The work of Nigel Kalton on greedy algorithms in Banach Spaces

July, 2011
References


Greedy Ordering

Let \((e_i)_{i=1}^{\infty}\) be a basis for a Banach space \(X\) with biorthogonal functionals \((e_i^*)_{i=1}^{\infty}\), i.e.

\[
x = \sum_{i=1}^{\infty} e_i^*(x)e_i \quad (x \in X).
\]

We assume that \((e_i)\) is semi-normalized, i.e. there exist constants \(0 < a \leq b\) such that

\[
a \leq \|e_i\| \leq b \quad (i \geq 1).
\]

For every \(x \in X\) we define the greedy ordering for \(x\) as the map \(\rho_x : \mathbb{N} \to \mathbb{N}\) which arranges the coefficients of \(x\) in decreasing order of absolute value.

The \(m\)-th greedy approximation is then given by

\[
G_m(x) = \sum_{j=1}^{m} e_{\rho_x(j)}^*(x)e_{\rho_x(j)}.
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The Thresholding Greedy Algorithm (TGA) converges, if $G_m(x) \rightarrow x$.

**Example**

Suppose $x = e_1 - 3e_2 - 4e_5 + 2e_7$.

Then

\[
G_1(x) = -4e_5; \quad G_2(x) = -4e_5 - 3e_2, \\
G_3(x) = -4e_5 - 3e_2 + 2e_7, \\
G_4(x) = -4e_5 - 3e_2 + 2e_7 + e_1,
\]

and

\[
\rho_x(1) = 5; \quad \rho_x(2) = 2; \quad \rho_x(3) = 7; \quad \rho_x(4) = 1.
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Convergence of the TGA

Definition (Konyagin-Temlyakov) 

$(e_i)$ is quasi-greedy (QG) if there exists a constant $K$ (the quasi-greedy constant) such that

$$\|G_n(x)\| \leq K\|x\| \quad (x \in X, \ n \geq 1).$$

Theorem (Wojtaszczyk)

$(e_i)$ is quasi-greedy (QG) if and only if the TGA converges, i.e.

$$\forall x \in X$$

$$x = \sum_{j=1}^{\infty} e_{\rho_x(j)}^*(x)e_{\rho_x(j)}.$$

• Note that a basis $(e_i)$ is unconditional if every rearrangement of $x = \sum e_i^*(x)e_i$ converges, so

unconditional $\Rightarrow$ QG.
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Existence of QG bases

Proposition
the Haar basis for $L_1[0, 1]$ and the Schauder basis for $c[0, 1]$ are not QG.

Question
Do $L_1[0, 1]$ and $C[0, 1]$ have a QG basis?

Definition
A basis $(e_n)$ is good if $\| \sum_{n \in A} e_n \| \to \infty$ as $|A| \to \infty$ (plus another more technical condition).

Example
Symmetric bases are good except for the unit vector bases of $c_0$ and $\ell_1$. 
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Theorem (DKK)
Suppose $X$ has a basis and $S$ has a good unconditional basis or $S = \ell_1$ then $S \oplus X$ has a QG basis.

Corollary (DKK)
$L_1[0, 1]$ and the Schatten ideals $S_p$ ($1 \leq p < \infty$) have a QG basis.
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QG bases and Grothendieck’s theorem

Definition

$X$ is a **GT space** if every bounded operator $T : X \to \ell_2$ is absolutely summing.

Theorem (DKK)

Suppose $X^*$ is a GT space. If $(e_n)$ is a QG basis for $X$ then $(e_n)$ is equivalent to the unit vector basis of $c_0$.

Corollary (DKK)

$C[0,1]$ and the disc algebra do not have a QG basis.

Corollary (DKK)

$c_0$ is the only Banach space to have a unique quasi-greedy basis up to basis equivalence.
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Corollary (DKK)

$c_0$ is the **only** Banach space to have a unique quasi-greedy basis up to basis equivalence.
Partial Unconditionality

Definition
(Elton) A semi-normalized basic sequence \((e_i)\) is nearly unconditional if for every \(\delta > 0\) there exists a constant \(K(\delta)\) such that for all coefficient sequences \((a_i)\), with \(\sup |a_i| \leq 1\), and all \(E \subset \{i : |a_i| \geq \delta\}\), we have

\[
\|P_E(\sum a_i e_i)\| \leq K(\delta) \| \sum a_i e_i \|.
\]

Remark
\((e_i)\) is unconditional if and only if

\[
\sup_{\delta > 0} K(\delta) < \infty
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Theorem (Elton)
Every Banach space contains a nearly unconditional basic sequence.

Theorem (Gowers-Maurey)
\[ \exists \text{ a Banach space GM without any unconditional basic sequence.} \]

Theorem
\[ \Rightarrow \quad \text{QG} \Rightarrow \text{nearly unconditional.} \]
\[ \exists \text{nearly unconditional sequences that are not QG.} \]

Problem
Does every Banach space contain a QG basic sequence?
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- QG \( \Rightarrow \) nearly unconditional.
- \( \exists \) nearly unconditional sequences that are not QG.

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Does every Banach space contain a QG basic sequence?
Type and Cotype

Definition

- Let $2 \leq q < \infty$. $X$ has **cotype** $q$ if $\exists C_q$
  $\forall n \geq 1, x_1, \ldots, x_n \in X$

\[
\left( \sum \|x_i\|^q \right)^{1/q} \leq C_q \text{Ave}_\pm \| \sum \pm x_i \|
\]

- Let $1 < p \leq 2$. $X$ has **type** $p$ if $\exists C_p$ $\forall n \geq 1, x_1, \ldots, x_n \in X$

\[
T_p\left( \sum \|x_i\|^p \right)^{1/p} \geq \text{Ave}_\pm \| \sum \pm x_i \|
\]

Example

$L_p[0, 1]$ has cotype $\max(2, p)$ and type $\min(2, p)$. 
Type and Cotype

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- Let $2 \leq q < \infty$. $X$ has **cotype $q$** if $\exists C_q$
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Theorem (DKK)

If $X$ has finite cotype (i.e. cotype $q$ for some $q < \infty$) then every semi-normalized weakly null sequence has a QG subsequence. So $X$ contains a QG sequence.

Theorem (DOSZ, 2009)

c_0 can be renormed so that the QG constant of every subsequence of the unit vector basis is at least $8/7$. 
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Best $n$-term approximation

For $x \in X$, the error in the best $n$-term approximation to $x$ is given by:

$$\sigma_n(x) = \inf\{\|x - \sum_{j \in A} \alpha_j e_j\| : |A| = n, \alpha_j \in \mathbb{R}\}.$$

Hence $\sigma_n(x) \leq \|x - G_n(x)\|$.

Definition (Konyagin-Temlyakov)

$(e_i)$ is greedy with greedy constant $C \geq 1$

$$\|x - G_n(x)\| \leq C \sigma_n(x) \quad (x \in X, n \in \mathbb{N}).$$

Example

- The unit vector basis of $\ell_p$ or $c_0$ is 1-greedy.
- Every symmetric basis is greedy.
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- The unit vector basis of $\ell_p$ or $c_0$ is 1-greedy.
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Theorem (Temlyakov)

The Haar basis of $L^p[0, 1]$ is greedy for $1 < p < \infty$.

Democratic Bases

Definition

• The fundamental function $\varphi : \mathbb{N} \to \mathbb{R}$ of $(e_i)$ is given by:

$$
\varphi(n) := \sup_{|A| \leq n} \left\| \sum_{i \in A} e_i \right\|.
$$

• $(e_i)$ is democratic with constant $\Delta$ if $\forall$ finite $A \subset \mathbb{N}$,

$$
\varphi(|A|) \leq \Delta \left\| \sum_{i \in A} e_i \right\|.
$$

Thus, if $|A| = |B|$, then

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\frac{1}{\Delta} \left\| \sum_{i \in B} e_i \right\| \leq \left\| \sum_{i \in A} e_i \right\| \leq \Delta \left\| \sum_{i \in B} e_i \right\|.
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Theorem (Konyagin-Temlyakov)

A basis is greedy if and only if it is unconditional and democratic
Almost greedy bases

Definition
The error in the best \( n \) term projection approximating \( x \) is given by
\[
\tilde{\sigma}_n(x) = \inf\{\|x - \sum_{j \in A} e^*_j(x)e_j\| : |A| \leq n\}.
\]

Theorem (DKKT)
The following are equivalent:

\( \exists C \) such that
\[
\|x - G_n(x)\| \leq C\tilde{\sigma}_n(x) \quad (x \in X, n \geq 1).
\]

\( (e_i) \) is \textit{QG and democratic}.

\( \exists C \) such that
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Theorem (DKKT)

The following are equivalent:

1. $\exists C$ such that $\| x - G_n(x) \| \leq C\tilde{\sigma}_n(x)$ $(x \in X, n \geq 1)$.

2. $(e_i)$ is QG and democratic.

3. $\exists C$ such that $\| x - G_{2n}(x) \| \leq C\sigma_n(x)$ $(x \in X, n \geq 1)$.
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Theorem (DKKT)

The following are equivalent:

1. $\exists C$ such that $\|x - G_n(x)\| \leq C\tilde{\sigma}_n(x)$ (for $x \in X, n \geq 1$).
2. $(e_i)$ is QG and democratic.
3. $\exists C$ such that $\|x - G_{2n}(x)\| \leq C\sigma_n(x)$ (for $x \in X, n \geq 1$).
Definition

\((e_i)\) is almost greedy if \((e_i)\) is QG and democratic

Theorem (DKK)

Suppose \(X\) has a basis and contains a complemented subspace with a symmetric basis that is not \(c_0\). Then \(X\) has an almost greedy basis.

Theorem (DKK)

Let \(X\) be a Banach space. The following are equivalent:

- \(X\) contains an almost greedy basic sequence;
- \(X\) contains \(c_0\) or \(\ell_1\) or \(X\) has a spreading model generated by a weakly null sequence that is not equivalent to the unit vector basis of \(c_0\).
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\end{align*}\)
Semi-greedy bases

For $x \in X$, let $G_n^C(x)$ be a best approximation to $x$ from $\text{span}\{e_{\rho(i)} : 1 \leq i \leq n\}$, i.e.

$$\|x - G_n^C(x)\| = \min\{\|x - \sum_{i=1}^{n} a_i e_{\rho(i)}\| : (a_i)_{i=1}^{n} \in \mathbb{R}^n\}.$$

**Definition**

$(e_i)$ is **semi-greedy** if there exists a constant $K$ such that

$$\|x - G_n^C(x)\| \leq K\sigma_n(x) \quad (n \geq 1, x \in X).$$
Theorem (DKK)

- *almost greedy* $\Rightarrow$ *semi-greedy*.
- *The converse holds if* $X$ *has finite cotype.*

In the general case we have the following:

*Semi-greedy* $\Rightarrow$ *democratic.*

**Problem**

*Does semi-greedy imply quasi-greedy in general?*
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Problem

*Does semi-greedy imply quasi-greedy in general?*
Duality

Duality fails in general:

- If \((e_i)\) is greedy then \((e_i^*)\) may fail to be democratic.
- If \((e_i)\) is QG then \((e_i^*)\) may fail to be QG.

Definition

A fundamental function \((\varphi(n))\) has the upper regularity property (URP) if \(\exists C > 0\) and \(0 < \beta < 1\) such that

\[
\varphi(m) \leq C(m/n)^\beta \varphi(n) \quad (m > n).
\]

Theorem (DKKT)

If \((e_n)\) is a greedy (resp. almost greedy) basis whose fundamental function has URP, then \((e_n^*)\) is a greedy (resp. almost greedy) basic sequence.
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If \((e_n)\) is a greedy (resp. almost greedy) basis whose fundamental function has URP, then \((e_n^*)\) is a greedy (resp. almost greedy) basic sequence.
Corollary (DKKT)
Suppose $X$ has type $p > 1$. If $(e_n)$ is a greedy basis for $X$ then $(e^*_n)$ is a greedy basis for $X^*$

Corollary (DKKT)
Let $(e_i)$ be a QG basis for a separable Hilbert space. Then both $(e_i)$ and $(e^*_i)$ are almost greedy bases for $H$.

Theorem (DKKT)
If $(\varphi(n))$ does not have URP then there exists a reflexive Banach space with a greedy basis whose fundamental function equivalent to $\varphi$ whose dual basis is not greedy.
Characterization of duality

Definition
Let \((e_n)\) be a basis for \(X\) with fundamental function \((\varphi_n)\). Let \((\varphi_n^*)\) be the fundamental function for \((e_n^*)\). Then \((e_n)\) is bidemocratic if \(\exists C\) such that

\[
\varphi(n)\varphi^*(n) \leq Cn \quad (n \in \mathbb{N}).
\]

Theorem
Let \((e_n)\) be a QG basis for \(X\). The following are equivalent:

\(\varphi_n\) is bidemocratic

Both \((e_i)\) and \((e_i^*)\) are almost greedy.