UNCONDITIONAL STRUCTURES OF TRANSLATES FOR $L_p(\mathbb{R}^d)$

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Abstract. We prove that a sequence $(f_i)_{i=1}^{\infty}$ of translates of a fixed $f \in L_p(\mathbb{R})$ cannot be an unconditional basis of $L_p(\mathbb{R})$ for any $1 \leq p < \infty$. In contrast to this, for every $2 < p < \infty$, $d \in \mathbb{N}$ and unbounded sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ we establish the existence of a function $f \in L_p(\mathbb{R}^d)$ and sequence $(g_n^*)_{n\in\mathbb{N}} \subset L_p^*(\mathbb{R}^d)$ such that $(T_{\lambda_n} f, g_n^*)_{n\in\mathbb{N}}$ forms an unconditional Schauder frame for $L_p(\mathbb{R}^d)$. In particular, there exists a Schauder frame of integer translates for $L_p(\mathbb{R})$ if (and only if) $2 < p < \infty$.

1. Introduction

If $d \in \mathbb{N}$ and $\lambda \in \mathbb{R}^d$, the *translation operator* T_λ is defined by $T_\lambda f(x) = f(x-\lambda)$ for all $x \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}^d$. Note that for the case $d = 1$ and $\lambda > 0$, the operator T_λ is simply translation of f by λ units to the right. Given $1 \leq p < \infty$, $f \in L_p(\mathbb{R})$, and $\Lambda \subset \mathbb{R}$, the resulting space $X_p(f,\Lambda) \equiv \overline{\text{span}\{T_\lambda f\}}_{\lambda\in\Lambda}$ and set $\{T_\lambda f\}_{\lambda\in\Lambda}$ have been studied in a variety of contexts and in particular arise in the study of wavelets and Gabor frames [HSWW, CDH].

Some of the natural problems to consider when studying translations of a fixed function f relate to characterizing when can $X_p(f,\Lambda) = L_p(\mathbb{R}^d)$ and when can $\{T_\lambda f\}_{\lambda\in\Lambda}$ be ordered to form a coordinate system such as a (unconditional) Schauder basis or (unconditional) Schauder frame for $L_p(\mathbb{R}^d)$. For $d = 1$, the cases when $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{N}$ are of particular interest. For $1 \leq p \leq 2$, a Fourier transform argument yields that there does not exist an $f \in L_p(\mathbb{R})$ such that $X_p(f,\mathbb{Z}) = L_p(\mathbb{R})$ [AO]. On the other hand, for all $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R} \setminus \mathbb{Z}$

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such that $\lim_{n\to\pm\infty}|\lambda_n - n| = 0$, there exists $f \in L_2(\mathbb{R})$ such that $X_2(f,(\lambda_n)_{n\in\mathbb{Z}}) = L_2(\mathbb{R})$ [O]. The case $2 < p < \infty$ is completely different, as for all $2 < p < \infty$ there exists $f \in L_p(\mathbb{R})$ such that $X_p(f,\mathbb{Z}) = L_p(\mathbb{R})$ and, moreover, $T_m f \notin X_p(f,\mathbb{Z}\setminus\{m\})$ for all $m \in \mathbb{Z}$ [AO].

Suppose that $f \in L_p(\mathbb{R})$ and that $\{T_\lambda f : \lambda \in \Lambda\}$ is an unconditional basic sequence in $L_p(\mathbb{R})$. What can be said about $X_p(f,\Lambda)$? Note that for a sequence (x_j) in a Banach space the property of being an unconditional basis does not depend on the order. We can therefore index unconditional bases by any countable set, for example by the elements of Λ, if we assume that $\{T_\lambda f : \lambda \in \Lambda\}$ is an unconditional basic sequence in $L_p(\mathbb{R})$, which of course implicitly includes the assumption that Λ is countable. In Section 2 we prove that if $(T_\lambda f)_{\lambda\in\Lambda}$ is an unconditional basic sequence in $L_p(\mathbb{R})$ with $2 < p < \infty$ such that $X_p(f,\Lambda)$ is complemented in $L_p(\mathbb{R})$ then $(T_\lambda f)_{\lambda\in\Lambda}$ must be equivalent to the unit vector basis of ℓ_p. In particular, $X_p(f,\Lambda) \neq L_p(\mathbb{R})$. Together with results already proven in [OSSZ] for the case $1 \le p < 2$ and [OZ] for $p = 2$, we will conclude for all $1 \le p < \infty$ that there is no function $f \in L_p(\mathbb{R})$ and countable set $\Lambda \subset \mathbb{R}$ so that $(T_\lambda f : \lambda \in \Lambda)$ is an unconditional basis for $L_p(\mathbb{R})$.

In Section 3 we consider frames consisting of translates of a single function. By Wiener's famous Tauberian Theorem [Wi] it follows for an $f \in L_2(\mathbb{R})$ that $X_2(f,\mathbb{R}) = L_2(\mathbb{R})$ if and only if the Fourier transform of f is almost everywhere non-zero. Thus, there are many cases in which $X_2(f,\Lambda) = L_2(\mathbb{R})$, but by [CDH, Section 4] there does not exist $\Lambda \subset \mathbb{R}$ and $f \in L_2(\mathbb{R})$ such that $\{T_\lambda f\}_{\lambda\in\Lambda}$ is a Hilbert frame for $L_2(\mathbb{R})$. In the special case that $\Lambda \subset \mathbb{N}$ and $f \in L_2(\mathbb{R})$, then the sequence $(T_\lambda f)_{\lambda\in\Lambda}$ is a Hilbert frame for $X_2(f,\Lambda)$ if and only if it is a Riesz basis for $X_2(f,\Lambda)$, i.e., $(T_\lambda f)$ must be equivalent to the unit vector basis of ℓ_2 [CCK]. In Section 3 we provide some background on Schauder frames for Banach spaces and prove that there exists a function $f \in L_p(\mathbb{R})$ and sequence $(g_n^*)_{n\in\mathbb{N}} \subset L_p^*(\mathbb{R})$ such that $(T_n f, g_n^*)_{n\in\mathbb{N}}$ forms an unconditional Schauder frame for $L_p(\mathbb{R})$ if (and thus, by the previously cited result of [AO], only if) $2 < p < \infty$. More generally, we prove that for every $2 < p < \infty$, $d \in \mathbb{N}$

and unbounded sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$, there exists a function $f \in L_p(\mathbb{R}^d)$ and sequence $(g_n^*)_{n\in\mathbb{N}} \subset L_p^*(\mathbb{R}^d)$ such that $(T_{\lambda_n} f, g_n^*)_{n\in\mathbb{N}}$ forms an unconditional Schauder frame for $L_p(\mathbb{R}^d)$.

For $2 < p < \infty$, if $L_p(\mathbb{R})$ embeds into $X_p(f,\Lambda)$ and $X_p(f,\Lambda)$ is complemented in $L_p(\mathbb{R})$ then $(T_\lambda f)_{\lambda\in\Lambda}$ cannot be an unconditional basic sequence in $L_p(\mathbb{R})$. However, we prove in Section 4 that for $2 < p < \infty$ there exists $f \in L_p(\mathbb{R})$ and $\Lambda \subset \mathbb{N}$ so that $X_p(f,\Lambda)$ is isomorphic to $L_p(\mathbb{R})$, $X_p(f,\Lambda)$ is complemented in $L_p(\mathbb{R})$, and $\{T_\lambda f\}_{\lambda\in\Lambda}$ can be blocked to form an unconditional finite-dimensional decomposition (FDD) for $X_p(f,\Lambda)$.

In Section 5, we study the restriction operator $R_I : L_p(\mathbb{R}) \to L_p(I)$ given by $x \mapsto x|_I$ where $I \subset \mathbb{R}$ is some bounded interval. Assuming $(T_{\lambda_i} f)$ is an unconditional basic sequence, we characterize for what values of $1 \le p < \infty$ must the map $R_I : X_p(f,(\lambda_i)) \to L_p(I)$ be compact for all bounded intervals $I \subset \mathbb{R}$. We prove as well other relationships between the restriction operator $R_I : X_p(f,(\lambda_i)) \to L_p(I)$ and the structure of $X_p(f,(\lambda_i))$. Lastly, in Section 6 we state some open problems.

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2. Unconditional bases of translates

The goal of this section is to prove for all $1 \le p < \infty$ that $L_p(\mathbb{R})$ does not have an unconditional basis which consists of translates of the same function $f \in L_p(\mathbb{R})$. Previously, the problem had been solved for $1 \le p \le 4$. For the case $p = 2$, this was first proved by Olson and Zalik using tools from Harmonic Analysis [OZ, Theorem 2]. An extension to the range $1 \le p \le 2$ was obtained in [OSSZ] using Banach space techniques. In [OSSZ, Corollary 2.10] it was shown that if $1 \le p \le 2$, and if $f \in L_p(\mathbb{R})$ and $\Lambda \subset \mathbb{R}$ is such that $(T_\lambda f)_{\lambda\in\Lambda}$ forms an unconditional basic sequence, then $(T_\lambda f)_{\lambda\in\Lambda}$ is equivalent to the ℓ_p unit vector basis. For $1 \le p < 2$, this immediately implies that $X_p(f,\Lambda) \neq L_p(\mathbb{R})$. For $p = 2$, Proposition 5.1 below (which is extracted from the proof of [OSSZ, Proposition 2.6(a)]) gives the theorem of Olson and Zalik as an immediate corollary. In the case that $2 < p \le 4$, it was shown in [OSSZ, Theorem 2.11] that the closed linear span of an unconditional basic sequence consisting of translates of some $f \in L_p(\mathbb{R})$ must embed into ℓ_p, and can therefore not be isomorphic to $L_p(\mathbb{R})$. However, this approach breaks down for $4 < p < \infty$. Indeed [OSSZ, Theorem 2.14] states that for any $4 < p < \infty$ there is a function $f \in L_p(\mathbb{R})$ and a subset $\Lambda \subset \mathbb{Z}$ so that $(T_\lambda f)_{\lambda\in\Lambda}$ is an unconditional basic sequence whose closed linear span contains a subspace isomorphic to $L_p(\mathbb{R})$.

Theorem 2.1. *Let $2 < p < \infty$, $f \in L_p(\mathbb{R})$ and $\Lambda \subset \mathbb{R}$ countable. If $(T_\lambda f : \lambda \in \Lambda)$ is an unconditional basis of $X_p(f,\Lambda)$ and $X_p(f,\Lambda)$ is complemented in $L_p(\mathbb{R})$, then $(T_\lambda f)_{\lambda\in\Lambda}$ is equivalent to the unit vector basis of ℓ_p.*

We will need the following result from [JO].

Proposition 2.2. [JO, Section 3, Lemma 2]. *Let $1 \le q \le 2$. Let $(g_i) \subset L_q(\mathbb{R})$ be semi-normalized and unconditional basic. Assume that for some $\varepsilon > 0$ there exists a sequence of disjoint measurable sets $(B_i)_{i=1}^\infty$ with $\|g_i|_{B_i}\|_q \ge \varepsilon$ for all i. Then $(g_i)_{i=1}^\infty$ is equivalent to the unit vector basis of ℓ_q.*

Proof of Theorem 2.1. Let Λ be ordered into $(\lambda_i)_{i\in\mathbb{N}}$. Put $f_i = T_{\lambda_i} f$, for $i \in \mathbb{N}$, and $X = X_p(f,\Lambda)$. Without loss of generality we can assume that $\|f_i\|_p = \|f\|_p = 1$ for all $i \in \mathbb{N}$. Denote the biorthogonals of (f_i) inside X^* by $(\overline{g}_i)$, and let $P : L_p(\mathbb{R}) \to X$ be a bounded projection. Thus, $P^* : X^* \to L_p(\mathbb{R})^*$ is an isomorphic embedding. Let $g_i = P^*\overline{g}_i$ for $i\in\mathbb{N}$.

Recall that if $\{T_\lambda f : \lambda \in \Lambda\}$ can be ordered into a basic sequence in $L_p(\mathbb{R})$ for some $f \in L_p(\mathbb{R})$ and $\Lambda \subset \mathbb{R}$, then Λ is uniformly discrete [OZ, Theorem 1]. Hence we may choose $\delta > 0$ such that

$$0 < \delta < \inf\{|\lambda - \mu| : \lambda, \mu \in \Lambda, \lambda \ne \mu\}. \tag{1}$$

For $j \in \mathbb{Z}$, we define the interval $I_j = [j\delta, (j+1)\delta)$.

Claim. There exist $N \in \mathbb{N}$, $\varepsilon > 0$, and a sequence of distinct integers $(l_i)_{i=1}^{\infty}$ such that for all $i \in \mathbb{N}$ there exists $j_i \in \{l_i, l_i+1, \dots, l_{i+N}\}$ with $\big\|g_i|_{I_{j_i}}\big\|_q > \varepsilon$.

Indeed, choose first $l_0 \in \mathbb{Z}$ and $N \in \mathbb{N}$ so that

$$\|f|_{\mathbb{R}\setminus\bigcup_{j=l_0}^{l_0+N-1} I_j}\|_p^p = \int_{-\infty}^{l_0\delta} \big|f(z)\big|^p dz + \int_{(l_0+N)\delta}^{\infty} \big|f(z)\big|^p dz < \frac{1}{2^p}\Big(\sup_{i\in\mathbb{N}} \|g_i\|_q^p\Big)^{-1}$$

Then for $i \in \mathbb{N}$ choose $l_i \in \mathbb{Z}$ such that

$$l_0\delta \le (l_i+1)\delta - \lambda_i < (l_0+1)\delta.$$

Note if $i \neq i'$, then by (1) $|\lambda_i - \lambda_{i'}| > \delta$, and, thus $l_i \neq l_{i'}$. Moreover,

$$\begin{aligned}
\|f_i|_{\mathbb{R}\setminus\bigcup_{j=l_i}^{l_i+N} I_j}\|_p^p &= \int_{-\infty}^{l_i\delta} \big|f(x-\lambda_i)\big|^p dx + \int_{(l_i+N+1)\delta}^{\infty} \big|f(x-\lambda_i)\big|^p dx \\
&= \int_{-\infty}^{l_i\delta-\lambda_i} \big|f(z)\big|^p dz + \int_{(l_i+N+1)\delta-\lambda_i}^{\infty} \big|f(z)\big|^p dz \\
&\le \int_{-\infty}^{l_0\delta} \big|f(z)\big|^p dz + \int_{(l_0+N)\delta}^{\infty} \big|f(z)\big|^p dz < \frac{1}{2^p}\Big(\sup_{i\in\mathbb{N}} \|g_i\|_q^p\Big)^{-1}.
\end{aligned}$$

Thus, by Hölder's Theorem and the fact that $\|f\|_p = 1$, it follows that

$$\big\|g_i|_{\bigcup_{j=l_i}^{l_i+N} I_j}\big\|_q \ge \int_{\bigcup_{j=l_i}^{l_i+N} I_j} g_i f_i dz = 1 - \int_{\mathbb{R}\setminus\bigcup_{j=l_i}^{l_i+N} I_j} g_i f_i \ge 1 - \big\|g_i\big\|_q \big\|f|_{\mathbb{R}\setminus\bigcup_{j=l_i}^{l_i+N} I_j}\big\|_p \ge \frac{1}{2}\ .$$

Letting $\varepsilon = \frac{1}{2(N+1)}$ we deduce our claim.

Since the l_i's are distinct, it follows that for each $k \in \mathbb{Z}$

$$\big|\{i \in \mathbb{N} : j_i = k\}\big| \le \big|\{i : k \in [l_i, l_i+N]\}\big| = \big|\{i : l_i \in [k-N, k]\}\big| \le N+1\ .$$

We can therefore partition $\mathbb{N}$ into infinite sets $K_1, K_2, \dots, K_m$, with $m \le N+1$, so that for each $s = 1, 2 \dots m$ the sequence $(j_i)_{i\in K_s}$ consists of distinct integers.

For each $s \le m$ it follows that the sequence $(g_i)_{i\in K_s}$, satisfies the condition of Proposition 2.2, with $B_i = I_{j_i}$, for $i \in K_s$, and must therefore be equivalent to the unit vector basis of ℓ_q. Thus, since $\overline{\text{span}\{g_i\}}_{i\in\mathbb{N}}$ is the direct sum of $[g_i : i \in K_s]$, $s = 1, 2, \dots m$, it follows that $(g_i)_{i\in\mathbb{N}}$ must be equivalent to the unit vector basis of ℓ_q. Since $P^* : X^* \to L_p(\mathbb{R})^*$ is an isomorphic embedding and $P^*(\overline{g}_i) = g_i$ for all $i \in \mathbb{N}$, it follows that $(\overline{g}_i)$ is equivalent to (g_i) and, thus, also equivalent to the unit vector basis of ℓ_q. But this implies that (f_i) is equivalent to the unit vector basis of ℓ_p. $\square$

Using now the results in [OSSZ] cited at the beginning of this section we conclude the following.

Corollary 2.3. *If $f \in L_p(\mathbb{R})$, $1 \le p < \infty$, and $(T_\lambda f)_{\lambda\in\Lambda}$ is an unconditional basis for $X_p(f,\Lambda)$, then $X_p(f,\Lambda) \neq L_p(\mathbb{R})$.*

Moreover, if $(T_\lambda f)_{\lambda\in\Lambda}$ is unconditional and $X_p(f,\lambda)$ is complemented in $L_p(\mathbb{R})$, then it must be equivalent to the ℓ_p unit vector basis.

3. Unconditional Schauder frames of translates

In Section 2, it was shown that for any value of p, $1 \le p < \infty$, there does not exist an unconditional basis for $L_p(\mathbb{R})$ consisting of translates of a single function. In contrast to this, we will show that there does exist an unconditional Schauder frame for $L_p(\mathbb{R})$ consisting of integer translates of a single function if and only if $2 < p < \infty$. Before proving this result, we will develop some basic theory of Schauder frames.

If X is a separable Banach space, then a sequence $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$ is called a *Schauder frame* for X if

$$x = \sum_{i=1}^{\infty} g_i^*(x) x_i \quad \text{for all } \; x \in X. \tag{2}$$

A Schauder frame $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$ is called an *unconditional Schauder frame* for X if the series (2) converges unconditionally for all $x \in X$. Recall that a series converges *unconditionally* if it converges for any ordering of the elements of the series.

Let X be a separable Banach space. Assume that a sequence $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$ satisfies that the operator $S : X \to X$ defined by $S(x) = \sum_{i=1}^\infty g_i^*(x) x_i$ is well defined (and hence bounded due to the uniform boundedness principle). S is called the *frame operator* for

$(x_i, g_i^*)_{i=1}^\infty$. Note that the sequence $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$ is a Schauder frame if and only if the frame operator is the identity. We define $(x_i, g_i^*)_{i=1}^\infty$ to be an *approximate Schauder frame* if the frame operator is bounded, one to one, and onto (hence has bounded inverse)[T], and we define $(x_i, g_i^*)_{i=1}^\infty$ to be an *unconditional approximate Schauder frame* if it is an approximate Schauder frame and the series $\sum_{i=1}^\infty g_i^*(x)x_i$ converges unconditionally for all $x \in X$.

Lemma 3.1. *Let X be a separable Banach space and let $(x_i, g_i^*)_{i=1}^\infty \subset X \times X^*$ be an approximate Schauder frame for X with frame operator S. Then $(x_i, (S^{-1})^* g_i^*)_{i=1}^\infty$ is a Schauder frame for X. Furthermore, if $(x_i, g_i^*)_{i=1}^\infty$ is an unconditional approximate Schauder frame for X, then $(x_i, (S^{-1})^* g_i^*)_{i=1}^\infty$ is an unconditional Schauder frame for X.*

Proof. Let $x \in X$. We have that S and S^{-1} are bounded. Thus,

$$x = S(S^{-1}x) = \sum_{i=1}^\infty g_i^*(S^{-1}x)x_i = \sum_{i=1}^\infty ((S^{-1})^* g_i^*)(x)x_i.$$

Hence, $(x_i, (S^{-1})^* g_i^*)_{i=1}^\infty \subset X \times X^*$ is a Schauder frame for X. Furthermore, the series $\sum_{i=1}^\infty g_i^*(S^{-1}x)x_i$ converges unconditionally if $(x_i, g_i^*)_{i=1}^\infty$ is an unconditional approximate Schauder frame for X, and thus $(x_i, (S^{-1})^* g_i^*)_{i=1}^\infty$ is then an unconditional Schauder frame for X. □

In particular, Lemma 3.1 implies that $L_p(\mathbb{R}^d)$ has a (unconditional) Schauder frame formed by translating a single function if and only if it has an (unconditional) approximate Schauder frame formed by translating a single function. This is important for us, as we will provide an explicit construction for an unconditional approximate Schauder frame of translates for $L_p(\mathbb{R}^d)$ and then apply Lemma 3.1 to obtain an unconditional Schauder frame of translates for $L_p(\mathbb{R}^d)$ for any $p > 2$.

Theorem 3.2. *Let $2 < p < \infty$ and $d \in \mathbb{N}$. If $(\lambda_n)_{n\in\mathbb{N}}$ is an unbounded sequence in $\mathbb{R}^d$ then there exists a function $f \in L_p(\mathbb{R}^d)$ and a sequence $(g_n^*)_{n\in\mathbb{N}} \subset L_p^*(\mathbb{R}^d)$ such that $(T_{\lambda_n} f, g_n^*)_{n\in\mathbb{N}}$ forms an unconditional Schauder frame for $L_p(\mathbb{R}^d)$.*

Proof. By Lemma 3.1 it is enough to construct an unconditional approximate Schauder frame of $L_p(\mathbb{R}^d)$ which is of the form $(T_{\lambda_n} f, g_n^*)_{n\in\mathbb{N}}$. Also, it is enough to construct for some infinite subsequence $(\lambda_m)_{m\in M} \subset (\lambda_n)$ a function $f \in L_p(\mathbb{R}^d)$ and a sequence $(g_m^*)_{m\in M} \subset L_p^*(\mathbb{R}^d)$ so that $(T_{\lambda_m} f, g_m^*)_{m\in M}$ is an unconditional approximate Schauder frame of $L_p(\mathbb{R}^d)$. Indeed, by letting $g_n^* \equiv 0$ in the case that $n \in \mathbb{N} \setminus M$, it follows that the sequence $(T_{\lambda_n} f, g_n^*)_{n\in\mathbb{N}}$ is also an unconditional approximate Schauder frame of $L_p(\mathbb{R}^d)$.

Let $(e_i)_{i=1}^\infty$ be a normalized unconditional Schauder basis for $L_p(\mathbb{R}^d)$ with biorthogonal functionals $(e_i^*)_{i=1}^\infty$ such that $e_i \in L_p(\mathbb{R}^d)$ is a function satisfying $\mathrm{diam}(\mathrm{supp}(e_i)) \leq 1$ for all $i \in \mathbb{N}$, where the diameter is measured in the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^d$. Let C_u be the constant of unconditionality of $(e_i)_{i=1}^\infty$. For each $k \in \mathbb{N}$, choose $N_k \in \mathbb{N}$ such that $\big(\sum_{k=1}^\infty N_k^{1-p/2}\big)^{1/p} < \frac{1}{2C_u}$.

As $(\lambda_n)_{n\in\mathbb{N}}$ is unbounded, we may choose $j_1^{(1)} < j_2^{(1)} < \ldots < j_{N_1}^{(1)} < j_1^{(2)} < \ldots < j_{N_2}^{(2)} < \ldots$ to increase rapidly enough so that $\|\lambda_{j_1^{(1)}}\|_2 > 1$ and for all $k \in \mathbb{N}$ and $1 \leq s \leq N_k$,

$$\|\lambda_{j_s^{(k)}}\|_2 > 3\max\{\|\lambda_{j_{s'}^{(k')}}\|_2 \,:\, j_{s'}^{(k')} < j_s^{(k)}\} + 2\max\{\|x\|_2 \,:\, x \in \overline{\mathrm{supp}}(e_j),\, 1 \leq j \leq k\}. \tag{3}$$

We let $J_k = \{j_s^{(k)} : 1 \leq s \leq N_k\}$, for $k \in \mathbb{N}$, and if $j \in J_k$, for some $k \in \mathbb{N}$, we put $k_j = k$. Thus $J_1, J_2, \ldots$ are pairwise disjoint subsets of $\mathbb{N}$ with $|J_k| = N_k$. After checking the separate cases, one obtains the following from (3), $\|\lambda_{j_1^{(1)}}\|_2 > 1$, and $\mathrm{diam}(\mathrm{supp}(e_i)) \leq 1$ for all $i \in \mathbb{N}$.

$$\begin{gathered}\mathrm{supp}(T_{\lambda_i - \lambda_j}(e_{k_j})) \cap \mathrm{supp}(T_{\lambda_{i'} - \lambda_{j'}}(e_{k_{j'}})) = \emptyset \\ \text{whenever } i, j, i', j' \in \bigcup_{l=1}^\infty J_l, \text{ with } i \neq j,\ i' \neq j', \text{ and } (i,j) \neq (i',j').\end{gathered} \tag{4}$$

Note that the case $i = i'$ in (4) reduces to

$$\mathrm{supp}(T_{-\lambda_j} e_{k_j}) \cap \mathrm{supp}(T_{-\lambda_{j'}} e_{k_{j'}}) = \emptyset \text{ for all distinct } j, j' \in \bigcup_{l=1}^\infty J_l. \tag{5}$$

We define our function f by

$$f := \sum_{k=1}^{\infty} \sum_{j\in J_k} N_k^{-1/2} T_{-\lambda_j} e_k.$$

Our first step is to show that $f \in L_p(\mathbb{R}^d)$.

$$\begin{aligned}
\int |f|^p \,\mathrm{d}\mu &= \int \Big| \sum_{k=1}^{\infty} \sum_{j\in J_k} N_k^{-1/2} T_{-\lambda_j} e_k \Big|^p \,\mathrm{d}\mu \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} N_k^{-p/2} \int |e_k|^p \,\mathrm{d}\mu \qquad \text{by (5)} \\
&= \sum_{k=1}^{\infty} N_k^{1-p/2} \qquad \text{as } \|e_k\| = 1 \text{ for all } k \in \mathbb{N} \\
&< \frac{1}{2^p C_u^p}.
\end{aligned}$$

Thus we have that $f \in L_p(\mathbb{R}^d)$. For each $j \in \mathbb{N}$, we define $g_j^* \in L_p^*(\mathbb{R}^d)$ by

$$g_j^* = \begin{cases} N_k^{-1/2} e_k^* & \text{if } j \in J_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that for any finite $A \subset \mathbb{N}$ and any $h \in L_p(\mathbb{R}^d)$, with $\|h\|_p \le 1$,

$$\begin{aligned}
(6) \quad \sum_{j\in A} g_j^*(h) T_{\lambda_j}(f) &= \sum_{k=1}^{\infty} \sum_{i\in J_k\cap A} N_k^{-1/2} e_k^*(h) T_{\lambda_i}(f) \\
&= \sum_{k=1}^{\infty} \sum_{i\in J_k\cap A} N_k^{-1/2} e_k^*(h) \sum_{l=1}^{\infty} \sum_{j\in J_l} N_l^{-1/2} T_{\lambda_i-\lambda_j} e_l \\
&= \sum_{k=1}^{\infty} \sum_{i\in J_k\cap A} N_k^{-1} e_k^*(h) e_k + \sum_{k,l=1}^{\infty} \sum_{i\in J_k\cap A} \sum_{j\in J_l, j\neq i} N_k^{-1/2} N_l^{-1/2} e_k^*(h) T_{\lambda_i-\lambda_j} e_l \\
&=: h_A + r_A.
\end{aligned}$$

In order to show that $\sum_{i=1}^{\infty} g_i^*(h) T_{\lambda_i} f$ converges unconditionally we let $\varepsilon > 0$ and choose $M \in \mathbb{N}$ such that $\| \sum_{i=M}^{\infty} e_i^*(h) e_i \| < \varepsilon / C_u$ and $\sum_{k=M}^{\infty} N_k^{1-p/2} < \varepsilon^p$. Let $A \subset \mathbb{N}$ such that

$\min(A) \geq j_1^{(M)}$. Then it follows that $\|h_A\| \leq \varepsilon$ and (4) yields

$$\begin{aligned}
\|r_A\| &= \Big\| \sum_{k,l=1}^{\infty} \sum_{i\in J_k\cap A} \sum_{j\in J_l, j\neq i} N_k^{-1/2} N_l^{-1/2} e_k^*(h) T_{\lambda_i-\lambda_j} e_l \Big\|_p \\
&= \Bigg(\sum_{k,l=1}^{\infty} \sum_{i\in J_k\cap A} \sum_{j\in J_l, j\neq i} N_k^{-p/2} N_l^{-p/2} |e_k^*(h)|^p \Bigg)^{1/p} \\
&\leq C_u \Big(\sum_{k=M}^{\infty} N_k^{1-p/2} \sum_{l=1}^{\infty} N_l^{1-p/2} \Big)^{1/p} \leq \varepsilon/2
\end{aligned}$$

(Recall that $|J_k| = N_k$, $(\sum_{l=1}^{\infty} N_l^{1-p/2})^{1/p} \leq \frac{1}{2C_u}$, and $|e_k^*(h)| \leq C_u\|h\| \leq C_u$, for $k \in \mathbb{N}$).

Since $\varepsilon > 0$ was arbitrary this implies our claim that the series $\sum_{i=1}^{\infty} g_i^*(h) T_{\lambda_i} f$ converges unconditionally. We can therefore let $A = \mathbb{N}$ in (6) and note that $h_{\mathbb{N}} = h$, and then the previous estimations yield that

$$\Big\| h - \sum_{j\in\mathbb{N}} g_j^*(h) T_{\lambda_j}(f) \Big\|_p = \|r_A\|_p \leq C_u \Big(\sum_{k=1}^{\infty} N_k^{1-p/2} \Big)^{2/p} < \frac{1}{4},$$

which implies that the frame operator is invertible and, thus, that $(T_{\lambda_j} f, g_j^*)$ is an approximate unconditional Schauder frame and finishes the proof. □

We now discuss some consequences of Theorem 3.2. Given a Schauder frame $(x_i, f_i)_{i=1}^{\infty} \subset X \times X^*$, let $H_n : X \to X$ be the operator $H_n(x) = \sum_{i\geq n} f_i(x) x_i$. A Schauder frame $(x_i, f_i)_{i=1}^{\infty}$ is called *shrinking* if $\|x^* \circ H_n\| \to 0$ for all $x^* \in X^*$. A Schauder frame $(x_i, f_i)_{i=1}^{\infty} \subset X \times X^*$ for a Banach space X is shrinking if and only if $(f_i, x_i)_{i=1}^{\infty} \subset X^* \times X^{**}$ is a Schauder frame for X^* [CL]. Furthermore, every unconditional Schauder frame for a reflexive Banach space is shrinking [CLS, L]. Thus the following corollary of Theorem 3.2 ensues.

Corollary 3.3. *Let $1 < q < 2$ and $d \in \mathbb{N}$. If $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is unbounded then there exists a function $f^* \in L_q^*(\mathbb{R}^d)$ and sequence $(g_n)_{n\in\mathbb{N}} \subset L_q(\mathbb{R}^d)$ such that $(g_n, T_{\lambda_n} f^*)_{n\in\mathbb{N}}$ forms an unconditional Schauder frame for $L_q(\mathbb{R}^d)$.*

Note that in Corollary 3.3, the dual functionals $(T_{\lambda_n} f^*)_{n\in\mathbb{N}}$ are translations of a single function as opposed to the vectors $(g_n)_{n\in\mathbb{N}}$.

4. Unconditional FDDs of translates

In Section 2, it was shown that for all $f \in L_p(\mathbb{R})$, $1 \le p < \infty$, and $\Lambda \subset \mathbb{R}$, if $(T_\lambda f)_{\lambda\in\Lambda}$ is an unconditional basic sequence and $X_p(f,\Lambda)$ is complemented in $L_p(\mathbb{R})$ then $(T_\lambda f)_{\lambda\in\Lambda}$ is equivalent to the unit vector basis for ℓ_p. Instead of considering when $(T_\lambda f)_{\lambda\in\Lambda}$ is an unconditional basic sequence, we now study the cases where $(T_\lambda f)_{\lambda\in\Lambda}$ can be blocked into an unconditional FDD. Given a Banach space X, recall that a sequence of finite dimensional spaces $(F_i)_{i=1}^\infty \subset X$ is called a *finite dimensional decomposition* or *FDD* for X if for every $x \in X$ there exists for all $i \in \mathbb{N}$ a unique $x_i \in F_i$ such that $x = \sum_{i=1}^\infty x_i$. An FDD is called unconditional if the series $x = \sum_{i=1}^\infty x_i$ converges unconditionally for all $x \in X$.

Theorem 4.1. *Let $2 < p < \infty$. There exists $f \in L_p(\mathbb{R})$ and a subsequence $(n_i)_{i=1}^\infty$ of $\mathbb{N}$ so that for $X = X_p(f, (-n_i)_{i=1}^\infty)$, i) X is isomorphic to $L_p(\mathbb{R})$, ii) X is complemented in $L_p(\mathbb{R})$, and iii) there exists a partition of $\mathbb{N}$ into successive intervals $(J_j)_{j=1}^\infty$ so that setting $F_j = \operatorname{span}\{T_{-n_i} f\}_{i\in J_j}$, $(F_j)_{j=1}^\infty$ forms an unconditional FDD for X.*

Proof of Theorem 4.1. Let $\varepsilon \in (0,1)$ and choose a subsequence $(N_k)_{k=1}^\infty$ of $\mathbb{N}$ so that $N_1 \ge 4$ and

$$\sum_{k=1}^\infty N_k^{\frac{1}{p}-\frac{1}{2}} < \varepsilon \quad \text{and hence} \quad \sum_{k=1}^\infty N_k^{1-\frac{p}{2}} < 1\ . \tag{7}$$

Let $(h_j^i)_{j=1}^\infty$ be the normalized Haar basis for $L_p[3^i, 3^i+1]$ for $i \in \mathbb{N}$. Partition $\mathbb{N}$ into successive intervals $J_1, J_2, \ldots$ so that $|J_k| = N_k$ for $j \in \mathbb{N}$.

Let

$$f = \sum_{k=1}^\infty \sum_{i\in J_k} \frac{1}{\sqrt{N_k}} h_k^i \quad \text{and let} \quad f_i = T_{-3^i} f\ , i \in \mathbb{N}\ .$$

Then $f \in L_p(\mathbb{R})$ since

$$\|f\|_p^p = \sum_{k=1}^{\infty} \sum_{i \in J_k} \Big(\frac{1}{\sqrt{N_k}}\Big)^p = \sum_{k=1}^{\infty} N_k \Big(\frac{1}{\sqrt{N_k}}\Big)^p = \sum_{k=1}^{\infty} N_k^{1-\frac{p}{2}} \leq 1 \text{ (by (7))}.$$

The choice of 3^i above yields, as in Section 3, that for $k \in \mathbb{N}$ and $i \in J_k$, f_i is of the form

$$f_i = \frac{1}{\sqrt{N_k}} h_k + g_i \tag{8}$$

where (h_k) is the normalized Haar basis for $L_p[0,1]$, and the g_i are bounded in norm by 1 and have supports that are pairwise disjoint and also disjoint from $[0,1]$. The latter fact follows since $\operatorname{supp} g_i \subset \bigcup_{l=1,\ell\neq i}^{\infty} [3^\ell - 3^i\,,\, 3^\ell - 3^i + 1]$ for all $i \in \mathbb{N}$. Thus, $\{h_k : k \in \mathbb{N}\} \cup \{g_i : i \in \mathbb{N}\}$ is an unconditional basis of its closed linear span Y, and hence statement (iii) of the theorem follows at once with $n_i = 3^i$ for each $i \in \mathbb{N}$ and with J_k as defined above. Moreover, Y is 1-complemented in $L_p(\mathbb{R})$, so in order to show that X, the closed linear span of (f_i), is complemented in $L_p(\mathbb{R})$, it is enough to show that X is complemented in Y.

We denote the biorthogonals of $\{h_k : k \in \mathbb{N}\} \cup \{g_i : i \in \mathbb{N}\}$ in Y^* by $\{h_k^* : k \in \mathbb{N}\} \cup \{g_i^* : i \in \mathbb{N}\}$ and define

$$f_j^* = \frac{1}{\sqrt{N_k}} h_k^* + g_j^* - \frac{1}{N_k} \sum_{i \in J_k} g_i^*, \text{ for } k \in \mathbb{N} \text{ and } j \in J_k.$$

It follows for $k, l \in \mathbb{N}$, $i \in J_k$, and $j \in J_l$ that

$$f_j^*(f_i) = \frac{1}{N_k} \delta_{(k,l)} + \delta_{(i,j)} - \frac{1}{N_l} \delta_{(k,l)} = \delta_{(i,j)}.$$

We define $P(y) = \sum_{j \in \mathbb{N}} f_j^*(y) f_j$ for $y \in Y$, and need to show that P is bounded. For $y = \sum_{l=1}^{\infty} a_l h_l + \sum_{l=1}^{\infty} \sum_{i \in J_l} b_i g_i \in Y$, and numbers $k \in \mathbb{N}$ and $j \in J_k$ we compute

$$f_j^*(y) = \frac{a_k}{\sqrt{N_k}} + b_j - \frac{1}{N_k} \sum_{i \in J_k} b_i.$$

It follows therefore that $P(y)$ is the sum of the following four terms:

$$\sum_{k=1}^{\infty} \sum_{j \in J_k} \frac{a_k}{\sqrt{N_k}} \frac{h_k}{\sqrt{N_k}} = \sum_{k=1}^{\infty} a_k h_k,$$

$$\sum_{k=1}^{\infty}\sum_{j\in J_k}\Big(b_j-\frac{1}{N_k}\sum_{i\in J_k}b_i\Big)\frac{h_k}{\sqrt{N_k}}=\sum_{k=1}^{\infty}\Big(\sum_{j\in J_k}b_j-\sum_{i\in J_k}b_i\Big)\frac{h_k}{\sqrt{N_k}}=0,$$

$$\sum_{k\in\mathbb{N}}\sum_{j\in J_k}\frac{a_k}{\sqrt{N}_k}g_j,$$

$$\sum_{k\in\mathbb{N}}\sum_{j\in J_k}\Big(b_j-\frac{1}{N_k}\sum_{i\in J_k}b_i\Big)g_j.$$

The norm of the first term is bounded by $\|y\|$, and for the third term it follows from the pairwise disjointness of the supports of the g_j that

$$\Big\|\sum_{k\in\mathbb{N}}\sum_{j\in J_k}\frac{a_k}{\sqrt{N_k}}g_j\Big\|_p\le\Big(\sum_{k=1}^{\infty}N_k^{1-p/2}|a_k|^p\Big)^{1/p}\sup_{j\in\mathbb{N}}\|g_j\|_p\le\Big(\sum_{k=1}^{\infty}N_k^{1-p/2}\Big)^{1/p}\sup_k|a_k|\le\sup_k|a_k|$$

and finally, using again the disjointness of the support of the g_j, the fourth term can be estimated as follows:

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{N}}\sum_{j\in J_k}\Big(b_j-\frac{1}{N_k}\sum_{i\in J_k}b_i\Big)g_j\Big\| &\le\Big\|\sum_{k\in\mathbb{N}}\sum_{j\in J_k}b_jg_j\Big\|+\Big\|\sum_{k\in\mathbb{N}}\sum_{j\in J_k}\Big(\frac{1}{N_k}\sum_{i\in J_k}|b_i|\Big)g_j\Big\|\\ &=\Big(\sum_{j=1}^{\infty}|b_j|^p\|g_j\|^p\Big)^{1/p}+\left(\sum_{k\in\mathbb{N}}\Big(\frac{1}{N_k}\sum_{i\in J_k}|b_i|\Big)^p\sum_{j\in J_k}\|g_j\|^p\right)^{1/p}\\ &\le\|y\|+\left(\sum_{k\in\mathbb{N}}N_k^{1-p}\Big(\sum_{i\in J_k}|b_i|\Big)^p\right)^{1/p}\\ &\le\|y\|+\Big(\sum_{k\in\mathbb{N}}N_k^{1-p}N_k^{p-1}\sum_{i\in J_k}|b_i|^p\Big)^{1/p}\le 3\|y\|.\end{aligned}$$

The last inequality uses that $\|g_j\|_p\ge\frac{1}{2}$ for all $j\in\mathbb{N}$, which follows from (8) and the fact that $N_1\ge 4$. This shows that P is a bounded projection from Y onto X. This completes the proof of statement (ii) of the theorem. Finally, consider

$$\bar{h}_k=\frac{1}{\sqrt{N_k}}\sum_{j\in J_k}f_j=h_k+\frac{1}{\sqrt{N_k}}\sum_{j\in J_k}g_j\quad\text{(by (8))}.$$

Then for $k \in \mathbb{N}$ we have

$$\|\bar{h}_k - h_k\| = \Big\|\frac{1}{\sqrt{N_k}}\sum_{j\in J_k} g_j\Big\|_p = \frac{1}{\sqrt{N_k}}\Big(\sum_{j\in J_k}\|g_j\|_p^p\Big)^{1/p} \leq N_k^{\frac{1}{p}-\frac{1}{2}}\|f\|_p\ ,\ (\text{since } \ \|g_j\|_p \leq \|f\|_p).$$

It follows from (7) and from the Small Perturbation Lemma (c.f [FHHMPZ, Theorem 6.18]) that, for ε sufficiently small, $(\bar{h}_k)$ is equivalent to (h_k), and so $L_p(\mathbb{R})$ embeds into X. Since the closed linear span of (h_k) (naturally embedded in $L_p(\mathbb{R})$) is complemented in Y, it also follows from the Small Perturbation Lemma that, for small enough $\varepsilon > 0$, the closed linear span of $(\overline{h_k})$ is complemented in Y and thus also complemented in X. Thus, $L_p(\mathbb{R})$ is isomorphic to a complemented subspace of X and X is a complemented subspace of $L_p(\mathbb{R})$. By Pełczyński's decomposition method (*cf.* [LT, Remarks after Theorem 2.a.3]) X is therefore isomorphic to $L_p(\mathbb{R})$. □

5. Compactness of restriction operators

Let $1 \leq p < \infty$ and let X be a subspace of $L_p(\mathbb{R})$ generated by translates of a single function in $L_p(\mathbb{R})$. In this section we consider when the restriction operators $R_I \colon X \to L_p(I)$, $x \longmapsto x|_I$, are compact for bounded intervals I, and what this tells us about the structure of X. The first three results can essentially be extracted from [OSSZ]; the presentation here simplifies some of their arguments. The last two results, Propositions 5.4 and 5.5, are new and they demonstrate yet again that in the range $2 < p < \infty$ a richer structure is possible.

In the case where $(T_{\lambda_i} f)_{i=1}^\infty$ is an unconditional basic sequence of translates of some $f \in L_p(\mathbb{R})$, $1 \leq p \leq 2$, the space $X_p(f, (\lambda_i))$ must be quite thin as the next proposition reveals.

Proposition 5.1. *Let $(\lambda_i)_{i=1}^\infty \subset \mathbb{R}$ and $f \in L_p(\mathbb{R})$, $1 \leq p \leq 2$. Let $f_i = T_{\lambda_i} f$ for $i \in \mathbb{N}$, and assume that (f_i) is unconditional basic. Let $I \subset \mathbb{R}$ be a bounded interval and $X = X_p(f, (\lambda_i))$. Then the map $R_I : X \to L_p(I)$, $x \longmapsto x|_I$, is a compact operator.*

Proof. For $p = 1$ this follows by the proof of [OSSZ, Corollary 2.4]. In fact this holds under the assumption that (f_i) is basic (and even less).

Suppose that $1 < p \le 2$ and $\varepsilon > 0$. Since $\sum_{i=1}^\infty \|f_i|_I\|_p^p < \infty$ (see [OSSZ, Proposition 2.1]), there exists $N \in \mathbb{N}$ so that $(\sum_{i=N}^\infty \|f_i|_I\|^p)^{1/p} < \varepsilon$. Let $x = \sum_{i=N}^\infty a_i f_i$, $\|x\|_p = 1$. Then

$$\|x|_I\| \le \sum_{i=N}^\infty |a_i|\,\|f_i|_I\| \le \Big(\sum_{i=N}^\infty |a_i|^q\Big)^{1/q}\Big(\sum_{i=N}^\infty \|f_i|_I\|_p^p\Big)^{1/p}$$

by Hölder's inequality ($\frac{1}{p}+\frac{1}{q}=1$). Since $q \ge 2$, $(\sum_{i=N}^\infty |a_i|^q)^{1/q} \le (\sum_{i=N}^\infty |a_i|^2)^{1/2}$. Furthermore, by the unconditionality of (f_i), there exists a constant K so that

$$\Big(\sum_{i=1}^\infty |a_i|^2\Big)^{1/2} \le K\|x\| = K\ .$$

K depends only on p, the unconditionality constant of (f_i) and $\|f\| = \|f_i\|$ for $i \in \mathbb{N}$. Thus $\|x|_I\| \le K\varepsilon$. This proves that R_I is a compact operator on X. □

We will show in Proposition 5.4 below that Proposition 5.1 fails for $p > 2$. However, in the range $2 < p \le 4$ we have the following result whose proof can be extracted from the proof of [OSSZ, Theorem 2.11].

Proposition 5.2. *Let $(\lambda_i)_{i=1}^\infty \subset \mathbb{R}$ and $f \in L_p(\mathbb{R})$, $2 < p \le 4$. Let $f_i = T_{\lambda_i} f$ be such that (f_i) is unconditional basic. Then there is a basic sequence (g_i) in $L_p(\mathbb{R})$ equivalent to (f_i) such that, for $Y = \overline{\mathrm{span}}\{g_i : i \in \mathbb{N}\}$ and any bounded interval $I \subset \mathbb{R}$, the map $R_I : Y \to L_p(I)$, $y \longmapsto y|_I$, is a compact operator.*

Proof. Let (h_j) be the normalized Haar basis for $L_p[0,1]$. For $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ let h_j^i be h_j translated to $[i, i+1]$. Thus (h_j^i) is a normalized unconditional basis of $L_p(\mathbb{R})$.

By approximating each f_i by a simple dyadic function we find a seminormalized block basis (g_i) of (h_j^i) such that

$$\sum_{i=1}^\infty \big\|\, |f_i| - |g_i|\, \big\|_p < \infty\ . \tag{9}$$

By a very useful observation of Schechtman [S] it follows that (f_i) is equivalent to (g_i).

Set $Y = \overline{\mathrm{span}}\{g_i : i \in \mathbb{N}\}$ and let I be a bounded interval. To show that $R_I : Y \to L_p(I)$ is compact we can assume that $I = [-M, M]$ for some $M \in \mathbb{N}$. It follows from (9) and [OSSZ, Proposition 2.1] that $\sum_{i=1}^{\infty} \|g_i|_I\|_p^p < \infty$. Fix $\varepsilon > 0$ and choose N with

$$\sum_{i=N}^{\infty} \|g_i|_I\|_p^p < \varepsilon \ . \tag{10}$$

We note that $(g_i|_I)$ is a block basis of $(h_j^i)_{(j\in\mathbb{N},\ -M\le i<M)}$ (after omitting zero vectors), and thus it is unconditional basic. Let $y = \sum_{i=N}^{\infty} a_i g_i \in Y$. Recalling that for $p > 2$, seminormalized unconditional basic sequences in $L_p(\mathbb{R})$ satisfy lower ℓ_p and upper ℓ_2 estimates, we obtain the following inequalities with some constant C (dependent only on p and the norm of f).

$$\begin{aligned}
\|y|_I\|_p &= \Big\| \sum_{i=N}^{\infty} a_i g_i|_I \Big\|_p \le C\bigg(\sum_{i=N}^{\infty} |a_i|^2 \|g_i|_I\|_p^2 \bigg)^{1/2} \\
&\le C\|(a_i)_{i=N}^{\infty}\|_{\ell_p} \bigg(\sum_{i=N}^{\infty} \|g_i|_I\|_p^{\frac{2p}{p-2}} \bigg)^{\frac{p-2}{2p}} \text{(using Hölder's inequality with } \tfrac{p}{2} \text{ and } \tfrac{p}{p-2}\text{)} \\
&\le C^2 \|y\|_p \bigg(\sum_{i=N}^{\infty} \|g_i|_I\|_p^p \bigg)^{1/p} \le C^2 \varepsilon^{1/p} \|y\|_p \qquad \text{(using } 2 < p \le 4 \text{ so } \tfrac{2p}{p-2} \ge p\text{)}\ .
\end{aligned}$$

This completes the proof. □

It is worth noting that when the operators R_I on some subspace $X \subset L_p(\mathbb{R})$ are compact for all bounded intervals I then X must embed into ℓ_p in a natural way as the next proposition reveals. This observation and Proposition 5.1 and 5.2 above simplify some arguments in [OSSZ].

If P is a partition of $\mathbb{R}$ into bounded intervals (I_j) we let $\mathbb{E}_P$ denote the *conditional expectation operator on* $L_p(\mathbb{R})$ given by

$$\mathbb{E}_P(f) = \sum_{k=1}^{\infty} \int_{I_k} f(\xi) d\xi \frac{\chi_{I_k}}{m(I_k)}.$$

Proposition 5.3. *Let X be a subspace of $L_p(\mathbb{R})$, $1 \le p < \infty$. If for all bounded intervals $I \subset \mathbb{R}$ the operator $R_I : X \to L_p(I), x \mapsto x|_I$ is compact, then for all $\varepsilon > 0$ there exists a* *partition P of $\mathbb{R}$ into bounded intervals so that for all $x \in S_X$, $\|x - \mathbb{E}_P(x)\| < \varepsilon$. Thus X embeds into ℓ_p.*

Proof. Let $\varepsilon > 0$. For $n \in \mathbb{N}$ let Q_n be the set of dyadic intervals of length 2^{-n} in $[0,1)$, i.e.

$$Q_n = \big\{[0, 2^{-n}), [2^{-n}, 2^{1-n}), \dots [1 - 2^{-n}, 1)\big\}.$$

Then $\mathbb{E}_{Q_n}$ converges in the strong operator topology to the identity on $L_p[0,1]$ and therefore there exists for every relatively compact set $K \subset L_p[0,1)$ and every $\delta > 0$ a large enough $k \in \mathbb{N}$ so that for all $x \in K$, $\|x - \mathbb{E}_{Q_k}(x)\| < \varepsilon$. Choose a sequence $(\varepsilon_n) \subset (0,1)$, with $\sum \varepsilon_n < \varepsilon$ and for each n choose a dyadic partition P_n of the interval $[n, n+1)$ so that for all $x \in S_X$, $\|x|_{[n,n+1)} - \mathbb{E}_{P_n}(x|_{[n,n+1)})\| \le \varepsilon_n$.

By taking P to be the union of all P_n we deduce our claim. □

Proposition 5.1 fails in the case $2 < p \le 4$, and of course for $p > 4$ as well, as shown by the next proposition.

Proposition 5.4. *Let $2 < p < \infty$. There exists $f \in L_p(\mathbb{R})$ and $(\lambda_i)_{i=1}^\infty \subset \mathbb{N}$ so that for $f_i = T_{-\lambda_i} f$, $(f_i)_{i=1}^\infty$ is equivalent to the unit vector basis of ℓ_p, and letting $I = [0,1]$ and $R_I : X_p(f, (-\lambda_i)) \to L_p(I)$, $x \mapsto x|_I$, R_I is not a compact operator.*

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ and let $(N_j)_{j=1}^\infty$ be a subsequence of $\mathbb{N}$ satisfying $\sum_{j=1}^\infty N_j^{q-p} < \infty$. Set m_j as the least integer greater than N_j^q for $j \in \mathbb{N}$ and let $(x_j)_{j=1}^\infty$ be a normalized sequence of disjointly supported elements in $L_p(I)$. Let $(J_j)_{j=1}^\infty$ be a partition of $\mathbb{N}$ into successive intervals so that $|J_j| = m_j$ for all j.

For $i \in J_j$, let x_j^i be x_j placed on the interval $[3^i, 3^i + 1]$ by right translation of 3^i units. Define

$$f = \sum_{j=1}^\infty \sum_{i \in J_j} \frac{1}{N_j}\, x_j^i\ .$$

Note that

$$\|f\|_p^p = \Big\| \sum_{j=1}^\infty \sum_{i\in J_j} \frac{1}{N_j} x_j^i \Big\|_p^p = \sum_{j=1}^\infty m_j \frac{1}{N_j^p} \le 2 \sum_{j=1}^\infty \frac{N_j^q}{N_j^p} < \infty$$

so $f \in L_p(\mathbb{R})$.

Setting $f_i = T_{-3^i} f$, for $i \in \mathbb{N}$, we have, as in the proof of Theorem 4.1,

$$f_i = \frac{1}{N_j} x_j + g_i \ , \quad \text{for} \ \ i \in J_j \ , \tag{11}$$

where the g_i's are disjointly supported, seminormalized and with supports disjoint from I. Therefore (g_i) is equivalent to the unit vector basis of ℓ_p. Thus, to see that (f_i) is equivalent to the unit vector basis of ℓ_p it is sufficient to prove that for all $(a_i)_{i=1}^\infty \in \ell_p$,

$$\Big\| \sum_{i=1}^\infty a_i f_i\big|_I \Big\|_p \le 2 \Big(\sum_{i=1}^\infty |a_i|^p \Big)^{1/p} \ . \tag{12}$$

First note that for $j \in \mathbb{N}$,

$$\frac{1}{N_j} \Big| \sum_{i\in J_j} a_i \Big| \le \frac{1}{N_j} \Big(\sum_{i\in J_j} |a_i|^p \Big)^{1/p} m_j^{1/q} \le 2 \Big(\sum_{i\in J_j} |a_i|^p \Big)^{1/p} \ .$$

Hence

$$\Big\| \sum_{i=1}^\infty a_i f_i\big|_I \Big\|_p^p = \Big\| \sum_{j=1}^\infty \sum_{i\in J_j} a_i \frac{1}{N_j} x_j \Big\|_p^p = \sum_{j=1}^\infty \Big| \sum_{i\in J_j} a_i \frac{1}{N_j} \Big|^p \le 2^p \sum_{i=1}^\infty |a_i|^p \ ,$$

which proves (12).

To see that R_I is not compact, define $y_j = \sum_{i\in J_j} f_i$. Then $\|y_j\|$ is of the order $m_j^{1/p}$ and $\|y_j|_I\| = \| \sum_{i\in J_j} \frac{1}{N_j} x_j \| = \frac{m_j}{N_j} \ge \frac{m_j}{m_j^{1/q}} = m_j^{1/p}$. Thus $m_j^{-1/p} y_j$ is seminormalized and weakly null in $L_p(\mathbb{R})$, but $\|R_I m_j^{-1/p} y_j\|_p \ge 1$ for all j. $\square$

Using much the same argument we have

Proposition 5.5. *Let $2 < p < \infty$. There exists $f \in L_p(\mathbb{R})$ and translations of f, $f_i = T_{-3^i} f$, $i \in \mathbb{N}$, so that*

i) *(f_i) is basic,*

ii) *$L_p(\mathbb{R})$ embeds isomorphically into $X_p(f, (-3^i))$,*

iii) (f_i) *can be blocked into an unconditional* FDD.

Sketch. Let (h_j) be the normalized Haar basis for $L_p[0,1]$. For $i,j\in\mathbb{N}$, let h_j^i be h_j translated to $[3^i,3^i+1]$. Set $f=\sum_j\sum_{i\in J_j}\frac{1}{N_j}h_j^i$ where (J_j) is a partition of $\mathbb{N}$ into successive intervals and $m_j=|J_j|$ is the least integer greater than N_j^q, for $j\in\mathbb{N}$. As above $f_i=\frac{1}{N_j}h_j+g_i$, for $i\in J_j$, where (g_i) is seminormalized and disjointly supported in $\mathbb{R}\setminus[0,1]$.

If $y_j=\sum_{i\in J_j}f_i$, it follows that $m_j^{-1/p}y_j\approx h_j+e_j$ where (e_j) is seminormalized and disjointly supported in $\mathbb{R}\setminus[0,1]$. Since (h_j) admits a lower ℓ_p-estimate, it follows that (h_j+e_j) is equivalent to (h_j), proving ii).

Set $F_j=\operatorname{span}\{f_i : i\in J_j\}$ and note that $F_j\subset\widetilde{F}_j=\operatorname{span}\{h_j,(g_i)_{i\in J_j}\}$. Since $(\widetilde{F}_j)$ is an unconditional FDD, so is (F_j).

To see that (f_i) is basic we need only note that $(f_i)_{i\in J_j}$ is uniformly equivalent, over j, to the unit vector basis of $\ell_p^{m_j}$, as demonstrated in the proof of Proposition 5.4. □

6. Open problems

We end with a collection of remaining open problems. Wavelets and Gabor frames are widely used coordinate systems formed by translating and applying a second operation (dilation or modulation) to a single function. We do not expect coordinate systems consisting solely of translates of a single function to be useful in practice, but it is of interest to know whether or not such coordinate systems are possible. There is still a large gap between the examples of fundamental systems for $L_p(\mathbb{R})$ consisting of translates of a single function and the results of non existence of certain coordinate systems of $L_p(\mathbb{R})$. We believe that the following problems 6.1, 6.2 and 6.3 have negative answers. We exclude in our problems the case $p=1$, since for that case [OSSZ, Corollary 2.4] provides a negative answer to all three questions.

Problem 6.1. Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, and let (f_i) be a sequence of translates of f. Can (f_i) ever be a basis of $L_p(\mathbb{R})$?

Asking whether or not there is a sequence (f_i) of translates of some $f \in L_p(\mathbb{R})$ which forms a basis of f requires a specific order. This might not be natural since there is already a natural order in $\mathbb{R}$. Therefore one could ask for the existence of coordinate systems which are weaker than Schauder bases and do not require an order. Recall that a sequence (x_j) in a Banach space is called *Markushevich basis* or *M-basis of X*, if (x_j) is *fundamental*, which means that the linear span of the x_j is dense in X, *minimal,* which says that for each $j \in \mathbb{N}$ x_j is not in the closed linear span of the other x_i, or equivalently that there is a unique sequence $(x_j^*) \subset X^*$, which is biorthogonal to (x_j), and *total,* which means that for $x \in X$, if $x_j^*(x) = 0$ for all $j \in \mathbb{N}$, then it follows that $x = 0$. We say that an M-basis (x_j) is *bounded* if $\sup_j \|x_j\| \cdot \|x_j^*\| < \infty$. It is clear that every Schauder basis is a bounded M-basis. However, note that for a sequence (x_j) the property of being an M-basis does not depend on any order.

Problem 6.2. Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, and let $\Lambda \subset \mathbb{R}$ be countable. Can $(T_\lambda f)_{\lambda \in \Lambda}$ ever be a bounded M- basis of $L_p(\mathbb{R})$?

We note that the examples of fundamental systems provided in [AO] consisting of translates of some $f \in L_p(\mathbb{R})$, for $2 < p < \infty$, are not bounded M-bases of $L_p(\mathbb{R})$, and, thus, are not positive answers to Problem 6.2. In Theorem 4.1 we constructed an unconditional frame for all of $L_p(\mathbb{R}^d)$, $2 < p < \infty$, of the form $(T_{\lambda_n} f, g_n^*) \in L_p(\mathbb{R}^d) \times L_q(\mathbb{R}^d)$, with $f \in L_p(\mathbb{R}^d)$, where (λ_n) could be any unbounded sequence in $(\mathbb{R}^d)$. This was possible because we allowed the (g_n^*) to be arbitrarily small.

Problem 6.3. Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, and let (f_i) be a sequence of translates of f. Is there a semi normalized sequence $(g_n^*) \subset L_q(\mathbb{R})$ so that (f_n, g_n^*) is an unconditional frame for $L_p(\mathbb{R})$?

For $2 < p < \infty$, we obtained a function $f \in L_p(\mathbb{R})$ and a subsequence $(n_i)_{i=1}^\infty$ of $\mathbb{N}$ so that $X_p(f, (-n_i)_{i=1}^\infty)$ is both isomorphic to $L_p(\mathbb{R})$ and complemented in $L_p(\mathbb{R})$, and $(T_{-n_i}f)_{i=1}^\infty$ can be blocked to form an unconditional FDD for $X_p(f, (n_i)_{i=1}^\infty)$.

Problem 6.4. Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$ and let (f_i) be a sequence of translates of f. Can (f_i) ever be blocked to be an (unconditional) FDD for $L_p(\mathbb{R})$?

In several of our examples we needed a restriction on p. We do not know whether or not some of these restrictions are necessary.

Problems 6.5. Let $f \in L_p(\mathbb{R})$, $1 < p < 2$, and let (f_i) be a sequence of translates of f.

i) Can (f_i) ever be basic such that $L_p(\mathbb{R})$ embeds into $\overline{\text{span}(f_i)}$?

ii) Can (f_i) ever be blocked into an (unconditional) FDD such that $L_p(\mathbb{R})$ embeds into $\overline{\text{span}(f_i)}$?

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