$ .

For the sake of clarity we also show how to define  $y_4^3$  as in (4.11) before proceeding with the inductive step. The array formed by the sequences of level 3 successor nodes

$$(x^3(1,2,l))_{l \geq 4}, (x^3(1,3,l))_{l \geq 4}, (x^3(2,3,l))_{l \geq 4}$$

can be refined to generate an asymptotic model  $K$ -equivalent to the unit vector basis of  $\ell_\infty^3$ . Thus by passing to subsequences, relabeling and ignoring tiny perturbations we get that for

all  $4 \leq l_1 < l_2 < l_3$ ,

$$(4.21) \quad \|x^3(1, 2, l_1) + x^3(1, 3, l_2) + x^3(2, 3, l_3)\| \leq K.$$

This will ensure the condition (4.14).

The second refinement is done as before. Fix a large  $N = N(K, \varepsilon_2/2)$  and using the  $c_0$  spreading models assumption pick sequences  $(x^3(1, 2, l))_{l=4}^{N+4}$ ,  $(x^3(1, 3, l))_{l=4}^{N+4}$ , and  $(x^3(2, 3, l))_{l=4}^{N+4}$  that are  $K$ -equivalent to the unit vector basis of  $\ell_\infty^N$ . Refine the tree by passing to subsequences of the successors of these so that the branch functionals  $f_{(1,2,l)\bar{j}}$ ,  $f_{(1,3,l)\bar{j}}$ , and  $f_{(2,3,l)\bar{j}}$  are stabilized. That is, their values on the chosen nodes are independent of  $\bar{j}$ . Then a similar combinatorial argument as before yields (see the Gap lemma below) a node from each sequence which we relabel as  $x^3(1, 2, 4)$ ,  $x^3(1, 3, 4)$ , and  $x^3(2, 3, 4)$  so that

$$|f_{(1,2,4)\bar{j}}(x^3(1, 3, 4))| < \varepsilon_2/2, \quad |f_{(1,2,4)\bar{j}}(x^3(2, 3, 4))| < \varepsilon_2/2,$$

$$|f_{(1,3,4)\bar{j}}(x^3(1, 2, 4))| < \varepsilon_2/2, \quad |f_{(1,3,4)\bar{j}}(x^3(2, 3, 4))| < \varepsilon_2/2, \quad \text{and}$$

$$|f_{(2,3,4)\bar{j}}(x^3(1, 2, 4))| < \varepsilon_2/2, \quad |f_{(2,3,4)\bar{j}}(x^3(1, 3, 4))| < \varepsilon_2/2.$$

Let

$$y_4^3 = x^3(1, 2, 4) + x^3(1, 3, 4) + x^3(2, 3, 4).$$

Then the branch functionals through these nodes satisfy the desired gap properties: For  $1 \leq t_1 < t_2 < 4$ , denoting  $\mathbf{x}_{(t_1, t_2, 4)} = x^1(t_1) + x^2(t_1, t_2) + x^3(t_1, t_2, 4)$  and  $\mathbf{y}_{(t_1, t_2, 4)} = y_{t_1}^1 + y_{t_2}^2 + y_4^3$  we have

$$\begin{aligned} \left| f_{(1,2,4)\bar{j}}(\mathbf{y}_{(1,2,4)} - \mathbf{x}_{(1,2,4)}) \right| &\leq \left| f_{(1,2,4)\bar{j}}(x^3(1, 3, 4)) \right| + \left| f_{(1,2,4)\bar{j}}(x^3(2, 3, 4)) \right| \\ &< \varepsilon_2/2 + \varepsilon_2/2, \end{aligned}$$

$$\begin{aligned} &\left| f_{(1,3,4)\bar{j}}(\mathbf{y}_{(1,3,4)} - \mathbf{x}_{(1,3,4)}) \right| \\ &\leq \left| f_{(1,3,4)\bar{j}}(x^2(2, 3)) \right| + \left| f_{(1,3,4)\bar{j}}(x^3(1, 2, 4)) \right| + \left| f_{(1,3,4)\bar{j}}(x^3(2, 3, 4)) \right| \\ &< \varepsilon_1 + \varepsilon_2/2 + \varepsilon_2/2, \end{aligned}$$

and

$$\begin{aligned} &\left| f_{(2,3,4)\bar{j}}(\mathbf{y}_{(2,3,4)} - \mathbf{x}_{(2,3,4)}) \right| \\ &\leq \left| f_{(2,3,4)\bar{j}}(x^2(1, 3)) \right| + \left| f_{(2,3,4)\bar{j}}(x^3(1, 2, 4)) \right| + \left| f_{(2,3,4)\bar{j}}(x^3(1, 3, 4)) \right| \\ &< \varepsilon_1 + \varepsilon_2/2 + \varepsilon_2/2, \end{aligned}$$

As before, the idea is that  $f_\beta$  is large on the branch part of  $y_{t_1}^1 + y_{t_2}^2 + y_4^3$  and is small on the off branch part where  $\beta = (t_1, t_2, 4)$ .

We now proceed inductively. Suppose that for  $k-1 \leq j < i$  and  $k \leq n$ ,

$$y_j^{k-1} = \sum_{m \in A_j^{k-1}} x^{k-1}(\bar{t}_m)$$

are defined where  $x^{k-1}(\bar{t}_m)$  are  $(k-1)$ -level nodes and  $A_j^{k-1} \subset \mathbb{N}$  is finite. For each  $\bar{t}_m = (t_1, \dots, t_{k-1})$  denote the sum of the initial segment of a diagonal sequence of the array constructed thus far by

$$\mathbf{y}_{\bar{t}_m} = \sum_{i=1}^{k-1} y_{t_i}^i,$$

and the sum of the initial segment of the tree by

$$\mathbf{x}_{\bar{t}_m} = \sum_{i=1}^{k-1} x^i(t_1, \dots, t_{k-1}).$$

For the induction hypothesis we also assume that the branch functionals  $f_{\bar{t}_m \hat{\ } \bar{j}}$  for the branches whose initial segments are  $\bar{t}_m$  satisfy the gap property:

$$(4.22) \quad \left| f_{\bar{t}_m \hat{\ } \bar{j}}(\mathbf{y}_{\bar{t}_m} - \mathbf{x}_{\bar{t}_m}) \right| < \sum_{i=1}^{k-1} \varepsilon_i.$$

Consider the array formed by the sequences of successor nodes

$$(x^k(\bar{t}_1 \hat{\ } l))_{l > \max \bar{t}_1}, (x^k(\bar{t}_2 \hat{\ } l))_{l > \max \bar{t}_2}, \dots, (x^k(\bar{t}_M \hat{\ } l))_{l > \max \bar{t}_M}$$

for  $m \in \bigcup_{j=k-1}^{i-1} A_j^{k-1}$  and where  $M = |\bigcup_{j=k-1}^{i-1} A_j^{k-1}|$ . The array is formed in the order the nodes appear in the support of  $y_{k-1}^{k-1}, \dots, y_{i-1}^{k-1}$ . By the main assumption the array generates an asymptotic model  $K$ -equivalent to the unit vector basis of  $\ell_\infty^M$ . Thus by passing to subsequences and relabeling we can assume that for all  $\max_{1 \leq m \leq M} \max \bar{t}_m < l_1 < l_2 < \dots < l_M$ ,

$$(4.23) \quad \left\| \sum_{m=1}^M x^k(\bar{t}_m \hat{\ } l_m) \right\| \leq K.$$

Fix a large  $N = N(K, \varepsilon_k/M)$  (determined by the lemma below). For each  $1 \leq m \leq M$ , using the fact that every sequence of successor nodes generates a  $c_0$  spreading model, pick  $(x^k(\bar{t}_m \hat{\ } l))_{l \in B_m}$ ,  $|B_m| = N$ , which is  $K$ -equivalent to the unit vector basis of  $\ell_\infty^N$ . For all  $m$  and  $l \in B_m$ , by passing to a subsequence of the successors  $(x^{k+1}(\bar{t}_m \hat{\ } l \hat{\ } j))_j$  of  $x^k(\bar{t}_m \hat{\ } l)$

we can assume that all the branch functionals  $f_{\bar{t}_m \wedge l \wedge \bar{j}}$  are stabilized on the chosen nodes. That is, for all  $\bar{j}$  and  $\bar{j}'$ , ignoring tiny perturbations, we have

$$f_{\bar{t}_m \wedge l \wedge \bar{j}}(x^k(\bar{t}_{m'} \wedge l')) = f_{\bar{t}_m \wedge l \wedge \bar{j}'}(x^k(\bar{t}_{m'} \wedge l'))$$

for all  $m \neq m'$ , and  $l \in B_m, l' \in B_{m'}$ . (Note: If  $k = n$ , the last level of the tree, then all the branch functionals are already determined.)

**Claim.** For all  $1 \leq m \leq M$  there exist  $l_m \in B_m$  such that for all  $m \neq m'$

$$(4.24) \quad \left| f_{\bar{t}_m \wedge l_m \wedge \bar{j}}(x^k(\bar{t}_{m'} \wedge l_{m'})) \right| < \varepsilon_k/M.$$

This is consequence of the following combinatorial lemma (for  $\varepsilon = \varepsilon_k/M$ .)

**Gap lemma.** Let  $\varepsilon > 0$ ,  $M \in \mathbb{N}$ . Then there exists  $N = N(\varepsilon, M, K)$  such that given sequences  $(x_l^1)_{l=1}^N, \dots, (x_l^M)_{l=1}^N$  each  $K$ -equivalent to the unit vector basis of  $\ell_\infty^N$  and functionals  $(f_l^1)_{l=1}^N, \dots, (f_l^M)_{l=1}^N$  of norm at most 1 there exists  $l_1, \dots, l_M$  such that

$$\left| f_{l_j}^j(x_{l_i}^i) \right| < \varepsilon, \text{ for all } i \neq j.$$

*Proof.* The proof is by induction on  $M$ . For the base case  $M = 2$  we prove the following which is a slight generalization of (4.18): For all  $N_0 \in \mathbb{N}$  there exists  $N = N(N_0, \varepsilon, K)$  so that whenever  $(x_l^1)_{l=1}^N, (x_l^2)_{l=1}^N$  and  $(f_l^1)_{l=1}^N, (f_l^2)_{l=1}^N$  are as in the statement there exist  $A_1, A_2 \subset \{1, \dots, N\}$  with  $|A_1|, |A_2| \geq N_0$  such that for all  $j \in A_1$  and  $i \in A_2$  we have  $|f_j^1(x_i^2)| < \varepsilon$  and  $|f_i^2(x_j^1)| < \varepsilon$ .

Fix  $N \geq N_0(1 + K/\varepsilon + K^2/\varepsilon^2)$ . For any functional  $f$  of norm at most 1 and sequence  $(x_i)_1^n$   $K$ -equivalent to the unit vector basis  $\ell_\infty^n$  we have, by (4.19),  $|\{i : |f(x_i)| \geq \varepsilon\}| \leq K/\varepsilon$ . Thus for  $N_0(1 + K/\varepsilon) \leq N_1 \leq N_0(1 + K/\varepsilon) + 1$ ,

$$\left| \bigcap_{l=1}^{N_1} \left\{ 1 \leq i \leq N : |f_l^1(x_i^2)| < \varepsilon \right\} \right| \geq N - N_1 K/\varepsilon \geq N_0.$$

Let  $A_2$  be a subset of  $\bigcap_{l=1}^{N_1} \{1 \leq i \leq N : |f_l^1(x_i^2)| < \varepsilon\}$  with cardinality  $N_0$  and we have

$$|A_1| = \left| \bigcap_{l \in A_2} \left\{ 1 \leq i \leq N_1 : |f_l^2(x_i^1)| < \varepsilon \right\} \right| \geq N_1 - N_0 K/\varepsilon \geq N_0,$$

as desired.

For the induction suppose that for all  $N_0 \in \mathbb{N}$  there exists  $N$  and  $A_1, \dots, A_m$  with  $|A_i| \geq N_0$  so that for all  $l_i \in A_i$

$$\left| f_{l_j}^j(x_{l_i}^i) \right| < \varepsilon, \text{ for all } 1 \leq i \neq j \leq m.$$

Fix  $N_0 \geq 1 + K/\varepsilon + K^2/\varepsilon^2$  and apply the argument in the base case for the pairs  $(x_l^i)_{l \in A_i}, (x_l^{m+1})_{l=1}^{N_0}$  and  $(f_l^i)_{l \in A_i}, (f_l^{m+1})_{l=1}^{N_0}$  for  $1 \leq i \leq m$  to get the desired  $(m+1)$ -tuple  $l_1, \dots, l_{m+1}$  so that

$$\left| f_{l_j}^j(x_{l_i}^i) \right| < \varepsilon, \text{ for all } 1 \leq i \neq j \leq m+1.$$

□

Consider  $l_1, \dots, l_M$  from the Claim. We discard the nodes  $x^l(\bar{t}_m \wedge l)$ ,  $l \in B_m$  and  $l \neq l_m$  and relabel the rest so that for all  $1 \leq m \leq M$ ,  $x^k(\bar{t}_m \wedge l_m) = x^k(\bar{t}_m \wedge i)$  where  $i = \max_m \max \bar{t}_m + 1$  and put

$$(4.25) \quad y_i^k = \sum_{j=k-1}^{i-1} \sum_{m \in A_j^{k-1}} x^k(\bar{t}_m \wedge i).$$

By (4.23)  $\|y_i^k\| \leq K$ . By the induction hypothesis (4.22) and the Claim (4.24) we have

$$(4.26) \quad \left| f_{(\bar{t}_m, i, \bar{j})} \left( (y_{\bar{t}_m}^k + y_i^k) - (x_{\bar{t}_m}^k + x^k(\bar{t}_m \wedge i)) \right) \right| < \sum_{i=1}^{k-1} \varepsilon_i + \sum_{m=1}^M \varepsilon_k/M = \sum_{i=1}^k \varepsilon_i$$

for all  $1 \leq m \leq M$  and  $\bar{j}$ , as desired. This concludes the construction of the array.

Now let  $\beta = (i_1, \dots, i_n)$  be arbitrary. Then by the construction and our main assumption we have

$$f_\beta \left( \sum_{k=1}^n y_{i_k}^k \right) \geq f_\beta \left( \sum_{k=1}^n x^k(i_1, \dots, i_k) \right) - \left| f_\beta \left( \sum_{k=1}^n x^k(i_1, \dots, i_k) - \sum_{k=1}^n y_{i_k}^k \right) \right| \geq C - \sum_{k=1}^n \varepsilon_k > C - \varepsilon.$$

The proof is completed.

□

## REFERENCES

- [AH] S. P. Argyros and R. G. Haydon, *A hereditarily indecomposable  $\mathcal{L}_\infty$ -space that solves the scalar-plus-compact problem*, Acta Math. 206 (2011), no. 1, 1-54.
- [AJO] D. Alspach, R. Judd, and E. Odell, *The Szlenk index and local  $l_1$ -indices*, Positivity 9 (2005), no. 1, 1-44.
- [AM] S. P. Argyros and P. Motakis, *A hereditarily indecomposable Banach space with rich spreading model structure*, Israel J. Math. 203 (2014), no. 1, 341-387.
- [AMS] S. P. Argyros, P. Motakis, and B. Sari, in preparation.
- [BD] J. Bourgain and F. Delbaen, *A class of special  $\mathcal{L}_\infty$  spaces*, Acta Math. 145 (1980), no. 3-4, 155-176.
- [BHO] S. F. Bellenot, R. Haydon, and E. Odell, Edward, *Quasi-reflexive and tree spaces constructed in the spirit of R. C. James*, Contemp. Math., 85, Amer. Math. Soc., Providence, RI, 1989.
- [BP] C. Bessaga and A. Pelczynski, *On subspaces of a space with an absolute basis*, Bull. Acad. Polon. Sci. Sr. Sci. Math. Astr. Phys. 6 1958 313-315.
- [BS] A. Brunel and L. Sucheston, *Equal signs additive sequences in Banach spaces*, J. Functional Anal. 21 (1976), no. 3, 286-304.
- [E] J. Elton, *Weakly null normalized sequences in Banach spaces*, Ph.D. Thesis, Yale University, New Haven, Ct. (1978).



- [EO] J. Elton and E. Odell, *The unit ball of every infinite-dimensional normed linear space contains a  $(1 + \varepsilon)$ -separated sequence*, Colloq. Math. 44 (1981), no. 1, 105-109.
- [FOS] D. Freeman, E. Odell, and Th. Schlumprecht, *The universality of  $\ell_1$  as a dual space*, Math. Ann. 351 (2011), no. 1, 149-186.
- [H] J. Hagler, *A counterexample to several questions about Banach spaces*, Studia Math. 60 (1977), 289-308.
- [HO] L. Halbeisen and E. Odell, *On asymptotic models in Banach spaces*, Israel J. Math. 139 (2004), 253-291.
- [LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 92, Springer-Verlag, Berlin (1977).
- [MMT] B. Maurey, V. D. Milman, and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, Geometric aspects of functional analysis (Israel, 1992-1994), 149-175, Oper. Theory Adv. Appl., 77, Birkhuser, Basel, 1995.
- [O] E. Odell, *Stability in Banach spaces*, Extracta Math. 17 (2002), no. 3, 385-425.
- [OS] E. Odell and Th. Schlumprecht, *A problem on spreading models*, J. Funct. Anal. 153 (1998), no. 2, 249-261
- [R] H. P. Rosenthal, *A characterization of Banach spaces containing  $\ell_1$* , Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411-2413.
- [Ro] H. P. Rosenthal *A Characterization of Banach Spaces Containing  $c_0$*  Journal of the American Mathematical Society, Vol. 7, No. 3. (Jul., 1994), pp. 707-748.
- [S] C. Stegall, *The Radon-Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc. 206 (1975), 213-223.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SAINT LOUIS UNIVERSITY , ST LOUIS, MO 63103 USA

*E-mail address:* `dfreema7@slu.edu`

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, 1 UNIVERSITY STATION C1200, AUSTIN, TX 78712-0257

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203-5017

*E-mail address:* `bunyamin@unt.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MEMPHIS, MEMPHIS, TN, 38152-3240

*E-mail address:* `bzheng@memphis.edu`