MOVING FINITE UNIT TIGHT FRAMES FOR S^n

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Abstract. Frames for $\mathfrak{R}^n$ can be thought of as redundant or linearly dependent coordinate systems, and have important applications in such areas as signal processing, data compression, and sampling theory. The word "frame" has a different meaning in the context of differential geometry and topology. A moving frame for the tangent bundle of a smooth manifold is a basis for the tangent space at each point which varies smoothly over the manifold. It is well known that the only spheres with a moving basis for their tangent bundle are S^1, S^3, and S^7. On the other hand, after combining the two separate meanings of the word "frame", we show that the n-dimensional sphere, S^n, has a moving finite unit tight frame for its tangent bundle if and only if n is odd. We give a procedure for creating vector fields on S^{2n-1} for all $n \in \mathfrak{N}$, and we characterize exactly when sets of such vector fields form a moving finite unit tight frame on S^{2n-1}. This gives as well a new method for constructing finite unit tight frames for Hilbert spaces.

1. Introduction

Bases and frames for Hilbert spaces give a method to linearly represent vectors as a sequence of coefficients. However, frames can be redundant in that different sequence of coefficients can be used to reconstruct a single vector, which can be useful in applications. For example, if a signal is transmitted as a sequence of basis coefficients and some coefficients are

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lost or corrupted, then an entire dimension is lost and cannot be recovered. The redundancy of a frame helps to mitigate this error by effectively spreading the loss over the whole space instead of localizing it to certain dimensions. Given $n \in \mathbb{N}$, a *tight frame for* $\mathbb{R}^n$ is a sequence $(f_i)_{i=1}^k \subset \mathbb{R}^n$ with $k \geq n$ such that there exists a constant $C > 0$, called the *frame bound*, satisfying

$$x = \frac{1}{C}\sum_{i=1}^{k}\langle f_i, x\rangle f_i \quad \text{ for all } x \in \mathbb{R}^n, \tag{1}$$

where, $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. Thus, a tight frame allows for the exact linear reconstruction of a vector from the sequence of frame coefficients $(\langle f_i, x\rangle)_{i=1}^k$. We call a tight frame $(f_i)_{i=1}^k$ a *finite unit tight frame, or FUNTF*, if $\|f_i\| = 1$ for all $1 \leq i \leq k$. Note that in the case $k = n$, a sequence $(f_i)_{i=1}^n \subset \mathbb{R}^n$ is a FUNTF if and only if it is an orthonormal basis. FUNTFs are the most useful frames for signal processing, as they minimize mean squared error due to noise [GKK]. This has motivated the study of FUNTFs and in particular has led to interest in finding ways to construct FUNTFs [CL]. In [BF], using a potential energy concept inspired from physics, it is proven that there exists a finite unit tight frame of k vectors for $\mathbb{R}^n$ for all natural numbers $k \geq n$. An algorithm for constructing FUNTFs is given in [DFKLOW], and in [CFHWZ] the method of spectral tetris is introduced to explicitly construct FUNTFs as well other types of frames.

In the context of differential geometry and topology, the word "frame" has a different meaning. A moving frame for the tangent bundle of a smooth manifold can be thought of as a basis for the tangent space at each point on the manifold which moves continuously over the manifold. To avoid confusion, we will refer to moving frames in this context as moving bases. If $n \in \mathbb{N}$ and M^n is an n-dimensional smooth manifold, then a *vector field* is a continuous function $f : M^n \to TM^n$ such that $f(p) \in T_pM^n$ for all $p \in M$. That is, a vector field continuously assigns to each point $p \in M^n$ a vector $f(p)$ in the tangent space

T_pM. Thus, a moving basis for the tangent bundle of a n-dimensional smooth manifold M^n is a sequence of vector fields $(f_i)_{i=1}^n$ such that $(f_i(p))_{i=1}^n$ is a basis for T_pM for all $p \in M^n$.

Of particular historical interest in the theory of vector fields and moving bases is the case where the underlying manifold M^n is the n-dimensional sphere S^n. Note that S^1 can be represented by the unit circle in $\mathbb{R}^2$ and that S^2 can be represented by the unit sphere in $\mathbb{R}^3$. The famous Hairy Ball Theorem states that if n is even then there does not exist a nowhere zero vector field for S^n. As a moving basis for S^n would consist of n nowhere zero vector fields, the manifold S^n trivially cannot have a moving basis when n is even. On the other hand, it can be simply proven that S^1, S^3, and S^7 all have a moving basis for their tangent space. To give a brief sketch of this proof, we identify S^1 with the unit circle of the complex numbers $\mathbb{C}$, S^3 with the unit sphere of the quaternions $\mathbb{H}$, and S^7 with the unit sphere of the octonions $\mathbb{O}$. Each of these division algebras contain the real numbers, so in particular we have that $1 \in S^1 \subset \mathbb{C}$, $1 \in S^3 \subset \mathbb{H}$, and $1 \in S^7 \in \mathbb{O}$. Now for each of the cases $n = 1, 3, 7$, we can pick a basis $(v_i)_{i=1}^n$ for the tangent space T_1S^n. To move this basis $(v_i)_{i=1}^n \subset T_1S^n$ from $1 \in S^n$ to a different point $p \in S^n$, we simply multiply each vector in the basis by p giving $(pv_i)_{i=1}^n \subset T_aS^n$. This gives a moving basis for S^n for the cases $n = 1, 3, 7$. This proof will not work for other values of $n \in \mathbb{N}$ as $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the only finite dimensional division algebras over $\mathbb{R}$ besides $\mathbb{R}$ itself. The question of does S^n have a moving basis for its tangent bundle for any other values of n was an important open problem in differential topology and was solved negatively in 1958 by Michel Kervaire and independently by Raoul Bott and John Milnor.

This motivates us to question what happens when we weaken the condition of basis to that of finite unit tight frame. By combining the two definitions of the word "frame", we are led to studying tight frames which vary smoothly over a manifold. As the reconstruction formula (1) uses an inner product, we need to consider Riemannian manifolds, where the Riemannian metric gives a smoothly varying inner product for the tangent space at each point on the manifold. Note that any smooth submanifold of $\mathbb{R}^n$ is naturally a Riemannian manifold where the Riemannian metric is given by the standard inner product on $\mathbb{R}^n$.

Definition 1.1. Let M^n be an n-dimensional Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle_a$ for each $a \in M^n$. Let $k \geq n$, and $f_i : M \to TM^n$ be a vector field for all $1 \leq i \leq k$. We say that $(f_i)_{i=1}^k$ is a *moving tight frame* for the tangent bundle of M^n if $(f_i(a))_{i=1}^k$ is a tight frame for $T_a M^n$ for all $a \in M^n$. That is, for all $a \in M^n$, there exists $C > 0$ such that

$$x = \frac{1}{C} \sum_{i=1}^{k} \langle x, f_i(a) \rangle_a f_i(a) \qquad \text{for all } x \in \pi^{-1}(a).$$

We say that $(f_i)_{i=1}^k$ is a *moving finite unit tight frame (FUNTF)* for the tangent bundle of M^n if it is a moving tight frame and $\|f_i(a)\| = \sqrt{\langle f_i(a), f_i(a) \rangle} = 1$ for all $1 \leq i \leq k$ and $a \in M^n$.

It is proven in [FPWW] that a sequence of vector fields $(f_i)_{i=1}^k$ over a Riemannian manifold M^n is a moving tight frame for TM^n if and only if M^n is a smooth submanifold of a k-dimensional Riemannian manifold N^k such that N^k has a moving orthonormal basis $(e_i)_{i=1}^k$ and $P_x e_i(x) = f_i(x)$ for all $x \in M^n$, where $P_x : T_x N^k \to T_x M^n$ is orthogonal projection. In particular, for all $n \in \mathbb{N}$, we may construct a moving tight frame for S^n by considering $S^n \subseteq \mathbb{R}^{n+1}$ and projecting the standard unit vector basis for R^{n+1} onto $T_x S^n$ for all $x \in S^n$. However, the vector fields obtained from this projection will not be nowhere zero, and the resulting frame will not be a FUNTF. Thus, moving tight frames are simple to create on S^n, but we will need a different approach to creating moving FUNTFs.

If $n \in \mathbb{N}$ is even, then the sphere S^n does not have a nowhere zero vector field, and hence no even dimensional sphere can have a moving FUNTF for its tangent bundle. On the other hand, we will show that every odd dimensional sphere has a moving FUNTF for its tangent bundle. To do this, we give a procedure for creating vector fields on S^{2n-1} for all $n \in \mathbb{N}$, and we characterize exactly when sets of such vector fields form a moving finite unit tight frame on S^{2n-1}. This gives as well a new method for constructing finite unit tight frames for Hilbert spaces, although the number of frame vectors created by this method can be very large relative to the dimension of the Hilbert space.

The concept of a moving tight frame was first applied in 2009 when P. Kuchment proved that particular vector bundles over the torus, which arise in mathematical physics, have natural moving tight frames but do not have moving bases [K]. In [FPWW], it was shown that the dilation theorem for tight frames in $\mathbb{R}^n$ can be extended to moving tight frames. The relationship between frames for Hilbert spaces and manifolds was also considered in a different context by Dykema and Strawn, who studied the manifold structure of collections of FUNTFs under certain equivalent classes [DySt].

For terminology and background on vector bundles and smooth manifolds see [L]. For terminology and background on frames for Hilbert spaces see [C] and [HKLW].

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2. Tight frames

Before discussing moving tight frames, we give some basic results about how to check whether or not a sequence of vectors is a tight frame for a Hilbert space. For $n \in \mathbb{N}$, we denote the unit vector basis for $\mathbb{R}^n$ by $(e_i)_{i=1}^n$. As the reconstruction formula (1) is linear, it holds for all $x \in \mathbb{R}^n$ if and only if it holds for each vector in the orthonormal basis $(e_i)_{i=1}^n$. Furthermore, if $1 \leq p \leq n$ and $x \in \mathbb{R}^n$, then $x = e_p$ if and only if $\langle x, e_q \rangle = \delta_{p,q}$ for all $1 \leq q \leq n$. This leads us to the following fact.

Fact 2.1. *Let $(f_i)_{i=1}^k \subset \mathbb{R}^n$. The sequence $(f_i)_{i=1}^k$ is a tight frame for $\mathbb{R}^n$ with frame bound $C > 0$ if and only if*

$$\frac{1}{C}\sum_{i=1}^{k}\langle f_i, e_p\rangle\langle f_i, e_q\rangle = \delta_{p,q} \qquad \text{for all } 1 \leq p, q \leq n.$$

Given $n \in \mathbb{N}$ and $a \in S^{2n-1} \subset \mathbb{R}^{2n}$, we will be interested in identifying when a sequence $\{f_i(a)\}_{i=1}^k$ is a tight frame for the subspace $T_a S^{2n-1} \subset \mathbb{R}^{2n}$. However, Fact 2.1 may only be used to check if a sequence of vectors is a tight frame for the entire space, not just a subspace. We will use the following fact to get around this problem.

Fact 2.2. *Let $n \in \mathbb{N}$ and $X, Y \subset \mathbb{R}^n$ such that $X = Y^\perp$. If $(x_i)_{i=1}^k \subset X$ is a tight frame for X with frame constant $C > 0$ and $(y_i)_{i=1}^\ell \subset Y$, then $(y_i)_{i=1}^\ell$ is a tight frame for Y with frame constant C if and only if $(x_1, ..., x_k, y_1, ..., y_\ell)$ is a tight frame for $\mathbb{R}^n$ with frame constant C.*

To apply Fact 2.2, we need to know the frame constant for a given finite unit tight frame. This is given by the following fact, which can be deduced from Fact 2.1 by summing over $1 \leq p = q \leq n$.

Fact 2.3. *Let $n, k \in \mathbb{N}$ such that $k \geq n$. If $(f_i)_{i=1}^k \subset \mathbb{R}^n$ is a finite unit tight frame for $\mathbb{R}^n$ then $\frac{k}{n}$ is the frame constant for $(f_i)_{i=1}^k$.*

3. A moving FUNTF for S^{2n-1}

We identify the sphere S^n with the set of unit vectors in R^{n+1}. That is, $S^n = \{(a_1, ..., a_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} a_i^2 = 1\}$. With this representation, it is very easy to identify the tangent space $T_a S^n$ for each $a \in S^n$. We first note that if $a \in S^n$, then the vector a is normal to the tangent space $T_a S^n$. Thus, as S^n is a manifold of 1 less dimension than R^{n+1}, if $a \in S^n$, then a vector $b \in \mathbb{R}^{n+1}$ is in the tangent space $T_a S^n$ if and only if b is orthogonal to a. That is, if $a = (a_1, ..., a_{n+1}) \in S^n$ and $b = (b_1, ..., b_{n+1}) \in \mathbb{R}^{n+1}$ then $b \in T_a S^n$ if and only if $\langle a,b\rangle = \sum_{i=1}^{n+1} a_i b_i = 0$. For example, in the case of the circle, if $(a_1,a_2)\in S^1\subset\mathbb{R}^2$ then $(-a_2,a_1)$ is the unit vector in the tangent line $T_{(a_1,a_2)}S^1$ which points counter-clockwise.

Thus, we can create a unit vector field for $S^1\subset\mathbb{R}^2$ by switching the two coordinates and including a negative sign in the first coordinate. This method can be used to create vector fields for higher dimensional spheres as well. Given any $n\in\mathbb{N}$, we create a unit vector field for $S^{2n-1}\subset\mathbb{R}^{2n}$ by switching pairs of coordinates and inserting a negative sign into one coordinate of each pair. For example, the map $(a_1,a_2,a_3,a_4)\mapsto(a_2,-a_1,a_4,-a_3)$ is a unit vector field on S^3. To formalize this, if $(\varepsilon_i)_{i=1}^{2n}\in\{-1,1\}^{2n}$ and $(k_i)_{i=1}^{2n}\in\{1,2,...,2n\}^{2n}$ then we define $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ to be the operator defined by $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a_1,...,a_{2n}) = (\varepsilon_{k_1}a_{k_1},...,\varepsilon_{k_{2n}}a_{k_{2n}})$. As we are only interested in $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}$ when it switches pairs of coordinates and multiplies one of the coordinates in each pair by -1, we restrict ourselves to the set,

$$\mathbf{A}_{2n} := \left\{U_{(\varepsilon_i,k_i)_{i=1}^{2n}}:\mathbb{R}^{2n}\to\mathbb{R}^{2n}\,\middle|\,\varepsilon_i=-\varepsilon_{k_i},\ i=k_{k_i}\text{ for all }1\le i\le 2n\right\}.$$

Note that the condition that $\varepsilon_i=-\varepsilon_{k_i}$ guarantees in particular that $i\neq k_i$ for all $1\le i\le 2n$. Thus, the set $\mathbf{A}_{2n}$ is exactly the collection of operators which switches pairs of coordinates and multiplies one of the coordinates in each pair by -1. The next lemma shows that the operators in $\mathbf{A}_{2n}$ can be used to construct unit vector fields.

Lemma 3.1. *Let $n\in\mathbb{N}$, $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}\in\mathbf{A}_{2n}$, and $a=(a_1,...,a_{2n})\in S^{2n-1}\subset\mathbb{R}^{2n}$. Then, $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a)\in T_aS^{2n-1}$ and $\|U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a)\|=1$.*

Proof. Recall that $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a)\in T_aS^{2n-1}$ if and only if $\langle U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a),a\rangle=0$. We have that,

$$2\langle U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a),a\rangle = 2\sum_{i=1}^{2n}\varepsilon_{k_i}a_{k_i}a_i = \sum_{i=1}^{2n}\varepsilon_{k_i}a_{k_i}a_i+\varepsilon_{k_{k_i}}a_{k_{k_i}}a_{k_i} = \sum_{i=1}^{2n}\varepsilon_{k_i}a_{k_i}a_i+\varepsilon_i a_i a_{k_i} = 0.$$

Thus, $\langle U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a),a\rangle=0$ and hence $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a)\in T_aS^{2n-1}$. The vector $U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a)$ is formed by permuting the coordinates of a and multiplying half the coordinates by -1. None of these operations changes the norm, and thus $\|U_{(\varepsilon_i,k_i)_{i=1}^{2n}}(a)\|=1$ as $\|a\|=1$. $\square$

By Lemma 3.1, if $U \in \mathbf{A}_{2n}$, then we may define a unit vector field $f_U : S^{2n-1} \to TS^{2n-1}$ by $f_U(a) := U(a)$ for all $a \in S^{2n-1}$. In the introduction we used division algebras over $\mathbb{R}$ to prove that S^1, S^3, and S^7 each have a moving basis for their tangent bundle. However, for $n \in \{1, 3, 7\}$ it is also possible to choose $A \subset A_{2n}$ such that $\{f_U\}_{U \in A}$ is a moving orthonormal basis for S^n. Indeed, if $(a_1, a_2) \in S^1$ then $\{(-a_2, a_1)\}$ is an orthonormal basis for $T_{(a_1,a_2)}S^1$, and if $(a_1, a_2, a_3, a_4) \in S^3$ then $\{(-a_2, a_1, a_4, -a_3), (-a_3, -a_4, a_1, a_2), (-a_4, a_3, -a_2, a_1)\}$ is an orthonormal basis for $T_{(a_1,a_2,a_3,a_4)}S^3$. Constructing a moving orthonormal basis for TS^7 can be done similarly. Thus, the simple method of constructing unit vector fields by switching pairs of coordinates and multiplying one element in each pair by -1 can be used to create a moving orthonormal basis for the tangent bundles of S^1, S^3, and S^7. As no other sphere has a moving basis for its tangent bundle, we now focus on moving FUNTFs. A FUNTF is able to reconstruct every vector in $\mathbb{R}^n$ exactly, and hence the vectors that comprise a FUNTF are in this respect, distributed uniformly across $\mathbb{R}_n$. Thus, it is natural to assume that whether or not a set of vector fields of the form $\{f_U\}_{U \in A}$ is a moving FUNTF depends on whether or not the set A in some respect uniformly switches pairs of coordinates and in some respect uniformly inserts negative signs. With this in mind, we introduce the following definition.

Definition 3.2. Let $n \in \mathbb{N}$ and $A \subset \mathbf{A}_{2n}$. We say that A is *balanced* if the following two conditions are satisfied for all $1 \le p, q \le 2n$ with $p \ne q$.

a) $\#\left\{U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in A \,\middle|\, k_p = q\right\} = \frac{\#A}{2n-1}$,
and for all $1 \le r, s \le 2n$ with $p, q, r,$ and s all being distinct integers,

b) $\#\left\{U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in A \,\middle|\, \varepsilon_p\varepsilon_q = -1,\ \{k_r, k_s\} = \{p, q\}\right\}$
$= \#\left\{U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in A \,\middle|\, \varepsilon_p\varepsilon_q = 1,\ \{k_r, k_s\} = \{p, q\}\right\}$.

For the sake of convenience, if $A \subset \mathbf{A}_{2n}$ and $1 \leq p, q \leq 2n$ are distinct integers then we denote $A_{p,q} = \left\{ U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in A \,\middle|\, k_p = q \right\}$, and if $\varepsilon \in \{-1, 1\}$ and $1 \leq p, q, r, s \leq 2n$ are distinct integers then we denote $A_{p,q,r,s,\varepsilon} = \left\{ U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in A \,\middle|\, \varepsilon_p \varepsilon_q = \varepsilon,\ \{k_r, k_s\} = \{p, q\} \right\}$.

Verifying that a particular set $A \subset \mathbf{A}_{2n}$ is balanced requires checking many different equations, and it is not immediately obvious that balanced sets always exist. However, the following lemma shows that the entire set $\mathbf{A}_{2n}$ itself is balanced.

Lemma 3.3. *For all $n \in \mathbb{N}$, the set $\mathbf{A}_{2n}$ is balanced.*

Proof. Let $n \in \mathbb{N}$. We first count the number of ways to choose an operator $U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in \mathbf{A}_{2n}$. We may count $(2n-1)$ ways to choose k_1 from $\{1, ..., 2n\} \setminus \{1\}$. If i_2 is the first element of $\{1, ..., 2n\} \setminus \{1, k_1\}$ then there are $(2n-3)$ choices for k_{i_2} from $\{1, ..., 2n\} \setminus \{1, k_1, i_2\}$. Continuing in this manner gives $(2n-1)(2n-3)...3 \cdot 1$ ways to choose $(k_i)_{i=1}^{2n}$ for an operator $U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in \mathbf{A}_{2n}$. For each $1 \leq i \leq 2n$, there are two ways to choose $(\varepsilon_i, \varepsilon_{k_i})$. As there are n of these pairs, we have that, $\#\mathbf{A}_{2n} = (2n-1)(2n-2)...3 \cdot 1 \cdot 2^n = \frac{(2n-1)(2n-2)...3\cdot 1\cdot 2^n n!}{n!} = \frac{(2n)!}{n!}$.

Let $1 \leq p, q \leq 2n$ such that $p \neq q$. We now count the elements in the set $A_{p,q}$. Given a choice of $(\varepsilon_q, \varepsilon_p)$ and setting $k_p = q$ and $k_q = p$, there are $\#\mathbf{A}_{2n-2}$ choices for $(\varepsilon_i, k_i)_{1 \leq i \leq 2n, i \neq p,q}$ such that $U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in \mathbf{A}_{2n}$. As there are 2 choices for $(\varepsilon_p, \varepsilon_q)$, we have that

$$\#A_{p,q} = 2\#\mathbf{A}_{2n-2} = 2\frac{(2n-2)!}{(n-1)!} = \frac{2n(2n-1)(2n-2)!}{n(2n-1)(n-1)!} = \frac{(2n)!}{(2n-1)n!} = \frac{\#\mathbf{A}_{2n}}{2n-1}.$$

Thus condition $i)$ is satified. We now show that condition $b)$ is satisfied. Let $1 \leq p, q, r, s \leq 2n$ such that $p, q, r,$ and s are all distinct integers. For all $1 \leq i \leq 2n$ and $U_{(\varepsilon_i,k_i)_{i=1}^{2n}} \in A_{p,q,r,s,-1}$, we let $\delta_i = -1$ if $i \in \{p, k_p\}$ and $\delta_i = 1$ otherwise. We define a map $\phi : A_{p,q,r,s,-1} \to A_{p,q,r,s,1}$ by $\phi(U_{(\varepsilon_i,k_i)_{i=1}^{2n}}) = U_{(\delta_i\varepsilon_i,k_i)_{i=1}^{2n}}$. This is a bijection, and hence $\#A_{p,q,r,s,-1} = \#A_{p,q,r,s,1}$. $\square$

Lemma 3.3 gives in particular that for all $n \in \mathbb{N}$ there exists a balanced subset of $\mathbf{A}_{2n}$. The following theorem then proves that there exists a moving FUNTF for S^{2n-1} for every $n \in \mathbb{N}$.

Theorem 3.4. *For all $n \in \mathbb{N}$, a subset $A \subset \mathbf{A}_{2n}$ is balanced if and only if $\{f_U\}_{U\in A}$ is a moving FUNTF for S^{2n-1}.*

Proof. Let $n \in \mathbb{N}$ and $A \subset \mathbf{A}_{2n}$. Note that for all $a \in S^{2n-1}$, we have that $a^\perp = T_a S^{2n-1}$. By Facts 2.2 and 2.3, for each $a \in S^{2n-1}$, $\{f_U(a)\}_{U\in A}$ is a FUNTF for $T_a S^{2n-1}$ if and only if $\big\{\sqrt{\frac{\#A}{(2n-1)}}a\big\} \cup \big\{f_U(a)\big\}_{U\in A}$ is a tight frame for $\mathbb{R}^{2n}$ with frame bound $\frac{\#A}{2n-1}$.

Assume that $A \subset \mathbf{A}_{2n}$ is not balanced. A fails either condition $a)$ or condition $b)$. We first assume that A fails condition $a)$. We have that there exists $1 \leq p, q \leq 2n$ such that $\#A_{p,q} \neq \frac{\#A}{2n-1}$. We will use Fact 2.1 to show that $\big\{\sqrt{\frac{\#A}{(2n-1)}}\frac{1}{\sqrt{2}}(e_p+e_q)\big\}\cup\big\{U(\frac{1}{\sqrt{2}}(e_p+e_q))\big\}_{U\in A}$ is not a tight frame for $\mathbb{R}^{2n}$ by showing that the reconstruction formula applied to e_p is not orthogonal to e_q. We let $a = \frac{1}{\sqrt{2}}(e_p + e_q)$.

$$
\begin{aligned}
\frac{\#A}{(2n-1)}\langle a, e_p\rangle\langle a, e_q\rangle + \sum_{U\in A}\langle Ua, e_p\rangle\langle Ua, e_q\rangle &= \frac{\#A}{(2n-1)}\frac{1}{2} + \sum_{U\in A_{p,q}} \varepsilon_{k_q}\varepsilon_{k_p}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\
&= \frac{\#A}{(2n-1)}\frac{1}{2} + \sum_{U\in A_{p,q}} -\frac{1}{2} \qquad \text{as } k_p = q \\
&= \frac{1}{2}\left(\frac{\#A}{(2n-1)} - \#A_{p,q}\right) \neq 0
\end{aligned}
$$

Thus, $\big\{\sqrt{\frac{\#A}{(2n-1)}}a\big\}\cup\big\{U(a)\big\}_{U\in A}$ is not a tight frame for $\mathbb{R}^{2n}$ by Fact 2.1 and hence $\big\{U(a)\big\}_{U\in A}$ is not a tight frame for $T_a S^{2n-1}$ by Fact 2.2. We now assume that A fails condition $b)$. There exist distinct integers $1 \leq p, q, r, s \leq 2n$ such that $\#A_{p,q,r,s,-1} \neq \#A_{p,q,r,s,1}$. We will show that $\Big\{\sqrt{\frac{\#A}{(2n-1)}}\frac{1}{\sqrt{2}}(e_p + e_q)\Big\} \cup \Big\{U(\frac{1}{\sqrt{2}}(e_p + e_q))\Big\}_{U\in A}$ is not a tight frame for $T_{\frac{1}{\sqrt{2}}(e_p+e_q)}S^{2n-1}$ by showing that the reconstruction formula applied to e_r is not orthogonal to e_s. We let $a = \frac{1}{\sqrt{2}}(e_p + e_q)$.

$$
\begin{aligned}
\frac{\#A}{(2n-1)}\langle a, e_r\rangle\langle a, e_s\rangle + \sum_{U\in A}\langle Ua, e_r\rangle\langle Ua, e_s\rangle &= 0 + \sum_{U\in A_{p,q,r,s,1}\cup A_{p,q,r,s,-1}} \langle Ua, e_r\rangle\langle Ua, e_s\rangle \\
&= \sum_{U\in A_{p,q,r,s,1}} \frac{1}{2} + \sum_{U\in A_{p,q,r,s,-1}} -\frac{1}{2}
\end{aligned}
$$

$$= \frac{1}{2}(\#A_{p,q,r,s,1} - \#A_{p,q,r,s,-1}) \neq 0$$

Thus, $\left\{\sqrt{\frac{\#A}{(2n-1)}}a\right\}\cup\left\{U(a)\right\}_{U\in A}$ is not a tight frame for $\mathbb{R}^{2n}$ by Fact 2.1 and hence $\left\{U(a)\right\}_{U\in A}$ is not a tight frame for T_aS^{2n-1} by Fact 2.2. Thus, if $A \subset \mathbf{A}_{2n}$ is not balanced then $(f_U)_{U\in A}$ is not a moving FUNTF for S^{2n-1}.

Assume that $A \subset \mathbf{A}_{2n}$ is balanced. We will show that $\left\{\sqrt{\frac{\#A}{(2n-1)}}a\right\} \cup \left\{U(a)\right\}_{U\in A}$ is a tight frame for R^{2n} for all $a \in S^{2n-1}$. Let $a = (a_i)_{i=1}^{2n} \in S^{2n-1} \subset \mathbb{R}^{2n}$ and $1 \leq p, q \leq 2n$ with $p \neq q$.

$$\begin{aligned}
\frac{\#A}{(2n-1)}\langle a, e_p\rangle\langle a, e_q\rangle + \sum_{U\in A}\langle Ua, e_p\rangle\langle Ua, e_q\rangle &= \frac{\#A}{(2n-1)}a_pa_q + \sum_{U\in A}\varepsilon_{k_p}\varepsilon_{k_q}a_{k_p}a_{k_q} \\
&= \frac{\#A}{(2n-1)}a_pa_q + \sum_{U\in A_{p,q}} -a_qa_p + \sum_{\substack{r\neq p, r\neq q\\ s\neq p, s\neq q}}\left(\sum_{U\in A_{p,q,r,s,1}} a_ra_s + \sum_{U\in A_{p,q,r,s,-1}} -a_ra_s\right) \\
&= \frac{\#A}{(2n-1)}a_pa_q - \#A_{p,q}a_qa_p + 0 = 0 \qquad \text{as } A \text{ is balanced.}
\end{aligned}$$

Thus the reconstruction formula applied to e_p is orthogonal to e_q. We now let $a = (a_i)_{i=1}^{2n} \in S^{2n-1} \subset \mathbb{R}^{2n}$ and $1 \leq p \leq 2n$.

$$\begin{aligned}
\frac{\#A}{2n-1}\langle a, e_p\rangle^2 + \sum_{U\in A}\langle Ua, e_p\rangle^2 &= \frac{\#A}{2n-1}a_p^2 + \sum_{U\in A}a_{k_p}^2 \\
&= \frac{\#A}{2n-1}a_p^2 + \sum_{q\neq p}\sum_{U\in A_{p,q}} a_q^2 \\
&= \frac{\#A}{2n-1}a_p^2 + \sum_{q\neq p}\frac{\#A}{2n-1}a_q^2 \qquad \text{as } A \text{ is balanced.} \\
&= \frac{\#A}{2n-1} \qquad \text{as } \|a\| = 1
\end{aligned}$$

Thus, $\left\{\sqrt{\frac{\#A}{(2n-1)}}a\right\} \cup \left\{U(a)\right\}_{U\in A}$ is a tight frame for R^{2n} with frame bound $\frac{\#A}{2n-1}$ by Fact 2.1. This gives us that $\{U(a)\}_{U\in A}$ is a tight frame for T_aS^n for all $a \in S^n$ by Fact 2.2, and hence $\{f_U\}_{U\in A}$ is a moving FUNTF for S^{2n-1}. $\square$

By Lemma 3.3 and Theorem 3.4, we have for all $n \in \mathbb{N}$, that $(f_U)_{U\in \mathbf{A}_{2n}}$ is a moving FUNTF for S^{2n-1}. Thus, as in the proof of Lemma 3.3, S^{2n-1} has a moving FUNTF of $\frac{(2n)!}{n!}$ vector fields for all $n \in \mathbb{N}$. The natural next step is to find FUNTFs comprised of fewer vector fields. That is, our goal is to now find a proper subset of $\mathbf{A}_{2n}$ which is balanced.

Theorem 3.5. *For all $n \in \mathbb{N}$, there exists a balanced subset $A \subset \mathbf{A}_{2n}$ with $\#A = (2n-1)2^{n-1}$. In particular, S^{2n-1} has a moving FUNTF of $(2n-1)2^{n-1}$ vector fields for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Our first goal is to create a subset $B \subset \{1, .., 2n\}^{2n}$ such that

a) If $(k_i)_{i=1}^{2n} \in B$ then $k_i \neq i$ and $k_{k_i} = i$ for all $1 \leq i \leq 2n$

b) If $1 \leq r, s \leq 2n$ and $r \neq s$ then there exists $(k_i)_{i=1}^{2n} \in B$ such that $k_r = s$

c) If $(k_i)_{i=1}^{2n}, (\ell_i)_{i=1}^{2n} \in B$ then $k_i \neq \ell_i$ for all $1 \leq i \leq 2n$

Essentially, B would be a set of ways to pair up the components of $\mathbb{R}^{2n}$ such that any two components are paired up by exactly one permutation from B. Assuming we have created such a set B, we may define $A \subset \mathbf{A}_{2n}$ by

$$A = \left\{U_{(\varepsilon_i, k_i)_{i=1}^{2n}} \in \mathbf{A}_{2n} \,\middle|\, \varepsilon_1 = 1,\, \varepsilon_i = -\varepsilon_{k_i},\, (k_i)_{i=1}^{2n} \in B. \right\}$$

That is, we define A by pairing up each permutation in B with all possible sequences $(\varepsilon_i)_{i=1}^{2n}$ such that $\varepsilon_1 = 1$. Due to the restrictions on B, it is straightforward to check that the resulting set A will be balanced. The set B contains $2n-1$ elements, and for each choice of

$(k_i)_{i=1}^{2n} \in B$ there are 2^{n-1} choices for $(\varepsilon_i)_{i=1}^{2n}$. Thus, $\#A = (2n-1)2^{n-1}$. All that remains is to show that such a set B exists.

To create B, we will construct a $2n \times 2n$ symmetric matrix M such that every row (and column) of M is a permutation of $\{0, 1, ..., 2n-1\}$ and the diagonal of M is constant 0. Given such a matrix $M = [m_{i,j}]_{1\le i,j\le 2n}$, for all $1 \le i, j < 2n$ we let k_i^j be the column number whose entry in the ith row of M equals j. Then we may set $B := \{(k_i^j)_{i=1}^{2n}\}_{1\le j<2n}$. We have that a) is satisfied as M has 0 diagonal and is symmetric. We have that b) is satisfied as if $1 \le r, s \le 2n$ with $r \ne s$ then $k_r^{m_{r,s}} = s$. We have that c) is satisfied as each row of M is a permutation of $\{0, 1, ..., 2n-1\}$.

Thus, to create B, we need to construct a $2n\times 2n$ symmetric matrix M such that every row (and column) of M is a permutation of $\{0, 1, ..., 2n-1\}$ and the diagonal of M is constant 0. We create $M = [m_{i,j}]_{1\le i,j\le 2n}$ by defining $m_{i,j}$ for each $1 \le i, j \le 2n$ by

$$m_{i,j} = \begin{cases} 0 & \text{if } i = j, \\ (i+j-2) \bmod(2n-1)+1 & \text{if } i \ne j \text{ and } 1 \le i, j < 2n, \\ (2i-2) \bmod(2n-1)+1 & \text{if } j = 2n \text{ and } 1 \le i < n, \\ (2j-2) \bmod(2n-1)+1 & \text{if } i = 2n \text{ and } 1 \le j < n. \end{cases}$$

If we were to write out the matrix, it would look like:

$$\begin{pmatrix}
0 & 2 & 3 & 4 & \cdots & & & \cdots & 2n-4 & 2n-3 & 2n-2 & 2n-1 & 1 \\
2 & 0 & 4 & 5 & \cdots & & & \cdots & 2n-3 & 2n-2 & 2n-1 & 1 & 3 \\
3 & 4 & 0 & 6 & \cdots & & & \cdots & 2n-2 & 2n-1 & 1 & 2 & 5 \\
\vdots & & & \ddots & & & & & & & & & \vdots \\
2n-3 & 2n-2 & 2n-1 & 1 & \cdots & & & \cdots & 2n-7 & 0 & 2n-5 & 2n-4 & 2n-6 \\
2n-2 & 2n-1 & 1 & 2 & \cdots & & & \cdots & 2n-6 & 2n-5 & 0 & 2n-3 & 2n-4 \\
2n-1 & 1 & 2 & 3 & \cdots & & & \cdots & 2n-5 & 2n-4 & 2n-3 & 0 & 2n-2 \\
1 & 3 & 5 & 7 & \cdots & 2n-1 & 2 & \cdots & 2n-8 & 2n-6 & 2n-4 & 2n-2 & 0
\end{pmatrix}$$

Switching the variables i and j in the definition of $m_{i,j}$ leaves $m_{i,j}$ unchanged and hence, M is symmetric. Setting $m_{i,j} = 0$ if $i = j$ guarantees that M has constant 0 diagonal. To show that each row of M is a permutation of $\{0, 1, ..., 2n-1\}$ we first consider $1 \leq i < 2n$. The sequence $\big((i+j-2) \bmod (2n-1)\big)_{j=1}^{2n-1}$ is a permutation of $(j)_{j=0}^{2n-2}$ and hence $\big((i+j-2) \bmod (2n-1)+1\big)_{j=1}^{2n-1}$ is a permutation of $(j)_{j=1}^{2n-1}$. The ith row of M is formed by concatenating 0 as the $2n$th element of the sequence $\big((i+j-2) \bmod (2n-1)+1\big)_{j=1}^{2n-1}$ then switching the ith and the $2n$th element. Thus the ith row of M is a permutation of $(j)_{j=0}^{2n-1}$ when $1 \leq i < 2n$. For the case $i = 2n$, we have that $\big((2j-2) \bmod (2n-1)\big)_{j=1}^{2n-1}$ is a permutation of $(j)_{j=0}^{2n-2}$ as 2 and $2n-1$ are relatively prime, and hence $\big((2j-2) \bmod (2n-1)+1\big)_{j=1}^{2n-1}$ is a permutation of $(j)_{j=1}^{2n-1}$. The $2n$th row of M is formed by concatenating 0 as the $2n$th element of the sequence $\big((2j-2) \bmod (2n-1)+1\big)_{j=1}^{2n-1}$, and hence is a permutation of $(j)_{j=0}^{2n-1}$. Thus, we have formed a matrix M satisfying all our desired properties, and the proof is complete.

$\square$

References

[BF] J. J. Benedetto and M. Fickus, *Finite normalized tight frames*, Adv. Comput. Math. **18** (2003), 357–385.

[CFHWZ] P. G. Casazza, M. Fickus, A. Heinecke, Y. Wang, and Z. Zhou, *Spectral Tetris Fusion Frame Constructions*, submitted.

[C] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, (2003).

[CL] P.G. Casazza and N. Leonhard, *Classes of finite equal norm Parseval Frames*, Contemp. Math **451** (2008), 11–31.

[DFKLOW] K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower, and E. Weber,*Ellipsoidal tight frames and projection decompositions of operators*, Illinois J. Math. **48** (2004), no. 2, 477–489.

[DySt] K. Dykema and N. Strawn, *Manifold structure of spaces of spherical tight frames*, Int. J. Pure Appl. Math. **28** (2006), 217–256.

[FPWW] D. Freeman, D. Poore, A. R. Wei, and M. Wyse, *Moving Pareseval frames for vector bundles*, Houston J. Math., to appear.

[HKLW] D. Han, K. Kornelson, D. Larson, and E. Weber, *Frames for Undergraduates*, Amer. Math. Soc. **145** (2007)

[GKK] V. K. Goyal, J. Kovačević and J. A. Kelner *Quantized frame expansions with erasures*, ACHA **10** no. 3 (2001), 203–233.

[K] P. Kuchment, *Tight frames of exponentially decaying Wannier functions*, J. Phys. A: Math. Theor. **42**, no. 2, (2009)

[L] J. M. Lee, *Introduction to Smooth Manifolds*, Springer, (2002).

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