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Weakly null sequences with upper estimates

by

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Abstract. We prove that if (v_i) is a seminormalized basic sequence and X is a Banach space such that every normalized weakly null sequence in X has a subsequence that is dominated by (v_i), then there exists a uniform constant $C \geq 1$ such that every normalized weakly null sequence in X has a subsequence that is C-dominated by (v_i). This extends a result of Knaust and Odell, who proved this for the cases in which (v_i) is the standard basis for ℓ_p or c_0.

1. Introduction. In some circumstances, local estimates give rise to uniform global estimates. An elementary example of this is that every continuous function on a compact metric space is uniformly continuous. Uniform estimates are especially pertinent in functional analysis, as one of the cornerstones to the subject is the Uniform Boundedness Principle. Because uniform estimates are always desirable, it is important to determine when they occur. In this paper, we are concerned with uniform upper estimates of weakly null sequences in a Banach space. Before stating precisely what we mean by this, we give some historical context.

For each $1 < p < \infty$, Johnson and Odell [JO] have constructed a Banach space X such that every normalized weakly null sequence in X has a subsequence equivalent to the standard basis for ℓ_p, and yet there is no fixed $C \geq 1$ such that every normalized weakly null sequence in X has a subsequence C-equivalent to the standard basis for ℓ_p. A basic sequence (x_i) is equivalent to the unit vector basis for ℓ_p if it has both a lower and an upper ℓ_p estimate. That is, there exist constants $C, K \geq 1$ such that

$$\frac{1}{K}\Big(\sum |a_i|^p\Big)^{1/p} \leq \Big\|\sum a_i x_i\Big\| \leq C\Big(\sum |a_i|^p\Big)^{1/p} \qquad \forall (a_i) \in c_{00}.$$

The examples of Johnson and Odell show that the upper constant C and the lower constant K cannot always both be chosen uniformly. It is somewhat surprising then that Knaust and Odell proved [KO2] that the upper

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estimate can always be chosen uniformly. Specifically, they proved that for every Banach space X, if each normalized weakly null sequence in X has a subsequence with an upper ℓ_p estimate, then there exists a constant $C \geq 1$ such that each normalized weakly null sequence in X has a subsequence with a C-upper ℓ_p estimate. They also proved earlier the corresponding theorem for upper c_0 estimates [KO1]. The standard bases for ℓ_p, $1 < p < \infty$, and c_0 enjoy many strong properties which Knaust and Odell employ in their papers. It is natural to ask what are necessary and sufficient properties for a basic sequence to have in order to guarantee the uniform upper estimate. In this paper we show that actually all seminormalized basic sequences give uniform upper estimates. We make the following definition to formalize this.

DEFINITION 1.1. Let $V = (v_n)_{n=1}^{\infty}$ be a seminormalized basic sequence. A Banach space X has *property* S_V if every normalized weakly null sequence (x_n) in X has a subsequence (y_n) such that for some constant $C < \infty$,

$$(1) \qquad \Big\| \sum_{n=1}^{\infty} \alpha_n y_n \Big\| \leq C \quad \text{for all } (\alpha_n) \in c_{00} \text{ with } \Big\| \sum_{n=1}^{\infty} \alpha_n v_n \Big\| \leq 1.$$

X has *property* U_V if C may be chosen uniformly. We say that (y_n) has a *C-upper V-estimate* (or that V *C-dominates* (y_n)) if (1) holds for C, and that (y_n) has an *upper V-estimate* (or that V *dominates* (y_n)) if (1) holds for some C.

Using these definitions, we can formulate the main theorem of our paper:

THEOREM 1.2. *A Banach space has property S_V if and only if it has property U_V.*

S_V and U_V are isomorphic properties of V, so it is sufficient to prove Theorem 1.2 for only normalized bimonotone basic sequences. This is because every seminormalized basic sequence is equivalent to a normalized bimonotone basic sequence. Indeed, if $0 < A \leq \|v_i\| \leq B$ for all $i \in \mathbb{N}$, then we can define a new norm $|||\cdot|||$ on $[v_i]$ by $|||x||| = B^{-1} \sup_{n<m} \|P_{[n,m]}x\| \vee \sup_{i\in\mathbb{N}} |v_i^*(x)|$ for all $x \in [v_i]$, where $P_{[n,m]}$ denotes the projection of $[v_i]$ onto the span of $\{v_n, \ldots, v_m\}$. The norm $|||\cdot|||$ is equivalent to $\|\cdot\|$ on $[v_i]$ and (v_i) is normalized and bimonotone in the new norm.

In Section 2 we present the necessary definitions and reformulate our main results. We break up the main proof into two parts which we give in Sections 3 and 4. In Section 5 we give some illustrative examples which show in particular that our result is a genuine extension of [KO2] and not just a corollary.

For a Banach space X we use the notation B_X to mean the closed unit ball of X and S_X to mean the unit sphere of X. If $F \subset X$ we denote by $[F]$

the closed linear span of F in X. If N is a sequence in $\mathbb{N}$, we denote by $[N]^\omega$ the set of all infinite subsequences of N.

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2. Main results. Here we introduce the main definitions and theorems of the paper. Many of our theorems and lemmas are direct generalizations of corresponding results in [KO2]. We specify when we are able to follow the same outline as a proof in [KO2], and also when we are able to follow a proof exactly.

DEFINITION 2.1. Let X be a Banach space and $V = (v_n)_{n=1}^\infty$ be a normalized bimonotone basic sequence. With the exception of (ii), the following definitions are adapted from [KO2].

(i) A sequence (x_n) in X is called a uV-sequence if $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$, (x_n) converges weakly to 0, and

$$\sup_{\|\sum_{n=1}^\infty \alpha_n v_n\| \leq 1} \Big\| \sum_{n=1}^\infty \alpha_n x_n \Big\| < \infty.$$

(x_n) is called a *C-uV-sequence* if

$$\sup_{\|\sum_{n=1}^\infty \alpha_n v_n\| \leq 1} \Big\| \sum_{n=1}^\infty \alpha_n x_n \Big\| < C.$$

(ii) A sequence (x_n) in X is called a *hereditary uV-sequence* if every subsequence of (x_n) is a uV-sequence, and is called a *hereditary C-uV-sequence* if every subsequence of (x_n) is a C-uV-sequence.

(iii) A sequence (x_n) in X is called an *M-bad uV-sequence* for a constant $M < \infty$ if every subsequence of (x_n) is a uV-sequence, and no subsequence of (x_n) is an M-uV-sequence.

(iv) An array $(x_i^n)_{i,n=1}^\infty$ in X is called a *bad uV-array* if each sequence $(x_i^n)_{i=1}^\infty$ is an M_n-bad uV-sequence for some constants M_n with $M_n \to \infty$.

(v) $(y_i^k)_{i,k=1}^\infty$ is called a *subarray* of $(x_i^n)_{i,n=1}^\infty$ if there is a subsequence (n_k) of $\mathbb{N}$ such that every sequence $(y_i^k)_{i=1}^\infty$ is a subsequence of $(x_i^{n_k})_{i=1}^\infty$.

(vi) A bad uV-array $(x_i^n)_{i,n=1}^\infty$ is said to satisfy the *V-array procedure* if there exists a subarray (y_i^n) of (x_i^n) and there exists $(a_n) \subseteq \mathbb{R}^+$ with $a_n \leq 2^{-n}$, for all $n \in \mathbb{N}$, such that the weakly null sequence (y_i) with $y_i := \sum_{n=1}^\infty a_n y_i^n$ has no uV-subsequence.

(vii) X satisfies the *V-array procedure* if every bad uV-array in X satisfies the V-array procedure. X satisfies the *V-array procedure for normalized bad uV-arrays* if every normalized bad uV-array in X satisfies the V-array procedure.

NOTE. A subarray of a bad uV-array is a bad uV-array. Also, a bad uV-array satisfies the V-array procedure if and only if it has a subarray which satisfies the V-array procedure.

Our Theorem 1.2 is now an easy corollary of the theorem below.

THEOREM 2.2. *Every Banach space satisfies the V-array procedure for normalized bad uV-arrays.*

Theorem 2.2 implies Theorem 1.2 because if a Banach space X has property S_V and not U_V then there exists a normalized bad uV-array, and the V-array procedure gives a weakly null sequence in B_X which has no uV-subsequence. The sequence must be seminormalized, so we could pass to a basic subsequence on which the norm of each element is essentially constant, and then renormalize. This would give a weakly null sequence with no uV-subsequence, contradicting X being U_V.

The proof for Theorem 2.2 will be given first for the following special case.

PROPOSITION 2.3. *Let K be a countable compact metric space. Then $C(K)$ satisfies the V-array procedure.*

The case of a general Banach space reduces to this special case by the following proposition.

PROPOSITION 2.4. *Let $(x_i^n)_{i,n=1}^\infty$ be a normalized bad uV-array in a Banach space X. Then there exists a subarray (y_i^n) of (x_i^n) and a countable w^*-compact subset K of B_{Y^*}, where $Y := [y_i^n]_{i,n=1}^\infty$, such that $(y_i^n|_K)$ is a bad uV-array in $C(K)$.*

Theorem 2.2 is an easy consequence of Propositions 2.3 and 2.4. Note that Proposition 2.4 is only proved for normalized bad uV-arrays. This makes the proof a little less technical.

Before we prove anything about subarrays though, we need to first consider just a single weakly null sequence. One of the many nice properties enjoyed by the standard basis for ℓ_p, which we denote by (e_i), is that (e_i) is 1-spreading. This is the property that every subsequence of (e_i) is 1-equivalent to (e_i). Spreading is of particular importance because it implies the following two properties which are implicitly used in [KO2]:

(i) If (e_i) C-dominates a sequence (x_i) then (e_i) C-dominates every subsequence of (x_i).

(ii) If a sequence (x_i) C-dominates (e_i) then (x_i) C-dominates every subsequence of (e_i).

Throughout the paper, we will be passing to subsequences and subarrays, so properties (i) and (ii) would be very useful for us. In our paper we have to get by without property (ii). On the other hand, for a given sequence that does not have property (i), we may use the following two results, which are both easy consequences of Ramsey's theorem (cf. [O]), and will be needed in subsequent sections.

LEMMA 2.5. *Let $V = (v_i)_{i=1}^{\infty}$ be a normalized bimonotone basic sequence. If $(x_i)_{i=1}^{\infty}$ is a sequence in the unit ball of some Banach space X such that every subsequence of $(x_i)_{i=1}^{\infty}$ has a further subsequence which is dominated by V, then there exists a constant $1 \le C < \infty$ and a subsequence $(y_i)_{i=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ so that every subsequence of $(y_i)_{i=1}^{\infty}$ is C-dominated by V.*

Proof. Let $A_n = \{(m_k)_{k=1}^{\infty} \in [\mathbb{N}]^{\omega} \mid (x_{m_k}) \text{ is } 2^n\text{-dominated by } V\}$. Since A_n is Ramsey, for all $n \in \mathbb{N}$ there exists a sequence $(m_i^n)_{i=1}^{\infty} = M_n \in [M_{n-1}]^{\omega}$ such that $[M_n]^{\omega} \subseteq A_n$ or $[M_n]^{\omega} \subseteq A_n^{\mathrm{c}}$. We claim that $[M_n]^{\omega} \subseteq A_n$ for some $n \in \mathbb{N}$, in which case we could choose $(y_i)_{i=1}^{\infty} = (x_{m_i^n})_{i=1}^{\infty}$. Every subsequence of $(y_i)_{i=1}^{\infty}$ is then 2^n-dominated by V.

If our claim were false, we let $(y_n)_{n=1}^{\infty} = (x_{m_n^n})_{n=1}^{\infty}$ and $(y_{k_n})_{n=1}^{\infty}$ be a subsequence of $(y_n)_{n=1}^{\infty}$ for which there exists $C < \infty$ such that $(y_{k_n})_{n=1}^{\infty}$ is C-dominated by V. Let $N \in \mathbb{N}$ be such that $2^N - 2N > C$ and set

$$l_i = \begin{cases} m_i^N & \text{if } i \le N, \\ m_{k_i}^{k_i} & \text{if } i > N. \end{cases}$$

Then $(l_i)_{i=1}^{\infty} \in [M_N]^{\omega} \subset A_N^{\mathrm{c}}$, which implies that some $(a_i)_{i=1}^{L} \subset [-1, 1]$ exists such that $\|\sum_{i=1}^{L} a_i v_i\| \le 1$ and $\|\sum_{i=1}^{L} a_i x_{l_i}\| > 2^N$. This yields

$$\begin{aligned} 2^N &< \Big\| \sum_{i=1}^{L} a_i x_{l_i} \Big\| \le \sum_{i=1}^{N} |a_i| + \Big\| \sum_{i=N+1}^{L} a_i x_{m_{k_i}^{k_i}} \Big\| \le N + \Big\| \sum_{i=N+1}^{L} a_i y_{k_i} \Big\| \\ &\le 2N - \Big\| \sum_{i=1}^{N} a_i y_{k_i} \Big\| + \Big\| \sum_{i=N+1}^{L} a_i y_{k_i} \Big\| \le 2N + \Big\| \sum_{i=1}^{L} a_i y_{k_i} \Big\|, \end{aligned}$$

which implies

$$C < 2^N - 2N < \Big\| \sum_{i=1}^{L} a_i y_{k_i} \Big\|.$$

Thus $(y_{k_n})_{n=1}^{\infty}$ being C-dominated by V is contradicted. ∎

The following lemma is used for a given (x_i) to find a subsequence (y_i) and a constant $C \ge 1$ such that (v_i) C-dominates every subsequence of (y_i) and that C is approximately minimal for every subsequence of (y_i).

Lemma 2.6. *Let $V=(v_n)_{n=1}^\infty$ be a normalized bimonotone basic sequence, $(x_n)_{n=1}^\infty$ be a sequence in the unit ball of some Banach space X, and $a_n\nearrow\infty$ with $a_1=0$. If every subsequence of $(x_n)_{n=1}^\infty$ has a further subsequence which is dominated by V then there exists a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ and an $N\in\mathbb{N}$ such that every subsequence of $(y_n)_{n=1}^\infty$ is a_{N+1}-dominated by V but not a_N-dominated by V.*

Proof. By the previous lemma, we may assume by passing to a subsequence that there exists $C<\infty$ such that every subsequence of $(x_n)_{n=1}^\infty$ is C-dominated by V. Let $M\in\mathbb{N}$ be such that $a_M<C\le a_{M+1}$. For $1\le n\le M$ let

$$A_n=\left\{(m_k)\in[\mathbb{N}]^\omega \;\middle|\; \begin{array}{l}(x_{m_k})_{k=1}^\infty \text{is } a_{n+1}\text{-dominated by } V\\ \text{and is not } a_n\text{-dominated by } V\end{array}\right\}.$$

Then A_n is Ramsey, and $\{A_n\}_{n=1}^M$ forms a finite partition of $[\mathbb{N}]^\omega$, which implies that there exist $N\le M$ and $(m_k)\in[\mathbb{N}]^\omega$ such that $[(m_k)_{k=1}^\infty]^\omega\subset A_N$. Every subsequence of $(y_n):=(x_{m_n})$ is a_{N+1}-dominated by V and not a_N-dominated by V. ∎

3. Proof of Proposition 2.3. Proposition 2.3 will be shown to follow easily from a characterization of countable compact metric spaces along with transfinite induction using the following result.

Lemma 3.1. *Let (X_n) be a sequence of Banach spaces each satisfying the V-array procedure. Then $(\sum_{n=1}^\infty X_n)_{c_0}$ satisfies the V-array procedure.*

To prove Lemma 3.1 we will need the following lemma which is stated in [KO2] for ℓ_p as Lemma 3.6. The proof for general V closely follows its proof.

Lemma 3.2. *Let (X_n) be a sequence of Banach spaces each satisfying the V-array procedure and let (x_i^n) be a bad uV-array in some Banach space X. Suppose that for all $m\in\mathbb{N}$ there is a bounded linear operator $T_m:X\to X_m$ with $\|T_m\|\le 1$ such that $(T_m x_i^m)_{i=1}^\infty$ is an m-bad uV-sequence in X_m. Then (x_i^n) satisfies the V-array procedure.*

Proof. Case 1: There exists $m\in\mathbb{N}$ and a subarray (y_i^n) of (x_i^n) such that $(T_m y_i^n)_{i,n=1}^\infty$ is a bad uV-array in X_m. Then $(T_m y_i^n)_{i,n=1}^\infty$ satisfies the V-array procedure because X_m does. Therefore, there exists a subarray $(T_m z_i^n)_{i,n=1}^\infty$ of $(T_m y_i^n)_{i,n=1}^\infty$ and $(a_n)\subset\mathbb{R}^+$ with $a_n\le 2^{-n}$ such that $(\sum_{n=1}^\infty a_n T_m z_i^n)_{i=1}^\infty$ has no uV-subsequence. Then $(\sum_{n=1}^\infty a_n z_i^n)_{i=1}^\infty$ has no uV-subsequence because $\|T_m\|\le 1$. Therefore $(y_i^n)_{i,n=1}^\infty$ and hence $(x_i^n)_{i,n=1}^\infty$ satisfies the V-array procedure.

CASE 2: Case 1 is not satisfied. Then for all $m \in \mathbb{N}$ and every subarray (y_i^n) of (x_i^n), we see that $(T_m y_i^n)$ is not a bad uV-array in X_m. We may assume by passing to a subarray and using Lemma 2.5 that there exists $(N_n)_{n=1}^\infty \in [\mathbb{N}]^\omega$ such that

(2) $(x_i^n)_{i=1}^\infty$ is a hereditary N_n-uV-sequence for all $n \in \mathbb{N}$.

By induction we choose for each $m \in \mathbb{N}_0$ a subarray $(z_{m,i}^n)_{i,n=1}^\infty$ of $(x_i^n)_{i,n=1}^\infty$ and an $M_m \in \mathbb{N}$ so that

(3) $(z_{m,i}^n)_{i,n=1}^\infty$ is a subarray of $(z_{m-1,i}^n)_{i,n=1}^\infty$ if $m \geq 1$,

(4) $z_{m,i}^n = z_{m-1,i}^n$ if $N_n \leq m$ and $i \in \mathbb{N}$,

(5) $(T_m z_{m,i}^n)_{i=1}^\infty$ is a hereditary M_m-uV-sequence $\forall n \in \mathbb{N}$ if $m \geq 1$.

For $m = 0$ let $(z_{0,i}^n)_{i,n=1}^\infty = (x_i^n)_{i,n=1}^\infty$. Now let $m \geq 1$. For each $n \in \mathbb{N}$ such that $N_n \leq m$ let $(z_{m,i}^n)_{i=1}^\infty = (z_{m-1,i}^n)_{i=1}^\infty$ and $K_n = m$. For each $n \in \mathbb{N}$ such that $N_n > m$, using Lemma 2.6, we let $(z_{m,i}^n)_{i=1}^\infty$ be a subsequence of $(z_{m-1,i}^n)_{i=1}^\infty$ for which there exists $K_n \in \mathbb{N}_0$ such that $(T_m z_{m,i}^n)_{i=1}^\infty$ is a K_n-bad uV-sequence and is also a hereditary $(K_n + 1)$-uV-sequence. The sequence $(K_n)_{n=1}^\infty$ is bounded because otherwise we are in Case 1. Let $M_m = \max_{n\in\mathbb{N}} K_n + 1$. This completes the induction.

For all $n, i \in \mathbb{N}$ we find by (4) that $(z_{m,i}^n)_{m=1}^\infty$ is eventually constant. Let $(z_i^n)_{i,n=1}^\infty = \lim_{m\to\infty}(z_{m,i}^n)_{i,n=1}^\infty$. Then $(z_i^n)_{i,n=1}^\infty$ is a subarray of $(x_i^n)_{i,n=1}^\infty$, and by (5),

(6) $(T_m z_i^n)_{i=1}^\infty$ is a hereditary M_m-uV-sequence for all $m, n \in \mathbb{N}$.

We will now inductively choose $(m_n) \in [\mathbb{N}]^\omega$ and $(a_n) \subset \mathbb{R}^+$ so that for all $n \in \mathbb{N}$:

$$(T_{m_n} z_i^{m_n})_{i=1}^\infty \text{ is an } m_n\text{-bad u}V\text{-sequence in } X_{m_n}, \tag{7}$$

$$a_n m_n > n, \tag{8}$$

$$\sum_{j=1}^{n-1} a_j N_{m_j} < \frac{a_n m_n}{4}, \tag{9}$$

$$0 < a_n < \min_{1\leq k<n} \left\{2^{-n}, 2^{-n}\,\frac{a_k m_k}{4M_{m_k}}\right\}. \tag{10}$$

Property (7) has been assumed in the statement of the lemma. For $n = 1$ let $a_1 = 1/2$ and $m_1 \in \mathbb{N}$ be such that $a_1 m_1 > 1$, so (8) is satisfied. (9) and (10) are vacuously true for $n = 1$, so all conditions are satisfied for $n = 1$.

Let $n > 1$ and assume $(a_j)_{j=1}^{n-1}$ and $(m_j)_{j=1}^{n-1}$ have been chosen to satisfy (8), (9) and (10). Choose $a_n > 0$ small enough such that $a_n < \min_{1\leq k<n}\{2^{-n}, 2^{-n} a_k m_k/4M_k\}$, thus satisfying (10). Choose $m_n > 0$ large enough to satisfy (8) and (9). This completes the induction.

By (10), for all $n \in \mathbb{N}$ we have

$$\sum_{j=n+1}^{\infty} a_j M_{m_n} < \frac{a_n m_n}{4}. \tag{11}$$

Also by (10), $a_j < 2^{-j}$ for all $j \in \mathbb{N}$, so $y_k := \sum_{j=1}^{\infty} a_j z_k^{m_j}$ is a valid choice for the V-array procedure. Let $C > 0$ and (y_{k_i}) be a subsequence of (y_k). We need to show that (y_{k_i}) is not a C-uV-sequence. Using (8), choose $n \in \mathbb{N}$ so that $a_n m_n > 2C$. Using (7) choose $l \in \mathbb{N}$ and $(\beta_i)_{i=1}^{l} \in B_{[v_i]_{i=1}^{l}}$ such that

$$\Big\| \sum_{i=1}^{l} \beta_i T_{m_n} z_{k_i}^{m_n} \Big\| > m_n. \tag{12}$$

We now have

$$\begin{aligned}
\Big\| \sum_{i=1}^{l} \beta_i y_{k_i} \Big\| &= \Big\| \sum_{i=1}^{l} \sum_{j=1}^{\infty} \beta_i a_j z_{k_i}^{m_j} \Big\| \\
&\geq \Big\| \sum_{i=1}^{l} \sum_{j=n}^{\infty} T_{m_n}(\beta_i a_j z_{k_i}^{m_j}) \Big\| \\
&\quad - \Big\| \sum_{i=1}^{l} \sum_{j=1}^{n-1} \beta_i a_j z_{k_i}^{m_j} \Big\| \quad \text{since } \|T_{m_n}\| \leq 1 \\
&\geq a_n \Big\| \sum_{i=1}^{l} \beta_i T_{m_n} z_{k_i}^{m_n} \Big\| - \sum_{j=n+1}^{\infty} a_j \Big\| \sum_{i=1}^{l} \beta_i T_{m_n} z_{k_i}^{m_j} \Big\| \\
&\quad - \sum_{j=1}^{n-1} a_j \Big\| \sum_{i=1}^{l} \beta_i z_{k_i}^{m_j} \Big\| \\
&> a_n m_n - \sum_{j=n+1}^{\infty} a_j M_{m_n} - \sum_{j=1}^{n-1} a_j N_{m_j} \quad \text{by (12), (6), and (2)} \\
&\geq a_n m_n - a_n m_n/4 - a_n m_n/4 \quad \text{by (9) and (11)} \\
&= a_n m_n/2 > C.
\end{aligned}$$

Therefore, (y_{k_i}) is not a C-uV-sequence. $(y_i)_{i=1}^{\infty} = (\sum_{j=1}^{\infty} a_j z_i^{m_j})_{i=1}^{\infty}$ has no uV-subsequence, so (x_i^n) satisfies the V-array procedure, which proves Lemma 3.2. ∎

Now we are prepared to give a proof of Lemma 3.1. We follow the outline of the proof of Lemma 3.5 in [KO2].

Proof of Lemma 3.1. Let (x_i^n) be a bad uV-array in $X = (\sum X_n)_{c_0}$ and $R_m : X \to X_m$ be the natural projections.

Claim. *For all $M < \infty$ there exist $n, m \in \mathbb{N}$ and a subsequence $(y_i)_{i=1}^\infty$ of $(x_i^n)_{i=1}^\infty$ such that $(R_m y_i)_{i=1}^\infty$ is an M-bad uV-sequence.*

Assuming the Claim, we can find $(N_n)_{n=1}^\infty \in [\mathbb{N}]^\omega$, $(m(n))_{n=1}^\infty \subset \mathbb{N}$, and subsequences $(y_i^n)_{i=1}^\infty$ of $(x_i^{N_n})_{i=1}^\infty$ such that $(R_{m(n)} y_i^n)_{i=1}^\infty$ is an n-bad uV-sequence for all $n \in \mathbb{N}$. By passing to a subsequence, we may assume either that $m(n) = m$ is constant, or that $(m(n))_{n=1}^\infty \in [\mathbb{N}]^\omega$. If $m(n) = m$, then $(R_m y_i^n)_{n,i=1}^\infty$ is a bad uV-array in X_m. Then $(R_m y_i^n)_{n,i=1}^\infty$ satisfies the V-array procedure, and thus $(y_i^n)_{n,i=1}^\infty$ satisfies the V-array procedure. If $(m(n))_{n=1}^\infty \in [\mathbb{N}]^\omega$, let $T_n := R_{m(n)}|_{[y_i^r]_{i,r=1}^\infty}$ and apply Lemma 3.2 to the array $(y_i^n)_{i,n=1}^\infty$ to finish the proof.

To prove the Claim, we assume it is false: there exists $M < \infty$ such that for all $m, n \in \mathbb{N}$ every subsequence of $(x_i^n)_{i=1}^\infty$ contains a further subsequence $(y_i)_{i=1}^\infty$ such that $(R_m y_i)_{i=1}^\infty$ is an M-uV-sequence.

By Ramsey's theorem, for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ every subsequence of $(x_i^n)_{i=1}^\infty$ contains a further subsequence $(y_i)_{i=1}^\infty$ such that $(R_m y_i)_{i=1}^\infty$ is a hereditary M-uV-sequence. Fix $n \in \mathbb{N}$ such that $(x_i^n)_{i=1}^\infty$ is an $(M+3)$-bad uV-sequence. We now construct a nested collection of subsequences $\{(y_{k,i})_{i=1}^\infty\}_{k=0}^\infty$ of $(x_i^n)_{i=1}^\infty$ (where $(y_{0,i})_{i=1}^\infty = (x_i^n)_{i=1}^\infty$) as well as $(m_i) \in [\mathbb{N}]^\omega$ so that for all $k \in \mathbb{N}$ we have

$$\sup_{m > m_k} \|R_m y_{k-1,k}\| \le 2^{-k}, \tag{13}$$

$$(y_{k,i})_{i=1}^\infty \text{ is a subsequence of } (y_{k-1,i})_{i=1}^\infty, \tag{14}$$

$$(R_m y_{k,i})_{i=1}^\infty \text{ is a hereditary } M\text{-u}V\text{-sequence } \forall m \le m_k. \tag{15}$$

For $k = 1$ we choose $m_1 \in \mathbb{N}$ such that $\sup_{m > m_1} \|R_m y_{0,1}\| \le 2^{-1}$. Pass to a subsequence $(y_{1,i})_{i=1}^\infty$ of $(y_{0,i})_{i=1}^\infty$ such that $(R_m y_{1,i})_{i=1}^\infty$ is a hereditary M-uV-sequence for all $m \le m_1$.

For $k > 1$, given $m_{k-1} \in \mathbb{N}$ and a sequence $(y_{k-1,i})_{i=1}^\infty$, choose $m_k > m_{k-1}$ so that $\sup_{m > m_k} \|R_m y_{k-1,k}\| \le 2^{-k}$, thus satisfying (13). Let $(y_{k,i})_{i=1}^\infty$ be a subsequence of $(y_{k-1,i})_{i=1}^\infty$ so that $(R_m y_{k,i})_{i=1}^\infty$ is a hereditary M-uV-sequence for all $m \le m_k$, thus satisfying (14) and (15). This completes the induction.

We define $y_k = y_{k-1,k}$ for all $k \in \mathbb{N}$. By (14), $(y_{k,i})_{i=1}^k \cup (y_i)_{i=k+1}^\infty$ is a subsequence of $(y_{k,i})_{i=1}^\infty$. Therefore, (15) shows that

$$(v_i)_{i=k+1}^\infty \ M\text{-dominates } (R_m y_{q_i})_{i=k+1}^\infty \ \forall m \le m_k,\ (q_i) \in [\mathbb{N}]^\omega,\ k \in \mathbb{N}. \tag{16}$$

Since $(x_i^n)_{i=1}^\infty$ is an $(M+3)$-bad uV-sequence, there exists $(\alpha_i) \in B_{[V]}$ such that

$$\Big\| \sum_{i=1}^\infty \alpha_i y_i \Big\| > M + 3. \tag{17}$$

For all $k \in \mathbb{N}$ and $m \in (m_{i-1}, m_i]$ (with $m_0 = 0$) we have

$$\Big\|\sum_{i=1}^{\infty} R_m(\alpha_i y_i)\Big\| \le \sum_{i=1}^{k-1} |\alpha_i|\,\|R_m y_i\| + \|R_m(\alpha_k y_k)\| + \Big\|\sum_{i=k+1}^{\infty} R_m(\alpha_i y_i)\Big\|$$
$$\le \sum_{i=1}^{k-1} 2^{-i} + 1 + \Big\|\sum_{i=k+1}^{\infty} \alpha_i R_m y_i\Big\| \quad \text{by (13)}$$
$$\le 1 + 1 + M \quad \text{by (16)},$$

which implies

$$\Big\|\sum_{i=1}^{\infty} \alpha_i y_i\Big\| = \sup_{m\in\mathbb{N}} \Big\|\sum_{i=1}^{\infty} R_m(\alpha_i y_i)\Big\| \le M + 2.$$

This contradicts (17), so the Claim, and hence Lemma 3.1, is proved. ∎

The proof for Proposition 2.3 now follows in exactly the same way as in [KO2].

Proof of Proposition 2.3. If K is a countable compact metric space then there is a countable limit ordinal α such that $C(K)$ is isomorphic to $C(\alpha)$ (see [BP]). Thus if the V-array procedure fails for $C(K)$, then there is a first limit ordinal α such that the V-array procedure fails for $C(\alpha)$. If α is the first infinite ordinal then $C(\alpha)$ is isomorphic to c_0 and satisfies the V-array procedure. Otherwise, we can find a sequence $\beta_n < \alpha$ of limit ordinals such that $C(\alpha)$ is isomorphic to $(\sum C(\beta_n))_{c_0}$. Thus $C(\alpha)$ satisfies the V-array procedure by Lemma 3.1. ∎

4. Proof of Proposition 2.4.

The proof of Theorem 2.2 will be complete once we have proven Proposition 2.4. To make notation easier, we now consider the triangulated version $(x_i^n)_{1\le n\le i<\infty}$ of the square array $(x_i^n)_{i,n=1}^{\infty}$. The benefit of using a triangular array is that a natural sequential order can be put on a triangular array. As the following proposition shows, we can then pass to a basic sequence in that order.

LEMMA 4.1. *For all $\varepsilon > 0$, a triangular bad uV-array $(x_i^n)_{n\le i}$ admits a triangular subarray $(y_i^n)_{n\le i}$ which is basic in its lexicographical order (where i is the first letter and n is the second letter), and its basis constant is not greater than $1+\varepsilon$. In other words, $y_1^1, y_2^1, y_2^2, y_3^1, y_3^2, y_3^3, y_4^1, \ldots$ is a basic sequence.*

Proof. The proof is an easy adaptation of the proof that a weakly null sequence has a basic subsequence. ∎

The following lemma shows that we need to prove Proposition 2.4 only for triangular arrays.

LEMMA 4.2. *A square array satisfies the V-array procedure if and only if its triangulated version does.*

Proof. If $(y_i^n)_{i,n=1}^\infty$ is a subarray of $(x_i^n)_{i,n=1}^\infty$ then $(y_i^n)_{1\le n\le i<\infty}$ is a triangular subarray of $(x_i^n)_{1\le n\le i<\infty}$. Also, if $(y_i^n)_{1\le n\le i<\infty}$ is a triangular subarray of $(x_i^n)_{1\le n\le i<\infty}$ then $(y_i^n)_{1\le n\le i<\infty}$ may be extended to a subarray of $(x_i^n)_{i,n=1}^\infty$ by letting $(y_i^n)_{i<n} = (x_i^{m_n})_{i<n}$, where $(m_n)\in[\mathbb{N}]^\omega$ is such that $(y_i^n)_{i=1}^\infty \subset (x_i^{m_n})_{i=1}^\infty$ for all $n\in\mathbb{N}$.

We now show that applying the V-array procedure to $(y_i^n)_{i,n=1}^\infty$ and $(y_i^n)_{1\le n\le i<\infty}$ yields sequences which either both satisfy the V-array procedure or both fail the V-array procedure. For all $n\in\mathbb{N}$ let $0\le|\alpha_n|\le 2^{-n}$, $z_i=\sum_{n=1}^i \alpha_n y_i^n$, and $y_i=\sum_{n=1}^\infty \alpha_n y_i^n$. For all $m\in\mathbb{N}$ if $(\beta_i)_{i=1}^\infty\in B_{[V]}$ then

$$\Big\|\sum_{i=1}^m \beta_i z_i - \sum_{i=1}^m \beta_i y_i\Big\| = \Big\|\sum_{i=1}^m \beta_i \sum_{n=i+1}^\infty \alpha_n y_i^n\Big\| \le \sum_{i=1}^m |\beta_i| \sum_{n=i+1}^\infty |\alpha_n| \le \sum_{i=1}^m 2^{-i} < 1.$$

Thus $\sup_{m\in\mathbb{N}}\|\sum_{i=1}^m \beta_i z_i\| = \infty$ if and only if $\sup_{m\in\mathbb{N}}\|\sum_{i=1}^m \beta_i y_i\| = \infty$, which implies the claim. ∎

We now assume that the given bad uV-array (x_i^n) is labeled triangularly and that it is a bimonotone basic sequence in its lexicographical order. This assumption is valid because the properties of "being a bad uV-array" and "satisfying the V-array procedure" are invariant under isomorphisms. We also assume that (x_i^n) is normalized.

The following theorem is our main tool used to construct the subarray (y_i^n) of (x_i^n) and the countable w^*-compact set $K\subset B_{[y_i^n]}$ for Proposition 2.4.

THEOREM 4.3. *Assume that $(x_i^n)_{1\le n\le i}$ is a normalized triangular array in X such that for every $n\in\mathbb{N}$ the sequence $(x_i^n)_{i=1}^\infty$ is weakly converging to 0. Let $V=(v_i)$ be a normalized basic sequence and let $(C_n)\subset[0,\infty)$ and $\varepsilon>0$. Then (x_i^n) has a triangular subarray (y_i^n) with the following property*:

For all $m,s\in\mathbb{N}$ and all $m\le m_1<\cdots<m_s$ all $(\alpha_j)_{j=1}^s\in B_V$ with $\|\sum_{j=1}^s \alpha_j y_{m_j}^m\|\ge C_n$ there is a $g\in(2+\varepsilon)B_{X^*}$ and $(\beta_j)_{j=1}^s\in B_V$ so that

$$\sum_{j=1}^s \beta_j g(y_{m_j}^m)\ge C_n, \tag{18}$$

$$g(y_j^{m'})=0 \text{ whenever } m'\le j \text{ and } j\notin\{m_1,\dots,m_s\}. \tag{19}$$

If we also assume that $(x_i^n)_{1\le n\le i}$ is a bimonotone basic sequence in its lexicographical order then there exists $(j_i)\in[\mathbb{N}]^\omega$ so that we may choose the *subarray (y_i^n) by setting $y_i^n = x_{j_i}^n$ for all $n \le i$. In this case we have the above conclusion for some $g \in (1+\varepsilon)B_{Y^*}$.*

Proof. After passing to a subarray using Lemma 4.1 we can assume that (x_i^n) is a basic sequence in its lexicographical order and that its basis constant does not exceed the value $1+\varepsilon$. We first renorm $Z = [x_i^n]$ by a norm $|||\cdot|||$ in the standard way so that $\|z\| \le |||z||| \le (2+2\varepsilon)\|z\|$ and so that (x_i^n) is bimonotone in Z. We can therefore assume that (x_i^n) is a bimonotone basis and need to show the claim of Theorem 4.3 for $(1+\varepsilon)B_{X^*}$ instead of $(2+\varepsilon)B_{X^*}$.

Let $(\varepsilon_k) \subset (0,1)$ with $\sum_{k=1}^\infty k\varepsilon_k < \varepsilon/4$. By induction on $k \in \mathbb{N}_0$ we choose $i_k \in \mathbb{N}$ and a sequence $L_k \in [\mathbb{N}]^\omega$, and define $y_j^m = x_{i_j}^m$ for $m \le k$ and $m \le j \le k$ so that the following conditions are satisfied:

(i) $i_k = \min L_{k-1} < \min L_k$ and $L_k \subset L_{k-1}$, if $k \ge 1$ ($L_0 = \mathbb{N}$).

(ii) For all $s, t \in \mathbb{N}_0$, all $1 \le m \le k$, all $m \le m_1 < \cdots < m_s \le k$ and $l_0 < l_1 < \cdots < l_t$ in L_k, if there is an $f \in B_{X^*}$ with

$$\sum_{j=1}^{s} \alpha_j f(y_{m_j}^m) + \sum_{j=1}^{t} \alpha_{j+s} f(x_{l_j}^m) \ge C_m \text{ for some } (\alpha_j)_{j=1}^{s+t} \in B_{[V]} \tag{20}$$

(21) then there exists $g \in B_{X^*}$ such that

(a) $\sum_{j=1}^s \beta_j g(y_{m_j}^m) + \sum_{j=1}^t \beta_{j+s} g(x_{l_j}^m) \ge C_m$ for some $(\beta_j)_{j=1}^{s+t} \in B_{[V]}$,

(b) $|g(y_j^{m'})| < \varepsilon_j$ if $m' \le k$ and $j \in \{m', \dots, k\} \setminus \{m_1, \dots, m_s\}$,

(c) $|g(x_{l_0}^{m'})| < \varepsilon_{k+1}$ if $m' \le k+1$

(in the case $s = 0$ condition (b) is defined to be vacuous; also note that in (c) we allow $m' = k+1$).

We first note for $(i_j) \in [\mathbb{N}]^\omega$ that $(x_{i_j}^n)_{n \le j}$ is a subsequence of $(x_j^n)_{n \le j}$ in their lexicographic orders. Thus $(x_{i_j}^n)_{n \le j}$ is a bimonotone basic sequence in its lexicographic order.

For $k = 0$, if $f \in B_{X^*}$ satisfies (20) then $g = P^*_{[x_{l_1}^n, \infty)} f$ satisfies (21) by our bimonotonicity assumption.

Assume $k \ge 1$ and that we have chosen $i_1 < \cdots < i_{k-1}$. We let $i_k = \min L_{k-1}$.

Fix an infinite $M \subset L_{k-1} \setminus \{i_k\}$, a positive integer $m \le k$, an integer $0 \le s \le k-m+1$, and positive integers $m \le m_1 < \cdots < m_s \le k$, and define

$$A = A(m, s, (m_j)_{j=1}^s) = \bigcap_{t \in \mathbb{N}_0} A_t, \quad \text{where}$$

$$A_t = \left\{ (l_j)_{j=0}^\infty \in [M]^\omega \;\middle|\; \begin{array}{l} \text{if } (m_j)_{j=1}^s \text{ and } (l_j)_{j=0}^t \text{ satisfy (20)} \\ \text{then they also satisfy (21)} \end{array} \right\}.$$

For $t \in \mathbb{N}$ the set A_t is closed as a subset of $2^{\mathbb{N}}$ in the product topology, thus A is closed, and thus Ramsey. We will show that there is an infinite $L \subset M$ so that $[L]^\omega \subset A$. Once we have verified that claim we can finish our induction step by applying that argument successively to all choices of $m \le k$, $0 \le s \le k$ and $m \le m_1 < \cdots < m_s \le k$, as there are only finitely many.

Assume our claim is wrong and, using Ramsey's theorem, we could find an $L = (l_j)_{j=1}^\infty$ so that $[L]^\omega \cap A = \emptyset$.

Let $n \in \mathbb{N}$ be fixed, and let $p \in \{1, \ldots, n\}$. Then $L^{(p)} = \{l_p, l_{n+1}, \ldots\}$ is not in A and we can choose $t_n \in \mathbb{N}_0$, $(\alpha_j^n)_{j=1}^{t_n+s}$ and $f_n \in B_{X^*}$ so that (20) is satisfied (for $(l_{n+1}, \ldots, l_{l+t})$ replacing $(l_1, \ldots, l_t)$ and l_p replacing l_0) but for no $(\beta_j)_{j=1}^{s+t_n} \in B_{[V]}$ does condition (21) hold. By choosing t_n to be minimal so that (20) is satisfied, we can have t_n, $(\alpha_j^n)_{j=1}^{t_n+s}$ and f_n independent of p.

We now show that there is a $g_n \in B_X$ satisfying (a) and (b) of (21).

Let $k' = \max\{m - 1 \le i \le k \mid i \notin \{m_1, \ldots, m_s\}\}$. If $k' \le m$ then $\{m_1, \ldots, m_s\} = \{k'+1, k'+2, \ldots, k\}$ and by our assumed bimonotonicity $g_n := P^*_{[y^m_{k'+1}, \infty)} f_n \in B^*_X$ satisfies (a) and (b) of (21). If $k' > m$ let $0 \le s' \le s$ such that $m_1 < \cdots < m_{s'} < k'$, and apply the $k' - 1$ step of the induction hypothesis to f_n, $(\alpha_j^n)_{j=1}^{t_n+s}$, $m \le m_1 < \cdots < m_{s'}$ (replacing $m \le m_1 < \cdots < m_s$), and $k' < k' + 1 < \cdots < m_s < l_{n+1} < \cdots < l_{t_n}$ (replacing $l_p < l_{n+1} < \cdots < l_{t_n}$) to obtain a functional $g_n \in B_{X^*}$ which satisfies (a) and (b) of (21).

Since g_n cannot satisfy all three conditions of (21) (for any choice of $1 \le p \le n$), we deduce that $|g_n(x_{l_p}^{m_p})| \ge \varepsilon_{k+1}$ for some choice of $m_p \in \{1, \ldots, k+1\}$.

Let g be a w^* cluster point of $(g_n)_{n\in\mathbb{N}}$. As the set $\{1, \ldots, k+1\}$ is finite, for all $p \in \mathbb{N}_0$ we have $|g(x_{l_p}^{m_p})| \ge \varepsilon_{k+1}$ for some $m_p \in \{1, \ldots, k+1\}$. This implies there exists $1 \le m \le k+1$ such that $|g(x_{l_p}^m)| \ge \varepsilon$ for infinitely many $p \in \mathbb{N}$. This is a contradiction with the sequence $(x_{l_i}^m)_{i=1}^\infty$ being weakly null. Our claim is verified, and we are able to fulfill the induction hypothesis.

The conclusion of our theorem now follows by the following perturbation argument. If we have $n \le i_1 < \cdots < i_q$ and $(\alpha_j)_{j=1}^q \in B_V$ with $\|\sum_{j=1}^q \alpha_j y_{i_j}^n\| \ge C_n$, then there exists $f \in B_{X^*}$ so that $\sum_{j=1}^q \alpha_j f(y_{i_j}^n) \ge C_n$. Our construction gives an $h \in B_{X^*}$ with $\sum_{j=1}^q \alpha_j h(y_{i_j}^n) \ge C_n$ and $|h(y_j^m)| < \varepsilon_j$ if $m \le q$ and $j \in \{m', \ldots, k\} \setminus \{i_1, \ldots, i_q\}$. Because (y_i^n) is bimonotone, we may assume that $h(y_i^n) = 0$ for all $i \ge n$ with $i > i_q$. We perturb h by small multiples of the biorthogonal functionals of (y_i^n) to achieve $g \in X^*$ with $g(y_i^n) = h(y_i^n)$ for $i \in \{i_1, \ldots, i_q\}$ and $g(y_i^n) = 0$ for $i \notin \{i_1, \ldots, i_q\}$. Thus g satisfies (18) and (19). All that remains is to check that $g \in (1+\varepsilon)B_{X^*}$. Because (y_i^n) is normalized and bimonotone, we can estimate $\|g\|$ as follows:

$$\|g\| \leq \|h\| + \|g-h\| \leq 1 + \sum_{j=1}^{i_q-1} j\varepsilon_j < 1 + \frac{\varepsilon}{4}. \blacksquare$$

We are now prepared to give the proof of Proposition 2.4. We follow the same outline as the proof given in [KO2] for Proposition 3.4.

Proof of Proposition 2.4. Let (x_i^n) be a normalized bad uV-array in X and let M_n, for $n \in \mathbb{N}$, be chosen so that the sequence $(x_i^n)_{i=n}^{\infty}$ is an M_n-bad uV-sequence and $\lim_{n\to\infty} M_n = \infty$. By Lemma 4.2 we just need to consider the triangular array $(x_i^n)_{n\leq i}$. By passing to a subarray using Lemma 4.1 and then renorming, we may assume that $(x_i^n)_{n\leq i}$ is a normalized bimonotone basic sequence in its lexicographical order.

We apply Theorem 4.3 for $\varepsilon = 1$ and $(C_n) = (M_n)$ to obtain a subarray $(y_i^n)_{n\leq i}$ that satisfies conditions (18) and (19). Moreover, (y_i^n) in its lexicographical order is a subsequence of (x_i^n) in its lexicographical order, and thus is bimonotone. Furthermore, $(y_i^n)_{i=n}^{\infty}$ is a subsequence of $(x_i^n)_{i=n}^{\infty}$ for all $n \in \mathbb{N}$. We write $Y = [y_i^n]_{n\leq i}$.

Let $F(n)$ be a finite $(1/2n2^n)$-net in $[-2, 2]$ which contains the points 0, -2, and 2. Whenever we have a functional $g \in 2B_{X^*}$ which satisfies conditions (18) and (19) we may perturb g by small multiples of the biorthogonal functions of $(y_i^n)_{n\leq i}$ to obtain $f \in 3B_{X^*}$ which satisfies (18), (19), and the following new condition:

$$f(y_i^n) \in F(n) \quad \text{for all } n \leq i. \tag{22}$$

We now start the construction of K. Let $Y = [y_i^n]_{n\leq i}$ and $m \in \mathbb{N}$. We define

$$L_m = \left\{ (k_1, \ldots, k_q) \;\middle|\; \begin{array}{l} m \leq k_1 < \cdots < k_q, \\ \|\sum_{i=1}^{q-1} \alpha_i y_{k_i}^m\| \leq M_m \text{ for all } (\alpha_i) \in B_V, \\ \|\sum_{i=1}^{q} \alpha_i y_{k_i}^m\| > M_m \text{ for some } (\alpha_i) \in B_V \end{array} \right\}.$$

It is important to note that if $(k_i) \in [\mathbb{N}]^{\omega}$ and $k_1 \geq m$ then there is a unique $q \in \mathbb{N}$ such that $(k_1, \ldots, k_q) \in L_m$.

Whenever $\vec{k} = (k_1, \ldots, k_q) \in L_m$, an application of Theorem 4.3 and then perturbation gives a functional $f \in 3B_{Y^*}$ which satisfies conditions (18), (19), and (22). In particular, $\sum_{i=1}^{q} f(\alpha_i y_{k_i}^m) > M_m$ for some $(\alpha_i) \in B_V$. We denote $f/3$ by $f_{\vec{k}}$ and let, for any $n \in \mathbb{N}$,

$$K_n = \{Q_m^* f_{\vec{k}} \mid m \in \mathbb{N}, \vec{k} \in L_n\}.$$

Here Q_m denotes the natural norm 1 projection from Y onto $[(y_i^n)]_{1\leq n\leq i\leq m}$. Finally, we define

$$K = \bigcup_{n=1}^{\infty} K_n \cup \{0\}.$$

We first show that $(y_i^n|_K)_{n\leq i}$ is a bad uV-array as an array in $C_{\rm b}(K)$. Fix an $n_0\in\mathbb{N}$. Then $(y_i^{n_0})_{i=n_0}^\infty$ is an M_{n_0}-bad uV-sequence. Consequently, given a subsequence $(y_{k_i}^{n_0})_{i=1}^\infty$ of $(y_i^{n_0})_{i=n_0}^\infty$ we have $\vec{k}:=(k_1,\ldots,k_q)\in L_{n_0}$ for some $q\in\mathbb{N}$. By (22), $f_{\vec{k}}=Q_{q+1}^*f_{\vec{k}}$ and thus $f_{\vec{k}}\in K_{n_0}\subset K$. Now, $\sum_{i=1}^q f_{\vec{k}}(\alpha_i y_{k_i}^{n_0})>M_{n_0}/3$ for some $(\alpha_i)\in B_V$, and so $(y_i^{n_0}|_K)_{i=n_0}^\infty$ is an $(M_{n_0}/3)$-bad sequence in $C_{\rm b}(K)$, thus proving that $(y_i^n|_K)_{n\leq i}$ is a bad uV-array.

K is obviously a countable subset of B_{Y^*}. Since Y is separable, K is w^*-metrizable. Thus we need to show that K is a w^*-closed subset of B_{Y^*} in order to finish the proof.

Let $(g_j)\subset K$ and assume that (g_j) converges w^* to some $g\in B_{Y^*}$. We have to show that $g\in K$. Every g_j is of the form $Q_{m_j}^*f_{\vec{k}_j}$ for some $m_j\in\mathbb{N}$, $\vec{k}_j\in L_{n_j}$, and some $n_j\in\mathbb{N}$.

By passing to a subsequence of (g_j), we may assume that either $n_j\to\infty$ as $j\to\infty$, or there is an $n\in\mathbb{N}$ such that $n_j=n$ for all $j\in\mathbb{N}$. We will start with the first alternative. Let i_j be the first element of $\vec{k}_j$. Since $i_j\geq n_j$, we have $i_j\to\infty$. Also, $f_{\vec{k}_j}(y_i^n)=0$ for all $n\leq i<i_j$. Thus $f_{\vec{k}_j}\to 0$ in the w^* topology as $j\to\infty$, so $g=0\in K$.

From now on we assume that there is an $n\in\mathbb{N}$ such that $\vec{k}_j\in L_n$ for all $j\in\mathbb{N}$. Since L_n is relatively sequentially compact as a subspace of $\{0,1\}^{\mathbb{N}}$ endowed with the product topology, we may assume by passing to a subsequence of (g_j) that $\vec{k}_j\to\vec{k}$ for some $\vec{k}\in\overline{L}_n$, the closure of L_n in $\{0,1\}^{\mathbb{N}}$.

We now show that $\vec{k}$ is finite. Suppose to the contrary that $\vec{k}=(k_i)_{i=1}^\infty$. As $\vec{k}\in\overline{L}_n$, for all $r\in\mathbb{N}$ there exists $N_r\in\mathbb{N}$ such that $\vec{k}_j=(k_1,\ldots,k_r, l_1,\ldots,l_s)$ for some $l_1,\ldots,l_s$ for all $j\geq N_r$. Because $\vec{k}_j\in L_n$, we have $k_1\geq n$, which implies that there exists $q\in\mathbb{N}$ such that $(k_1,\ldots,k_q)\in L_n$. By uniqueness, L_n does not contain any sequence extending $(k_1,\ldots,k_q)$. Therefore, $\vec{k}_{N_{q+1}}=(k_1,\ldots,k_{q+1},l_1,\ldots,l_s)\notin L_n$, a contradiction.

Since B_{Y^*} is w^*-sequentially compact, we may assume that $f_{\vec{k}_j}$ converges w^* to some $f\in B_{Y^*}$. We claim that $f\in K$. To prove this we first show that $Q_m^*f\in K$ for all $m\in\mathbb{N}$. By (19) and (22) the set $\{Q_m^*f_{\vec{k}_j}(y_i^n)\mid j\in\mathbb{N}, 1\leq n\leq i\}$ has only finitely many elements. Since $Q_m^*f_{\vec{k}_j}\to Q_m^*f$ as $j\to\infty$ we obtain $Q_m^*f_{\vec{k}_j}=Q_m^*f$ for $j\in\mathbb{N}$ large enough. In particular, $Q_m^*f\in K$. Next let $q=\max\vec{k}$. Since $\vec{k}_j\to\vec{k}$ and $\vec{k}$ is finite, we have $Q_q^*f=f$ and thus $f\in K$.

Now we show that $g\in K$. By passing again to a subsequence of (g_j) we can assume that either $m_j\geq\max\vec{k}$ for all $j\in\mathbb{N}$, or there exists $m<\max\vec{k}$ such that $m_j=m$ for all $j\in\mathbb{N}$. If the first case occurs, then $g_j=Q_{m_j}^*f_{\vec{k}_j}$

converges w^* to f, and hence $g = f \in K$. If the second case occurs, then $g_j = Q_m^* f_{\vec{k}_j}$ converges w^* to $Q_m^* f$, and hence $g = Q_m^* f \in K$. ■

5. Examples. In previous sections, we introduced for any seminormalized basic sequence (v_i) the property $U_{(v_i)}$, and then proved that if a Banach space X is $U_{(v_i)}$ then there exists a constant $C \geq 1$ such that X is C-$U_{(v_i)}$. As Knaust and Odell proved that result for the cases in which (v_i) is the standard basis for c_0 or ℓ_p with $1 \leq p < \infty$, we need to show that our result is not a corollary of theirs. For example, if (v_i) is a basis for $\ell_p \oplus \ell_q$ with $1 < q < p < \infty$ which consists of the union of the standard bases for ℓ_p and ℓ_q then a Banach space is $U_{(v_i)}$ or C-$U_{(v_i)}$ if and only if X is U_{ℓ_p} or C-U_{ℓ_p} respectively. Thus the result for this particular (v_i) follows from [KO2]. We make this idea more formal by defining the following equivalence relation:

DEFINITION 5.1. If (v_i) and (w_i) are normalized basic sequences then we write $(v_i) \sim_U (w_i)$ (or $(v_i) \sim_{CU} (w_i)$) if each reflexive Banach space is $U_{(v_i)}$ (or C-$U_{(v_i)}$) if and only if it is $U_{(w_i)}$ (or C-$U_{(w_i)}$).

We define the equivalence relation strictly in terms of reflexive spaces to avoid the unpleasant case of ℓ_1. Because ℓ_1 does not contain any normalized weakly null sequence, ℓ_1 is trivially $U_{(v_i)}$ for every (v_i). This is counter to the spirit of what it means for a space to be $U_{(v_i)}$. By considering reflexive spaces, we avoid ℓ_1, and we also make the propositions included in this section formally stronger. Reflexive spaces are also especially nice when considering properties of weakly null sequences because the unit ball of a reflexive space is weakly sequentially compact. That is, every sequence in the unit ball of a reflexive space has a weakly convergent subsequence.

In order to show that our result is not a corollary of the theorem of Knaust and Odell, we give an example of a basic sequence (v_i) such that $(v_i) \not\sim_U (e_i)$ where (e_i) is the standard basis for c_0 or ℓ_p with $1 \leq p < \infty$. To this end we consider a basis (v_i) for a reflexive Banach space X with the property that ℓ_p is not $U_{(v_i)}$ for any $1 < p < \infty$, but that X is $U_{(v_i)}$ and not U_{c_0}. In particular, we will be interested in the dual of the following space.

DEFINITION 5.2. *Tsirelson's space*, T, is the completion of c_{00} under the norm satisfying the implicit relation:

$$\|x\| = \|x\|_\infty \vee \sup_{n \in \mathbb{N},\, (E_i)_{i=1}^n \subset [\mathbb{N}]^\omega,\, n \leq E_1 < \cdots < E_n} \frac{1}{2} \sum_{i=1}^n \|E_i(x)\|.$$

(t_i) is the unit vector basis of T and (t_i^*) are the biorthogonal functionals to (t_i).

Tsirelson constructed the dual of T as the first example of a Banach space which does not contain c_0 or ℓ_p for any $1 \leq p < \infty$ [T]. Though we are more interested in T^* and (t_i^*), we use the implicit definition of T (which was formulated by Figiel and Johnson in [FJ]) as it is nice to work with. The properties of (t_i^*) that will be most useful for us are that (t_i^*) dominates all of its normalized block bases, and has a spreading model equivalent to the standard basis for c_0. The sequences (t_i) and (t_i^*) have the further interesting property of being block stable. Casazza, Johnson, and Tzafriri showed in [CJT] that (t_i) has the property that if (x_i) is a normalized block basis of (t_i) then (x_i) is equivalent to (t_{n_i}) where $n_i \in \mathrm{supp}(x_i)$ for all $i \in \mathbb{N}$. The corresponding statement for (t_i^*) follows from the result for (t_i). As we have defined T, but wish to know about sequences in T^*, we need the following proposition which relates sequences in a space to sequences in its dual.

PROPOSITION 5.3. *If (v_i) and (x_i) are normalized basic sequences, then:*

(i) *(v_i) dominates (x_i) if and only if (v_i^*) is dominated by (x_i^*).*
(ii) *If (v_i) is unconditional, then (v_i) dominates all of its normalized block bases if and only if (v_i^*) is dominated by all of its normalized block bases.*

Proof. Without loss of generality we may assume that (v_i) and (x_i) are bimonotone. We assume that (v_i) C-dominates (x_i) and let $(a_i) \in c_{00}$. Because (v_i) is bimonotone, there exists $(b_i) \in c_{00}$ such that $\sum a_i v_i^*(\sum b_i v_i) = \|\sum a_i v_i^*\|$ and $\|\sum b_i v_i\| = 1$. We have

$$\Big\|\sum a_i v_i^*\Big\| = \sum a_i b_i = \sum a_i x_i^*\Big(\sum b_i x_i\Big) \leq C\Big\|\sum a_i x_i^*\Big\|.$$

Thus (v_i^*) is C-dominated by (x_i^*). The converse is true by duality in the sense that we replace the roles of (v_i) and (x_i) by (x_i^*) and (v_i^*) respectively. We find that (x_i^{**}) is equivalent to (x_i) and (v_i^{**}) is equivalent to (v_i) and thus the converse follows and hence (i) is proven.

After possibly renorming, we may assume that (v_i) is 1-unconditional. For the first direction in (ii), we assume that (v_i) C-dominates all of its normalized block bases. Let $a_i \in c_{00}$ and (w_i^*) be a normalized block basis of (v_i^*). As (v_i) is bimonotone, there exists a normalized block basis (w_i) of (v_i) be such that $w_i^*(w_j) = \delta_{ij}$. Let $x \in S_{[v_i]}$ be such that $\sum a_i v_i^*(x) = \|\sum a_i v_i^*\|$. We now have

$$\begin{aligned}\Big\|\sum a_i v_i^*\Big\| &= \sum a_i v_i^*(x) = \sum a_i w_i^* \sum v_j^*(x) w_j \\ &\leq \Big\|\sum a_i w_i^*\Big\| \Big\|\sum v_j^*(x) w_j\Big\| \\ &\leq C\Big\|\sum a_i w_i^*\Big\| \Big\|\sum v_j^*(x) v_j\Big\| = C\Big\|\sum a_i w_i^*\Big\|.\end{aligned}$$

Thus (v_i^*) is C-dominated by (w_i^*), and we have proven the first direction.

For the converse, assume that (v_i^*) is C-dominated by all of its normalized block bases. Let $(a_i) \in c_{00}$ and (w_i) be a normalized block basis of (v_i). There exists $f \in B_{[v_i]^*}$ such that $f(\sum a_i w_i) = \|\sum a_i w_i\|$. Choose $(k_n) \in [\mathbb{N}]^\omega$ such that $\mathrm{supp}(w_n) \subset [k_n, k_{n+1})$ for all $n \in \mathbb{N}$. There is a normalized block basis (f_i) of (v_i^*) and $(b_i) \in c_{00}$ such that $f = \sum b_i f_i$ and $\mathrm{supp}(f_n) \subset [k_n, k_{n+1})$ for all $n \in \mathbb{N}$. As (v_i) is 1-unconditional, we may assume that $a_i, b_i, f_i(w_i) \geq 0$. This gives $\sum a_i b_i f_i(w_i) \leq \sum a_i b_i$, as $f_i(w_i) \leq 1$. We now have

$$\Big\|\sum a_i w_i\Big\| = \Big(\sum b_i f_i\Big)\Big(\sum a_i w_i\Big) \leq \Big(\sum b_i v_i^*\Big)\Big(\sum a_i v_i\Big) \leq C\Big\|\sum a_i v_i\Big\|.$$

Hence,(v_i) C-dominates (w_i) and (ii) is proven. ■

We will use Proposition 5.3 together with some basic properties of (t_i) to prove the following proposition.

PROPOSITION 5.4. *$(t_i^*) \not\sim_U (e_i)$, where (e_i) is the standard basis for c_0 or ℓ_p for $1 \leq p < \infty$.*

Proof. It easily follows from the definition that (t_i) is an unconditional normalized basic sequence and that (t_i) is dominated by each of its normalized block bases. Also, the spreading model for (t_i) is isomorphic to the standard ℓ_1 basis. By Proposition 5.3, (t_i^*) is an unconditional basic sequence that dominates all of its block bases and has its spreading model isomorphic to the standard basis for c_0. Furthermore, T^* is reflexive because (t_i^*) is unconditional and T^* does not contain an isomorphic copy of c_0 or ℓ_1. As (t_i^*) has the standard basis for c_0 as its spreading model, ℓ_p is not $U_{(t_i^*)}$ for all $1 < p < \infty$. Therefore $(t_i^*) \not\sim_U \ell_p$ for all $1 \leq p < \infty$. As (t_i^*) dominates all of its normalized block bases and every normalized weakly null sequence in T^* has a subsequence equivalent to a normalized block basis of (t_i^*), it follows T^* is $U_{(t_i^*)}$. Since T^* does not contain c_0 isomorphically, T^* is not U_{c_0}. Therefore, $(t_i^*) \not\sim_U c_0$. ■

We have shown that $(t_i^*) \not\sim (e_i)$ where (e_i) is the usual basis for c_0 or ℓ_p for $1 \leq p < \infty$, but we can actually show something much stronger than this. One of the main properties of ℓ_p used in [KO2] is that ℓ_p is subsymmetric. If for each basic sequence (v_i) there existed a constant $C \geq 1$ and a subsymmetric basic sequence (w_i) such that $(v_i) \sim_{CU} (w_i)$ then actually the first half of [KO2] would apply to all basic sequences without changing anything. The following example shows in particular that this is not true even for the weaker condition of spreading (the property that all subsequences are equivalent).

PROPOSITION 5.5. *If (v_i) is a normalized spreading basic sequence, then $(v_i) \not\sim_U (t_i^*)$.*

In general, it can be fairly difficult to check if a Banach space is $U_{(v_i)}$, as every normalized weakly null sequence in the space needs to be checked. In contrast to this, it is very easy to check if T^* is $U_{(v_i)}$. This is because (t_i) is dominated by all of its block bases, and thus by Proposition 5.3, T^* is $U_{(v_i)}$ if and only if (v_i) dominates a subsequence of (t_i^*). In proving Proposition 5.5 we will carry this idea further by considering a class of spaces each of which has a subsymmetric basis (e_i) such that (e_i) is dominated by all of its normalized block bases. The additional condition of subsymmetry implies that $[e_i^*]$ is $U_{(v_i)}$ if and only if (v_i) dominates (e_i^*). Hence, we need to check only one sequence instead of all weakly null sequences in $[e_i^*]$.

We consider generalizations of the spaces introduced by Schlumprecht [S] as the first known arbitrarily distortable Banach spaces. We put less restriction on the function f given in the following proposition, but we also infer less about the corresponding Banach space. The techniques from [S] are used to prove the following proposition.

PROPOSITION 5.6. *Let $f : \mathbb{N} \to [1, \infty)$ increase to ∞, $f(1) = 1 < f(2)$, and $\lim_{n\to\infty} n/f(n) = \infty$. If X is defined as the closure of c_{00} under the norm $\|\cdot\|$ which satisfies the implicit relation*

$$\|x\| = \|x\|_\infty \vee \sup_{m\geq 2,\, E_1<\cdots<E_m} \frac{1}{f(m)} \sum_{j=1}^{m} \|E_j(x)\| \quad \textit{for all } x \in c_{00},$$

then X is reflexive.

Proof. Let (e_n) denote the standard basis for c_{00}. It is straightforward to show that the norm $\|\cdot\|$ as given in the statement of the proposition exists, as well as that (e_n) is a normalized, 1-subsymmetric and 1-unconditional basis for X. Furthermore, (e_n) is 1-dominated by all of its normalized block bases. We will prove that X is reflexive by showing that (e_n) is boundedly complete and shrinking.

We first prove that (e_n) is boundedly complete. As (e_n) is unconditional, if (e_n) is not boundedly complete then it has some normalized block basis which is equivalent to the standard c_0 basis. However, (e_n) is 1-dominated by all its normalized block bases, so (e_n) is also equivalent to the standard c_0 basis. Hence $\sup_{N\in\mathbb{N}} \|\sum_{n=1}^N e_n\| < \infty$. This contradicts the fact that $\|\sum_{n=1}^N e_n\| \geq N/f(N) \to \infty$. Thus (e_n) is boundedly complete.

We now assume that (e_n) is not shrinking. As (e_n) is unconditional, it has a normalized block basis (x_n) which is equivalent to the standard basis for ℓ_1. We will use James' blocking lemma [J] to show that this leads to a contradiction. In one of its more basic forms, James' blocking lemma states that if (x_n) is equivalent to the standard basis for ℓ_1 and $\varepsilon > 0$ then (x_n) has a normalized block basis which is $(1+\varepsilon)$-equivalent to the standard basis for ℓ_1. Let $0 < \varepsilon < \frac{1}{2}(f(2) - 1)$. By passing to a normalized block basis using James' blocking lemma, we may assume that (x_n) is $(1+\varepsilon)$-equivalent to the standard basis for ℓ_1, and thus any normalized block basis of (x_n) will also be $(1+\varepsilon)$-equivalent to the standard basis for ℓ_1. Let $\varepsilon_n > 0$ be such that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$.

We denote by $\|\cdot\|_m$ the norm on X which satisfies

$$\|x\|_m = \sup_{E_1<\cdots<E_m} \frac{1}{f(m)} \sum_{j=1}^{m} \|E_j(x)\| \quad \text{for all } x \in c_{00}.$$

We will construct by induction on $n \in \mathbb{N}$ a normalized block basis (y_i) of (x_i) such that for all $m \in \mathbb{N}$ we have

$$\text{if } \|y_j\|_m > \varepsilon_j \text{ for some } 1 \le j < n, \text{ then } \|y_n\|_m < \frac{1+\varepsilon_n}{f(m)}. \tag{23}$$

For $n = 1$ we let $y_1 = x_1$, and note that (23) is vacuously satisfied.

We now assume that we are given $n \ge 1$ and a finite block sequence $(y_i)_{i=1}^n$ of (x_i) which satisfies (23). We have

$$\lim_{m\to\infty} \|y_i\|_m \le \lim_{m\to\infty} \frac{\#\operatorname{supp}(y_i)}{f(m)} = 0$$

(where $\operatorname{supp}(y_i)$ denotes the support of y_i). Thus, there exists $N > \operatorname{supp}(y_n)$ such that $\|y_i\|_m < \varepsilon_i$ for all $1 \le i \le n$ and all $m \ge N$. Using James' blocking lemma, we block $(x_i)_{i=N}^{\infty}$ into $(z_i)_{i=1}^{\infty}$ such that $(z_i)_{i=1}^{\infty}$ is $(1+\varepsilon_{n+1}/3)$-equivalent to the standard ℓ_1 basis. Let $M \ge 6N/\varepsilon_{n+1}$ and define

$$y_{n+1} = \frac{1}{\|\sum_{i=1}^{M} z_i\|} \sum_{i=1}^{M} z_i.$$

Let $m \in \mathbb{N}$ be such that $\|y_j\|_m > \varepsilon_j$ for some $1 \le j \le n$. By our choice of $N \in \mathbb{N}$, we have $m < N$. There exist disjoint intervals $E_1 < \cdots < E_m$ in $\mathbb{N}$ and integers $1 = k_0 \le k_1 \le \cdots \le k_m$ such that

$$\begin{aligned}
f(m)\|y_{n+1}\|_m &= \frac{1}{\|\sum_{i=1}^{M} z_i\|} \sum_{i=1}^{m} \Big\| E_i \sum_{j=k_{i-1}}^{k_i} z_j \Big\| \\
&\le \frac{1+\varepsilon_{n+1}/3}{M} \sum_{i=1}^{m} \Big(\|E_i z_{k_{i-1}}\| + \Big\| \sum_{j=k_{i-1}+1}^{k_i-1} z_j \Big\| + \|E_i z_{k_i}\| \Big) \\
&\le \frac{1+\varepsilon_{n+1}/3}{M} (M+2m) < (1+\varepsilon_{n+1}/3)(1+2N/M) \\
&\le (1+\varepsilon_{n+1}/3)(1+\varepsilon_{n+1}/3) < 1+\varepsilon_{n+1}.
\end{aligned}$$

Hence, the induction hypothesis is satisfied.

We now show that property (23) leads to a contradiction with (y_i) being $(1+\varepsilon)$-equivalent to the standard ℓ_1 basis. Let $n \in \mathbb{N}$. For some $m \ge 2$ we have $\|\sum_{i=1}^{n} y_i/n\| = \|\sum_{i=1}^{n} y_i/n\|_m$. By (23) there exists $1 \le j \le n+1$ such that $\|y_i\|_m < \varepsilon_i$ for all $1 \le i < j$ and $f(m)\|y_i\|_m < 1+\varepsilon_i$ for all $j < i \le n$.

We have

$$\Big\|\sum_{i=1}^{n}\frac{y_i}{n}\Big\| = \Big\|\sum_{i=1}^{n}\frac{y_i}{n}\Big\|_m \le \frac{1}{n}\sum_{i=1}^{j-1}\|y_i\|_m + \frac{1}{n}\|y_j\|_m + \frac{1}{n}\sum_{i=j+1}^{n}\|y_i\|_m$$
$$< \frac{1}{n}\sum_{i=1}^{j-1}\varepsilon_i + \frac{1}{n} + \frac{1}{nf(m)}\sum_{i=j+1}^{n} 1+\varepsilon_i < \frac{\varepsilon}{n} + \frac{1}{n} + \frac{1}{f(2)} + \frac{\varepsilon}{nf(2)}$$
$$< \frac{\varepsilon}{n} + \frac{1}{n} + \frac{1}{1+2\varepsilon} + \frac{\varepsilon}{n(1+2\varepsilon)}.$$

Thus $\inf_{n\in\mathbb{N}}\|\sum_{i=1}^{n} y_i/n\| < 1/(1+2\varepsilon)$. This contradicts the fact that (y_i) is $(1+\varepsilon)$-equivalent to the standard ℓ_1 basis. Hence (e_i) is shrinking, and X is reflexive. ∎

Using the reflexive spaces presented in Proposition 5.6, we can prove the following lemma. Proposition 5.5 will then follow easily.

LEMMA 5.7. *If (v_i) is a 1-suppression unconditional normalized basic sequence such that (v_{k_i}) dominates (v_i) for all $(k_i) \in [\mathbb{N}]^\omega$ and (v_i) is not equivalent to the unit vector basis for c_0, then there exists a reflexive Banach space X which is $U_{(v_i)}$ and not $U_{(t_i^*)}$.*

Proof. There exists $K \ge 1$ such that (v_{k_i}) K-dominates (v_i) for all $(k_i) \in [\mathbb{N}]^\omega$. We define $\langle\cdot\rangle$ to be the norm on (v_i) determined by

$$\Big\langle \sum_{i\in\mathbb{N}} a_i v_i^* \Big\rangle = \sup_{(k_i)\in[\mathbb{N}]^\omega} \Big\|\sum_{i\in\mathbb{N}} a_i v_{k_i}^*\Big\| \quad \text{for all } (a_i) \in c_{00},$$

where (v_i^*) is the sequence of biorthogonal functionals to (v_i). The norm $\langle\cdot\rangle$ is K-equivalent to the original norm $\|\cdot\|$. Furthermore, under the new norm (v_{k_i}) 1-dominates (v_i) for all $(k_i) \in [\mathbb{N}]^\omega$. Thus after possibly renorming, we may assume that $K=1$.

Let $\varepsilon > 0$ and $\varepsilon_i \searrow 0$ be such that $\prod(1-\varepsilon_i)^{-1} < 1+\varepsilon$. Since (v_i) is unconditional and is not equivalent to the unit vector basis of c_0, there exists $(N_k) \in [\mathbb{N}]^\omega$ such that for all $k \in \mathbb{N}$ we have $N_k \ge k^2$ and

$$\Big\|\sum_{i=1}^{N_k} v_i\Big\| > \frac{k+1}{\varepsilon_{k+1}}. \tag{24}$$

We define the function $f : \mathbb{N} \to [1,\infty)$ by

$$f(n) = \begin{cases} 1 & \text{if } n = 1,\\ 1/(1-\varepsilon_1) & \text{if } 1 < n \le N_1,\\ k+1 & \text{if } N_k < n \le N_{k+1} \text{ for } k \in \mathbb{N}.\end{cases}$$

We denote by $|||\cdot|||$ the norm on c_{00} determined by the following implicit relation:

$$|||x||| = \|x\|_\infty \vee \sup_{m\geq 2,\, E_1<\cdots<E_m} \frac{1}{f(m)} \sum_{j=1}^{m} |||E_j(x)||| \quad \text{for all } x \in c_{00}.$$

The completion of c_{00} under the norm $|||\cdot|||$ is denoted by X, and its standard basis is denoted by (e_i). We have $N_k > k^2$, which implies that $\lim_{k\to\infty} k/f(k) = \infty$ and hence X is reflexive by Proposition 5.5.

We now show by induction on $k \in \mathbb{N}$ that if $(a_i)_{i=1}^{N_k} \in c_{00}$ then

$$\left(\prod_{i=1}^{k} \frac{1}{1-\varepsilon_i}\right) \Big|\Big|\Big| \sum_{i=1}^{N_k} a_i e_i \Big|\Big|\Big| \geq \Big\| \sum_{i=1}^{N_k} a_i v_i^* \Big\|. \tag{25}$$

For $k = 1$, we have

$$\frac{1}{1-\varepsilon_1} \Big|\Big|\Big| \sum_{i=1}^{N_1} a_i e_i \Big|\Big|\Big| \geq \sum_{i=1}^{N_1} |a_i| \geq \Big\| \sum_{i=1}^{N_1} a_i v_i^* \Big\|.$$

Thus (25) is satisfied. Now we assume that $k \in \mathbb{N}$ and that (25) holds for k.

Let $(a_i)_{i=1}^{N_{k+1}} \subset \mathbb{R}$ be such that $\|\sum_{i=1}^{N_{k+1}} a_i v_i^*\| = 1$. There exists $(\beta_i)_{1=1}^{N_{k+1}} \subset \mathbb{R}$ such that $\sum \beta_i a_i = \|\sum \beta_i v_i\| = 1$. Let $I = \{j \in \mathbb{N} \mid |\beta_j| < \varepsilon_{k+1}/(k+1)\}$. If $\sum_{i\in I} |a_i| \geq k+1$ then

$$\Big|\Big|\Big| \sum_{i=1}^{N_{k+1}} a_i e_i \Big|\Big|\Big| \geq \frac{1}{k+1} \sum_{i\in I} |a_i| \geq 1 = \Big\| \sum a_i v_i^* \Big\|$$

and we are done. Therefore we assume that $\sum_{i\in I} |a_i| < k+1$, and thus

$$\sum_{i\in I} \beta_i a_i \leq \sum_{i\in I} \frac{\varepsilon_{k+1}}{k+1} |a_i| < \varepsilon_{k+1}.$$

We let $\{j_i\}_{i=1}^{\sharp I^c} = I^c$, and claim that $\sharp I^c \leq N_k$. Indeed, if we assume to the contrary that $\sharp I^c > N_k$, then

$$1 \geq \Big\| \sum_{i=1}^{\sharp I^c} \beta_{j_i} v_{j_i} \Big\| \geq \Big\| \sum_{i=1}^{\sharp I^c} \beta_{j_i} v_i \Big\| \geq \frac{\varepsilon_{k+1}}{k+1} \Big\| \sum_{i=1}^{N_k} v_i \Big\| > \frac{\varepsilon_{k+1}}{k+1} \frac{k+1}{\varepsilon_{k+1}} = 1.$$

The first inequality is due to (v_i) being 1-suppression unconditional, and the second to (v_i) being 1-dominated by (v_{j_i}). Thus we have a contradiction and our claim that $\sharp I^c \leq N_k$ is proven. Now

$$\begin{aligned}
1 = \sum \beta_i a_i &= \sum_I \beta_i a_i + \sum_{I^c} \beta_i a_i \\
&< \varepsilon_{k+1} + \Big\| \sum_{i=1}^{\sharp I^c} a_{j_i} v_{j_i}^* \Big\| \leq \varepsilon_{k+1} + \Big\| \sum_{i=1}^{\sharp I^c} a_{j_i} v_i^* \Big\| \\
&\leq \varepsilon_{k+1} + \left(\prod_{i=1}^{k} \frac{1}{1-\varepsilon_i}\right) \Big|\Big|\Big| \sum_{i=1}^{\sharp I^c} a_{j_i} e_i \Big|\Big|\Big| \quad \text{by induction hypothesis} \\
&\leq \varepsilon_{k+1} + \left(\prod_{i=1}^{k} \frac{1}{1-\varepsilon_i}\right) \Big|\Big|\Big| \sum_{i=1}^{N_{k+1}} a_i e_i \Big|\Big|\Big| \quad \text{by 1-subsymmetry.}
\end{aligned}$$

The last inequality gives

$$1 \leq \Big(\prod_{i=1}^{k+1} \frac{1}{1-\varepsilon_i} \Big) \Big\|\Big| \sum_{i=1}^{N_{k+1}} a_i e_i \Big\|\Big|.$$

Thus the induction hypothesis is satisfied.

We see that (e_i) dominates (v_i^*), and hence (v_i) dominates (e_i^*). As (e_i^*) is subsymmetric and dominates all its block bases, $[e_i^*]$ is $U_{(v_i)}$. Since (e_i^*) is weakly null and is not equivalent to the unit vector basis of c_0, we deduce that $[e_i^*]$ is not $U_{(t_i^*)}$. ■

The proof of Proposition 5.5 now follows easily.

Proof of Proposition 5.5. Let (v_i) be a normalized C-spreading basic sequence. Because (v_i) is spreading, Rosenthal's ℓ_1 theorem implies that (v_i) must be either equivalent to the standard basis for ℓ_1, or weakly Cauchy. In the first case, it is obvious that $(v_i) \not\sim_U (t_i^*)$ as every Banach space is (U_{ℓ_1}). Thus we assume that (v_i) is weakly Cauchy. The difference sequence defined by $(w_i) = (v_{2i-1} - v_{2i})$ is weakly null. (w_i) is weakly null and spreading, and is thus unconditional. For all $(a_i) \in c_{00}$ we have

$$\Big\| \sum a_i w_i \Big\| \leq \Big\| \sum a_i v_{2i-1} \Big\| + \Big\| \sum a_i v_{2i} \Big\| \leq 2C \Big\| \sum a_i v_i \Big\|.$$

Thus, (v_i) dominates (w_i). If (w_i) is not equivalent to the standard basis for c_0 then, by Lemma 5.7, there exists a Banach space which is $U_{(w_i)}$ and hence $U_{(v_i)}$, but is not $U_{(t_i^*)}$. If (w_i) is equivalent to the standard basis for c_0 then

$$\sup_n \Big\| \sum_{i=1}^{n} (-1)^{n-1} v_i \Big\| = \sup_n \Big\| \sum_{i=1}^{n} w_i \Big\| < \infty.$$

However, $\sup_n \|\sum_{i=1}^n (-1)^n t_{k_i}^*\| = \infty$ for all $(k_i) \in [\mathbb{N}]^\omega$. Thus T^* is not $U_{(v_i)}$, and $(v_i) \not\sim_U (t_i^*)$. ■

We also considered the question: "Does there exist a basic sequence (v_i) such that $(v_i) \not\sim_U (w_i)$ for any unconditional (w_i)?". This is a much harder question, which is currently open. Neither the summing basis for c_0 nor the standard basis for James' space give a solution, as these are covered by the following proposition:

PROPOSITION 5.8. *If (v_i) is a basic sequence such that* $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^n \varepsilon_i v_i\| < D$ *for some $(\varepsilon_i) \in \{-1,1\}^{\mathbb{N}}$ and constant $D < \infty$, then $(v_i) \sim_U c_0$.*

Proof. Let X be a C-U_V Banach space, and let $(x_i) \in S_X$ be weakly null. By Ramsey's theorem, we may assume by passing to a subsequence that (v_i) C-dominates every subsequence of (x_i). By a theorem of John Elton [E], there exists $K < \infty$ and a subsequence (y_i) of (x_i) such that if $(a_i)_{i=1}^\infty \in [-1,1]^{\mathbb{N}}$ and $I \subset \{i \mid |a_i| = 1\}$ is finite then $\|\sum_I a_i y_i\| \leq K \sup_{n\in\mathbb{N}} \|\sum_{i=1}^n \varepsilon_i y_i\|$. Thus for all $A \in [\mathbb{N}]^{<\omega}$ we have

$$\Big\|\sum_{i\in A}\varepsilon_i y_i\Big\| \le K\sup_{n\in\mathbb{N}}\Big\|\sum_{i=1}^{n}\varepsilon_i y_i\Big\| \le KC\sup_{n\in\mathbb{N}}\Big\|\sum_{i=1}^{n}\varepsilon_i v_i\Big\| < KCD.$$

As this is true for all $A \in [N]^{<\omega}$, (y_i) is equivalent to the unit vector basis of c_0. Every normalized weakly null sequence in X has a subsequence equivalent to c_0, so X is U_{c_0}. ∎

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