CLOSED IDEALS OF OPERATORS BETWEEN THE CLASSICAL SEQUENCE SPACES

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Abstract. We prove that the spaces $\mathcal{L}(\ell_p, c_0)$, $\mathcal{L}(\ell_p, \ell_\infty)$ and $\mathcal{L}(\ell_1, \ell_q)$ of operators with $1 < p, q < \infty$ have continuum many closed ideals. This extends and improves earlier works by Schlumprecht and Zsák [20], by Wallis [22] and by Sirotkin and Wallis [19]. Several open problems remain. Key to our construction of closed ideals are matrices with the Restricted Isometry Property that come from Compressed Sensing.

1. Introduction

Closed ideals of operators between Banach spaces is a very old subject going back to at least the 1940s when Calkin proved that the compact operators are the only non-trivial, proper closed ideal of the algebra $\mathcal{L}(\ell_2)$ of all bounded linear maps on Hilbert space [4]. Two decades later Gohberg, Markus and Fel′dman [9] showed that the same result holds for the algebras $\mathcal{L}(\ell_p)$, $1 \leqslant p < \infty$, and $\mathcal{L}(c_0)$. It is perhaps surprising that the situation is vastly more complicated for the spaces $\ell_p \oplus \ell_q$ and $\ell_p \oplus c_0$, $1 \leqslant p < q < \infty$. We refer the reader to Pietsch's book 'Operator Ideals' [15, Chapter 5] and to sections 1 and 2.3 of [20] for a detailed history of the study of closed ideals on these spaces. Here we shall be fairly brief and concentrate on developments that best place our new results in context.

Let $X = \ell_p$ and let Y be either ℓ_q or c_0, where $1 \leqslant p < q < \infty$. We recall [21] that $\mathcal{L}(X \oplus Y)$ has exactly two maximal ideals: the closures of the ideals of operators factoring through X and Y, respectively. Moreover, the set of all non-maximal, proper closed ideals of $\mathcal{L}(X \oplus Y)$ is in a one-to-one, inclusion-preserving correspondence with the set of all closed ideals of $\mathcal{L}(X, Y)$ (see [15, Theorem 5.3.2] or Section 2 of [18]). Here an ideal of $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ with the *ideal property*: $ATB \in \mathcal{J}$ whenever $A \in \mathcal{L}(Y), T \in \mathcal{J}$ and $B \in \mathcal{L}(X)$. In his book, Pietsch raised the problem whether $\mathcal{L}(X, Y)$ has infinitely many closed ideals in the case $X = \ell_p$ and $Y = \ell_q$ with $1 \leqslant p < q < \infty$. This problem remained open for nearly forty years. It is straightforward that the smallest non-trivial closed ideal is the ideal of compact operators and that every other non-trivial closed ideal must contain the closed ideal generated by the formal inclusion map $I_{X,Y} \colon X \to Y$. However, it is not even obvious if there are any other non-trivial, proper closed ideals besides the compact operators. The first results in this direction are due to Milman [13] who first proved that $I_{X,Y}$ is finitely strictly singular, and then exhibited in the case $1 < p < q < \infty$ an operator in $\mathcal{L}(\ell_p, \ell_q)$ that is not finitely strictly singular. (Definitions will be given in Section 2 below.) More recently, further closed ideals, but still only finitely many, were found by Sari, Tomczak-Jaegermann, Troitsky and Schlumprecht [17] and by Schlumprecht [18].

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Finally, Schlumprecht and Zsák found a positive answer to Pietsch's question in the reflexive range $1 < p < q < \infty$. In fact, they showed [20] that in that range, $\mathcal{L}(\ell_p, \ell_q)$ contains continuum many closed ideals which with respect to inclusion are order-isomorphic to $\mathbb{R}$. Shortly after this, Wallis observed [22] that the techniques of [20] extend to prove the same result for $\mathcal{L}(\ell_p, \mathrm{c}_0)$ in the range $1 < p < 2$, and for $\mathcal{L}(\ell_1, \ell_q)$ in the range $2 < q < \infty$. The answer to Pietsch's question was completed by Sirotkin and Wallis [19] who proved that there are uncountably many closed ideals in $\mathcal{L}(\ell_1, \ell_q)$ for $1 < q \leqslant \infty$ as well as in $\mathcal{L}(\ell_1, \mathrm{c}_0)$ and in $\mathcal{L}(\ell_p, \ell_\infty)$ for $1 \leqslant p < \infty$. The techniques used in [20] and [19] are completely different and we shall have more to say about this in the final section of this paper. The method of Sirotkin and Wallis produces chains of closed ideals of size ω_1, the first uncountable ordinal. Moreover, their method can not tackle (for good reasons as we shall see later) the question whether $\mathcal{L}(\ell_p, \mathrm{c}_0)$ has infinitely many closed ideals in the range $2 \leqslant p < \infty$. The main focus of this paper is *this* remaining open problem regarding closed ideals of operators between classical sequence spaces. Our main result is a solution of this problem as well as an extension of the result of Wallis for $1 < p < 2$.

Theorem A. *For all $1 < p < \infty$ there are continuum many closed ideals in $\mathcal{L}(\ell_p, \mathrm{c}_0)$.*

Our techniques are somewhat similar to those used in [20]. However, instead of using independent sequences of 3-valued, symmetric random variables spanning Rosenthal spaces, here we are going to rely on matrices with the Restricted Isometry Property (RIP for short) to generate ideals. Their appearance in this problem was somewhat of a surprise to us. In fact, we shall prove considerably more by showing that the distributive lattice of closed ideals of $\mathcal{L}(\ell_p, \mathrm{c}_0)$ has a rich structure (see Theorem 2 below). From Theorem A it is a short step to extend the results of Sirotkin and Wallis as follows.

Theorem B. *For all $1 < p < \infty$ there are continuum many closed ideals in $\mathcal{L}(\ell_p, \ell_\infty)$ and $\mathcal{L}(\ell_1, \ell_p)$.*

As before, we will present more precise statements about the lattice of closed ideals for these spaces. We do not know if $\mathcal{L}(\ell_1, \mathrm{c}_0)$ and $\mathcal{L}(\ell_1, \ell_\infty)$ have continuum many closed ideals. Thus, in these cases the best known result is the aforementioned theorem of Sirotkin and Wallis [19] that shows the existence of an ω_1-chain of closed ideals.

The paper is organized as follows. Section 2 begins with definitions, notations and the introduction of RIP vectors. We then present the main results about ideals in $\mathcal{L}(\ell_p, \mathrm{c}_0)$. In Section 3 we present a proof of Theorem B. In the final section we first explain the method of Sirotkin and Wallis and how it differs from ours. We then prove some further results of interest about the ideal structure in $\mathcal{L}(\ell_p, \mathrm{c}_0)$. These explain why the techniques of [19] cannot work here. We conclude with further remarks and open problems.

To conclude this introduction, we mention two pieces of notation used throughout the paper. Firstly, we denote by $X \cong Y$ if the Banach spaces X and Y are isometrically isomorphic, whereas $X \sim Y$ indicates that they are merely isomorphic. Secondly, the action of a functional $f \in X^*$ on a vector $x \in X$ will be written as $\langle x, f\rangle$. Our convention is that the vector always appears on the left and the functional on the right.

2. Closed ideals in $\mathcal{L}(\ell_p, \mathrm{c}_0)$

Given Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the space of all operators from X to Y. By an operator we shall always mean a bounded linear map. When

$X = Y$ we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$. When X is formally a subset of Y, we write $I_{X,Y}$ for the formal inclusion map $X \to Y$ provided this is bounded. We shall sometimes write I instead of $I_{X,Y}$ when X and Y are clear from the context. Assume now that $X = \big(\bigoplus_{n=1}^\infty X_n\big)_{\ell_p}$ for some $1 \leqslant p < \infty$ and that either $Y = \big(\bigoplus_{n=1}^\infty Y_n\big)_{\ell_q}$ for some $p \leqslant q \leqslant \infty$ or $Y = \big(\bigoplus_{n=1}^\infty Y_n\big)_{c_0}$. Then, given a uniformly bounded sequence of operators $T_n \colon X_n \to Y_n$, $n \in \mathbb{N}$, we write $T = \operatorname{diag}(T_n)_{n\in\mathbb{N}}$ for the diagonal operator $T\colon X \to Y$ defined by $(x_n) \mapsto (T_n x_n)$.

By an *(operator) ideal of* $\mathcal{L}(X,Y)$ we mean a subspace $\mathcal{J}$ of $\mathcal{L}(X,Y)$ such that $ATB \in \mathcal{J}$ for all $A \in \mathcal{L}(Y)$, $T \in \mathcal{J}$ and $B \in \mathcal{L}(X)$. A *closed (operator) ideal* is an ideal that is closed in the operator norm. Note that the closure of an ideal is a closed ideal. We shall denote by $\mathfrak{H}(X,Y)$ the set of all closed ideals of $\mathcal{L}(X,Y)$. Note that $\mathfrak{H}(X,Y)$ is a distributive lattice under inclusion: the *meet* and *join* operations are given by

$$\mathcal{I} \wedge \mathcal{J} = \mathcal{I} \cap \mathcal{J}\ , \quad \mathcal{I} \vee \mathcal{J} = \overline{\mathcal{I} + \mathcal{J}} \qquad \text{for } \mathcal{I}, \mathcal{J} \in \mathfrak{H}(X,Y)\ .$$

We next recall some standard operator ideals. We denote by $\mathcal{K}(X,Y)$, $\mathcal{FS}(X,Y)$, and $\mathcal{S}(X,Y)$ the closed ideals of, respectively, compact, finitely strictly singular, and strictly singular operators from X to Y. Recall that $T \in \mathcal{L}(X,Y)$ is *finitely strictly singular* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every subspace $E \subset X$ of dimension at least n, there is a vector $x \in E$ with $\|Tx\| < \varepsilon\|x\|$. We say T is *strictly singular* if no restriction of T to an infinite-dimensional subspace of X is an isomorphism. It is clear that $\mathcal{K}(X,Y) \subset \mathcal{FS}(X,Y) \subset \mathcal{S}(X,Y)$. When $X = Y$ these ideals become $\mathcal{K}(X) \subset \mathcal{FS}(X) \subset \mathcal{S}(X)$. We shall also sometimes write $\mathcal{K}$, $\mathcal{FS}$ and $\mathcal{S}$ when X and Y are clear from the context. Recall that if $X = \ell_p$, $1 \leqslant p < \infty$, and either $Y = \ell_q$, $p < q < \infty$, or $Y = c_0$, then $\mathcal{L}(X,Y) = \mathcal{S}(X,Y)$ and $\mathcal{L}(Y,X) = \mathcal{K}(Y,X)$.

We next introduce notation for closed ideals generated by a fixed operator. Let $T\colon W \to Z$ be a bounded linear map. For any pair (X,Y) of Banach spaces we denote by $\mathcal{J}^T(X,Y)$ the closed ideal of $\mathcal{L}(X,Y)$ generated by T. Thus,

$$\mathcal{J}^T(X,Y) = \overline{\operatorname{span}}\{ATB :\ A \in \mathcal{L}(Z,Y), B \in \mathcal{L}(X,W)\}$$

is the closed linear span of operators factoring through T. As usual, we write $\mathcal{J}^T(X)$ instead of $\mathcal{J}^T(X,Y)$ when $X = Y$. We shall also write $\mathcal{J}^T$ instead of $\mathcal{J}^T(X,Y)$ or $\mathcal{J}^T(X)$ when X and Y are clear from the context.

We now begin our study of closed ideals in $\mathcal{L}(\ell_p, c_0)$ where $1 \leqslant p < \infty$ is fixed for the rest of this section. As mentioned in the Introduction, we know of the following ideals:

$$\{0\} \subsetneq \mathcal{K} \subsetneq \mathcal{J}^{I_{\ell_p,c_0}} \subset \mathcal{FS} \subsetneq \mathcal{S} = \mathcal{L}(\ell_p, c_0)\ .$$

To see that $\mathcal{FS}$ is a proper ideal, fix for each $n \in \mathbb{N}$ an embedding $T_n \colon \ell_p^n \to \ell_\infty^{M_n}$ for a sufficiently large $M_n \in \mathbb{N}$ satisfying, say, $\|T_n x\| \leqslant \|x\| \leqslant 2\|T_n x\|$ for all $x \in \ell_p^n$. Then the diagonal operator $\operatorname{diag}(T_n)_{n\in\mathbb{N}}$ from $\ell_p \cong \big(\bigoplus_{n\in\mathbb{N}} \ell_p^n\big)_{\ell_p}$ to $c_0 \cong \big(\bigoplus_{n\in\mathbb{N}} \ell_\infty^{M_n}\big)_{c_0}$ is clearly not finitely strictly singular. As noted in [22], the results of [20] extend without much difficulty to show that in the range $1 < p < 2$, there is a chain of closed ideals order-isomorphic to $\mathbb{R}$ between $\mathcal{J}^{I_{\ell_p,c_0}}$ and $\mathcal{FS}$. However, for $2 \leqslant p < \infty$ it was not even known if there are any other non-trivial, proper closed ideals besides $\mathcal{K}$ and $\mathcal{FS}$. We remedy this situation with our main result, Theorem 2 below. The main ingredient will be certain RIP vectors which we introduce next.

Fix $\delta \in (0,1)$ and positive integers $k < n \leqslant N$. We say that the $n \times N$ matrix A satisfies the *Restricted Isometry Property (RIP) of order k with error δ* if for all

$x \in \mathbb{R}^N$ with at most k non-zero entries the following holds:

$$(1-\delta)\|x\|_{\ell_2^N} \leqslant \|Ax\|_{\ell_2^n} \leqslant (1+\delta)\|x\|_{\ell_2^N} \ .$$

Candès and Tao [5] introduced this property and established the important rôle it plays in Compressed Sensing. Briefly, such matrices allow recovery of sparse data from few measurements. Candès and Tao also established the existence of matrices with the RIP: roughly speaking, they proved that a random $n \times N$ matrix with Gaussian entries satisfies RIP with overwhelming probability. In particular, and this is what we will use below, for any $\delta \in (0,1)$ and $k \in \mathbb{N}$, there exists a constant $c = c(\delta, k) > 0$ such that for all $n, N \in \mathbb{N}$ satisfying $\frac{\log N}{n} < c$ there exists an $n \times N$ matrix with the RIP of order k and error δ. Before continuing, let us mention that the RIP phenomenon has also been fundamental in asymptotic finite-dimensional Banach space theory. Indeed, all the tools necessary for constructing RIP matrices were already available in the 1970s and 1980s, for example, in Kashin's decomposition of ℓ_1^{2n} into two n-dimensional Euclidean sections [11], and in the well known Johnson–Lindenstrauss lemma concerning certain Lipschitz mappings of a finite set in Euclidean space onto a space of dimension logarithmic in the size of the set [10]. We refer the reader to the article of Baraniuk, Davenport, DeVore and Waking [1] where they discuss these connections and provide a simple proof of the RIP for random matrices whose entries satisfy certain concentration inequalities (this includes matrices with independent Gaussian or Rademacher entries as well as others). The book of Foucart and Rauhut [8] also discusses these topics in detail as well as many other aspects of the mathematics of Compressed Sensing.

The following statement is a straightforward consequence of the above discussion. There exists a sequence $u_1 < v_1 < u_2 < v_2 < \dots$ of positive integers such that setting

$$s_n = \begin{cases} (2u_n)^{p/2} \cdot n^p & \text{if } 2 \leqslant p < \infty \\ 2u_n \cdot n^2 & \text{if } 1 \leqslant p \leqslant 2 \end{cases} \tag{1}$$

the following properties hold:

$$u_n \geqslant 19n^3 \cdot (6n+1)^{u_1+u_2+\dots+u_{n-1}} \ , \tag{2}$$

$$v_n \geqslant 9n^3 \cdot s_n \ , \tag{3}$$

and there exist unit vectors $\big(g_i^{(n)}\big)_{i=1}^{v_n}$ in $\ell_2^{u_n}$ (the normalized columns of an RIP matrix) such that for every $J \subset \{1, 2, \dots, v_n\}$ with $|J| \leqslant (s_{n-1}+1) \vee 19n^2$ we have

$$\frac{1}{2}\sum_{i\in J}|a_i|^2 \leqslant \Big\|\sum_{i\in J} a_i g_i^{(n)}\Big\|^2 \leqslant 2\sum_{i\in J}|a_i|^2 \qquad \text{for all } (a_i)_{i\in J} \subset \mathbb{R} \ , \tag{4}$$

and

$$\sum_{i\in J}|\langle x, g_i^{(n)}\rangle|^2 \leqslant 2\|x\|^2 \qquad \text{for all } x \in \ell_2^{u_n} \ . \tag{5}$$

We next define, for each $n \in \mathbb{N}$, operators $T_n \colon \ell_2^{u_n} \to \ell_\infty^{v_n}$ by $x \mapsto \big(\langle x, g_i^{(n)}\rangle\big)_{i=1}^{v_n}$. Note that $\|T_n\| = 1$, and hence we may use these maps to define certain diagonal operators. Set $U = \big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{\ell_p}$ and $V = \big(\bigoplus_{n=1}^\infty \ell_\infty^{v_n}\big)_{c_0}$. For an infinite set $M \subset \mathbb{N}$ define $T_M \colon U \to V$ by $(x_n) \mapsto (y_n)$, where $y_n = T_n(x_n)$ if $n \in M$ and $y_n = 0$ otherwise. The following is the main result of this section.

Theorem 1. *Let M, N be infinite subsets of $\mathbb{N}$.*

(i) If $M \setminus N$ is infinite, then $T_M \notin \mathcal{J}^{T_N}(U,V)$.

(ii) If $N \setminus M$ is finite, then $\mathcal{J}^{T_N}(U,V) \subset \mathcal{J}^{T_M}(U,V)$.

As a corollary we will deduce that the lattice of closed ideals of $\mathcal{L}(U,V)$ has a rich structure in the following precise sense. Define $\mathfrak{B}$ to be the Boolean algebra obtained as the quotient of the power set $\mathcal{P}\mathbb{N}$ of $\mathbb{N}$ by the equivalence relation $\sim$ defined by $M\sim N$ if and only if $M\bigtriangleup N$ is finite. By Theorem 1(ii) above, the ideal $\mathcal{J}^{T_M}(U,V)$ depends only on the equivalence class $[M]$ of the infinite set M. Thus, we have a well-defined map

$$\begin{aligned} \varphi\colon \mathfrak{B} &\to \mathfrak{H}(U,V)\\ [M] &\mapsto \begin{cases} \mathcal{J}^{T_M} & \text{if } M \text{ is infinite,}\\ \mathcal{J}^{I_{U,V}} & \text{if } M \text{ is finite.}\end{cases}\end{aligned}$$

Here $I_{U,V}$ is the formal inclusion of U into V. Note that U can indeed be viewed as a subset of V in the following way. Every element of $U=\big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{\ell_p}$ can be thought of as a sequence $(x_i)_{i=1}^\infty$ with $\sum_{n=1}^\infty \Big(\sum_{i=t_{n-1}+1}^{t_n}|x_i|^2\Big)^{p/2}<\infty$, where $t_n=u_1+\dots+u_n$ for all $n\in\mathbb{N}$. On the other hand, V is nothing else but c_0 with the particular choice $\big(\bigoplus_{n=1}^\infty \ell_\infty^{v_n}\big)_{\mathrm{c}_0}$ of blocking of its unit vector basis. It will also be helpful to think of $I_{U,V}$ as a diagonal operator: write $V=\mathrm{c}_0$ as $\big(\bigoplus_{n=1}^\infty \ell_\infty^{u_n}\big)_{\mathrm{c}_0}$ using a different blocking of the unit vector basis. Then $I_{U,V}\colon U\to V$ is the diagonal operator whose n^{th} diagonal entry is the formal identity $\ell_2^{u_n}\to\ell_\infty^{u_n}$. At the very end of this section we will comment on the choice of $\varphi([M])$ for M finite.

Theorem 2. *The map φ defined above is injective, monotone and preserves the join operation. In particular, $\mathcal{L}(U,V)$ has continuum many closed ideals between $\mathcal{J}^{I_{U,V}}$ and $\mathcal{FS}$.*

Since $V=\mathrm{c}_0$ and for $1<p<\infty$ the space U is isomorphic to ℓ_p by Pełczyński's Decomposition Theorem (see *e.g.,* [6, Theorem 6.24]), our first main result, Theorem A, stated in the Introduction follows immediately.

We begin with the proof of Theorem 1. Of course, part (ii) is trivial since if $N\setminus M$ is finite, then a finite-rank perturbation of T_N factors through T_M. Let us now explain why part (i) has a chance of being true. An obvious way for (i) to fail would be the existence of an injection $m\mapsto n_m$ from M into N such that T_m factors through T_{n_m} for all $m\in M$. Indeed, then T_M would factor through T_N and we would immediately obtain $T_M\in\mathcal{J}^{T_N}$. However, for $m\in M\setminus N$ and for any $n\in N$, it is hard for T_m to factor through T_n. Indeed, T_m preserves the norm of the RIP vectors $g_i^{(m)}$, $1\leqslant i\leqslant v_m$. When $m>n$, the number v_m of these vectors is massive compared to the dimension u_n of the domain of T_n, and an easy pigeonhole argument will show that one cannot have a map $\ell_2^{u_m}\to\ell_2^{u_n}$ preserving the norm of so many RIP vectors. The case $m<n$ is slightly more complicated. In this case, T_m factoring through T_n says something about the action of T_n on a subspace of small dimension (small compared to the dimension u_n of the domain of T_n). It will then follow from (4) and (5) that the restriction of T_n to this small-dimensional subspace, and hence T_m, factors through the formal identity $I\colon\ell_2^r\to\ell_\infty^r$, where $r\leqslant s_m$, and in particular r is still much smaller than the number v_m of RIP vectors whose norm T_m preserves. Another simple pigeonhole argument will again show that the formal identity I cannot preserve the norm of so many RIP vectors, but this time it will be crucial that I is mapping into an ℓ_∞-space.

We begin the proof of our main theorem with a beefed up version of the claim above that the restriction of T_n to a small-dimensional subspace factors through the formal identity $\ell_2^r\to\ell_\infty^r$ with a suitable bound on r.

Lemma 3. *Fix $m \in \mathbb{N}$, a subset N of $\{n \in \mathbb{N} : n > m\}$ and an operator*

$$B\colon \ell_2^{u_m} \to \Big(\bigoplus_{n\in N} \ell_2^{u_n}\Big)_{\ell_p}$$

with $\|B\| \leqslant 1$. Let $D\colon \big(\bigoplus_{n\in N} \ell_2^{u_n}\big)_{\ell_p} \to \big(\bigoplus_{n\in N} \ell_\infty^{v_n}\big)_{c_0}$ be the diagonal operator $\operatorname{diag}(T_n)_{n\in N}$. Then there exist subsets $J_n \subset \{1, 2, \dots, v_n\}$, $n \in N$, such that $\sum_{n\in N} |J_n| \leqslant s_m$ and there is an approximate factorization

$$\begin{tikzcd}
\ell_2^{u_m} \arrow[r, "B"] \arrow[d, "P"'] & \Big(\bigoplus_{n\in N} \ell_2^{u_n}\Big)_{\ell_p} \arrow[r, "D"] & \Big(\bigoplus_{n\in N} \ell_\infty^{v_n}\Big)_{c_0} \\
\Big(\bigoplus_{n\in N} \ell_2^{J_n}\Big)_{\ell_p} \arrow[rr, "I"] & & \Big(\bigoplus_{n\in N} \ell_\infty^{J_n}\Big)_{c_0} \arrow[u, "R"']
\end{tikzcd}$$

with $\|DB - RIP\| \leqslant \frac{1}{m}$, where I is the formal inclusion, $\|P\| \leqslant 2$ and $\|R\| \leqslant 1$.

Proof. Let us first observe that we need only consider the case $2 \leqslant p < \infty$. This is because the case $1 \leqslant p \leqslant 2$ reduces to the case $p = 2$. Indeed, if $1 \leqslant p \leqslant 2$, then $D\colon \big(\bigoplus_{n\in N} \ell_2^{u_n}\big)_{\ell_p} \to \big(\bigoplus_{n\in N} \ell_\infty^{v_n}\big)_{c_0}$ factors through D viewed as a map from $\big(\bigoplus_{n\in N} \ell_2^{u_n}\big)_{\ell_2} \to \big(\bigoplus_{n\in N} \ell_\infty^{v_n}\big)_{c_0}$ via the formal identity $\big(\bigoplus_{n\in N} \ell_2^{u_n}\big)_{\ell_p} \to \big(\bigoplus_{n\in N} \ell_2^{u_n}\big)_{\ell_2}$. The case $p = 2$ then provides the required approximate factorization via the formal identity $I\colon \ell_2^r \to \ell_\infty^r$ for some $r \leqslant s_m$. This completes the proof by taking $J_n = \{1, \dots, r\} \subset \{1, \dots, v_n\}$ for $n = \min N$, and $J_{n'} = \emptyset$ for all $n' \in N \setminus \{n\}$.

Let us now assume that $2 \leqslant p < \infty$, and we may of course also take $N \neq \emptyset$. Set $t = s_m + 1$ and let H be a subset of $\{(n, j) : n \in N,\ 1 \leqslant j \leqslant v_n\}$ of size t such that

$$\big\|B^* g_j^{(n)}\big\| \geqslant \big\|B^* g_{j'}^{(n')}\big\| \qquad \text{for all } (n, j) \in H,\ (n', j') \notin H\ .$$

Set $J = \big\{(n, j) \in H : \big\|B^* g_j^{(n)}\big\| \geqslant \frac{1}{m}\big\}$. For each $n \in N$ letting $H_n = \{j : (n, j) \in H\}$ and $J_n = \{j : (n, j) \in J\}$, we obtain the following inequalities.

$$\begin{aligned}
\tfrac{1}{m^2}|J_n| &\leqslant \sum_{j\in H_n} \big\|B^* g_j^{(n)}\big\|^2 = \sum_{j\in H_n} \sum_{i=1}^{u_m} \big|\big\langle B^* g_j^{(n)}, e_i\big\rangle\big|^2 \\
&= \sum_{i=1}^{u_m} \sum_{j\in H_n} \big|\big\langle g_j^{(n)}, Be_i\big\rangle\big|^2 \leqslant 2 \cdot \sum_{i=1}^{u_m} \|P_n B e_i\|^2 \\
&\leqslant 2 \cdot u_m^{\frac{p-2}{p}} \left(\sum_{i=1}^{u_m} \|P_n B e_i\|^p\right)^{2/p}
\end{aligned}$$

where $(e_i)_{i=1}^{u_m}$ stands for the unit vector basis of $\ell_2^{u_m}$ and P_n denotes the canonical projection of $\big(\bigoplus_{n'\in N} \ell_2^{u_{n'}}\big)_{\ell_p}$ onto $\ell_2^{u_n}$. The inequality in the second line follows from (5) and from $|H_n| \leqslant |H| = t = s_m + 1 \leqslant s_{n-1} + 1$. Finally, the last inequality uses Hölder's inequality. Raising the above to the power $p/2$ yields

$$|J_n| \leqslant |J_n|^{p/2} \leqslant 2^{p/2} \cdot u_m^{\frac{p-2}{2}} \cdot m^p \cdot \sum_{i=1}^{u_m} \|P_n B e_i\|^p\ .$$

We next sum over $n \in N$ to obtain

$$\sum_{n\in N} |J_n| \leqslant 2^{p/2} \cdot u_m^{\frac{p-2}{2}} \cdot m^p \cdot \sum_{i=1}^{u_m} \|Be_i\|^p \leqslant 2^{p/2} \cdot u_m^{p/2} \cdot m^p = s_m \ ,$$

using the assumption that $\|B\| \leqslant 1$. In particular, we have $|J| = \sum_{n\in N} |J_n| \leqslant s_m < |H|$. It follows from the choice of H that $\big\|B^* g_j^{(n)}\big\| < \frac{1}{m}$ for all $(n,j) \notin J$. From this we immediately obtain the claimed approximate factorization of DB by setting $P(x) = \big(\big\langle Bx, g_j^{(n)}\big\rangle_{j\in J_n}\big)_{n\in N}$ for $x \in \ell_2^{u_m}$, and setting R equal the diagonal operator whose n^{th} diagonal entry for $n \in N$ is the canonical embedding $\ell_\infty^{J_n} \to \ell_\infty^{v_n}$ sending $(x_j)_{j\in J_n}$ to $(y_j)_{j=1}^{v_n}$, where $y_j = x_j$ for $j \in J_n$ and $y_j = 0$ otherwise. Note that $\|P\| \leqslant 2$ follows from (5) and from $|J_n| \leqslant |J| \leqslant s_m < s_{n-1} + 1$. □

Before moving on to the proof of Theorem 1, we observe the following consequence of Lemma 3 that will be needed in the proof of Theorem 2.

Corollary 4. *For each infinite set $M \subset \mathbb{N}$, the operator $T_M \colon U \to V$ is finitely strictly singular.*

Proof. Fix an infinite set $M \subset \mathbb{N}$ and $\varepsilon > 0$. We need to find $d \in \mathbb{N}$ such that every subspace $E \subset U$ of dimension at least d contains a vector x with $\|T_M x\| < \varepsilon\|x\|$. Set $q = \max\{2, p\}$ and choose $m \in \mathbb{N}$ with $m^{-1} + 2m^{-1/q} < \varepsilon/2$. Next choose a very large $d \in \mathbb{N}$ so that every Banach space of dimension at least $d - (u_1 + \cdots + u_m)$ has a subspace 2-isomorphic to $\ell_2^{u_m}$. Such a d exists by Dvoretzky's theorem and depends only on m. We will show that this d works.

Let E be an arbitrary d-dimensional subspace of U. Denote by J the inclusion map from E into U and by Q the canonical projection of U onto $\big(\bigoplus_{i=1}^m \ell_2^{u_i}\big)_{\ell_p}$. Since the kernel of QJ has dimension at least $d - (u_1 + \cdots + u_m)$, it contains a subspace F that is 2-isomorphic to $\ell_2^{u_m}$. Let B be the restriction of J to F. We need to show that for some $x \in F$ we have $\|T_M Bx\| < \varepsilon\|x\|$. It is clearly sufficient to show that if $N = \{n \in M : n > m\}$ and B is *any* operator from $\ell_2^{u_m}$ to $\big(\bigoplus_{n\in N} \ell_2^{u_n}\big)_{\ell_p}$ with $\|B\| \leqslant 1$, then for some $x \in \ell_2^{u_m}$ we have $\|DBx\| < \frac{\varepsilon}{2}\|x\|$, where $D = \operatorname{diag}(T_n)_{n\in N}$. At this point we can apply Lemma 3 to obtain the approximately commuting diagram

$$\begin{tikzcd}
\ell_2^{u_m} \arrow[r, "B"] \arrow[d, "P"'] & \Big(\bigoplus\limits_{n\in N} \ell_2^{u_n}\Big)_{\ell_p} \arrow[r, "D"] & \Big(\bigoplus\limits_{n\in N} \ell_\infty^{v_n}\Big)_{c_0} \\
\Big(\bigoplus\limits_{n\in N} \ell_2^{J_n}\Big)_{\ell_p} \arrow[rr, "I"] & & \Big(\bigoplus\limits_{n\in N} \ell_\infty^{J_n}\Big)_{c_0} \arrow[u, "R"']
\end{tikzcd}$$

with $\|DB - RIP\| \leqslant \frac{1}{m}$, where I is the formal inclusion, $\|P\| \leqslant 2$ and $\|R\| \leqslant 1$.

Now, if x is a non-zero vector in $\ker P$, then $\|DBx\| \leqslant \frac{1}{m}\|x\| < \frac{\varepsilon}{2}\|x\|$, and we are done. So we may assume that the image of P has dimension at least u_m, and so at least m. A result of Milman [13] (see also [17, Lemma 3.4]) states that every m-dimensional subspace of c_0 contains a non-zero vector which has at least m co-ordinates of largest magnitude. Thus, there exists such a non-zero vector y in the image of P. It follows that $\|y\| \geqslant \|y\|_{\ell_q} \geqslant m^{1/q}\|Iy\|$. Choosing $x \in \ell_2^{u_m}$ with $Px = y$ we have

$$\|DBx\| \leqslant \frac{1}{m}\|x\| + \|RIPx\| \leqslant \frac{1}{m}\|x\| + 2m^{-1/q}\|x\| < \frac{\varepsilon}{2}\|x\| \ ,$$

as required. □

Proof of Theorem 1. As mentioned earlier, we only need to prove (i) as (ii) is trivial. We shall construct a functional Φ in $\mathcal{L}(U,V)^*$ that separates T_M from $\mathcal{J}^{T_N}(U,V)$.

For each $m \in \mathbb{N}$ and $S \in \mathcal{L}(U,V)$ define

$$\Phi_m(S) = \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\langle S g_i^{(m)}, e_i^{(m)} \big\rangle$$

where $(e_i^{(m)})_{i=1}^{v_m}$ denotes the unit vector basis of $\ell_1^{v_m}$ viewed in the obvious way as elements of $V^* \cong \big(\bigoplus_{n=1}^\infty \ell_1^{v_n}\big)_{\ell_1}$. Clearly, $\Phi_m \in \mathcal{L}(U,V)^*$ with $\|\Phi_m\| \leqslant 1$ for all $m \in \mathbb{N}$, and $\Phi_m(T_M) = 1$ for all $m \in M$. We are going to show that

$$\lim_{m\to\infty,\ m\in M\setminus N} \Phi_m(AT_NB) = 0 \qquad \text{for all } A \in \mathcal{L}(V),\ B \in \mathcal{L}(U)\ . \tag{6}$$

It then follows that if Φ is a weak*-accumulation point of $(\Phi_m)_{m\in M\setminus N}$ in $\mathcal{L}(U,V)^*$, then $\Phi(T_M) = 1$ and $\mathcal{J}^{T_N}(U,V) \subset \ker\Phi$. The proof of the theorem is then complete.

To see (6), let us fix $m \in M \setminus N$ and operators $A \in \mathcal{L}(V)$ and $B \in \mathcal{L}(U)$. We shall as we may assume that $\|A\| \leqslant 1$ and $\|B\| \leqslant 1$. We first observe that

$$\big|\Phi_m(AT_NB)\big| = \Big|\tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\langle T_N B g_i^{(m)}, A^* e_i^{(m)} \big\rangle\Big| \leqslant \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\|T_N B g_i^{(m)}\big\|\ . \tag{7}$$

We then split the right-hand side of (7) into two sums as follows. Let $n_0 = \min\{n \in N : n > m\}$. Let $B^{(1)}$ be the restriction of B to $\ell_2^{u_m}$ followed by the canonical projection of U onto $\big(\bigoplus_{n\in N, n<n_0} \ell_2^{u_n}\big)_{\ell_p}$. Similarly, let $B^{(2)}$ be the restriction of B to $\ell_2^{u_m}$ followed by the canonical projection of U onto $\big(\bigoplus_{n\in N, n\geqslant n_0} \ell_2^{u_n}\big)_{\ell_p}$. We also let $D^{(1)} = \operatorname{diag}(T_n)_{n\in N, n<n_0}$ and $D^{(2)} = \operatorname{diag}(T_n)_{n\in N, n\geqslant n_0}$. Continuing (7), we next obtain

$$\big|\Phi_m(AT_NB)\big| \leqslant \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\|D^{(1)} B^{(1)} g_i^{(m)}\big\| + \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\|D^{(2)} B^{(2)} g_i^{(m)}\big\|\ . \tag{8}$$

We shall now estimate the two terms above separately. Beginning with the first term, let us fix a $\frac{1}{3m}$-net F in the unit ball of $\big(\bigoplus_{n\in N, n<n_0} \ell_2^{u_n}\big)_{\ell_p}$. Note that the dimension of this space is $\sum_{n\in N, n<n_0} u_n \leqslant u_1+u_2+\dots+u_{m-1}$. By standard volume estimate, we can find such an F with $|F| \leqslant (6m+1)^{u_1+\dots+u_{m-1}}$ by taking it to be a maximal $\frac{1}{3m}$-separated subset of the ball. Now, set $H = \big\{i : \big\|B^{(1)} g_i^{(m)}\big\| > \frac{1}{m}\big\}$, and assume for a contradiction that $|H| > \frac{v_m}{m}$. Then by the pigeon-hole principle we find $x \in F$ and $J \subset H$ such that $|J| \geqslant \frac{v_m}{m|F|}$ and $\big\|B^{(1)} g_i^{(m)} - x\big\| \leqslant \frac{1}{3m}$ for all $i \in J$. It follows from (2) that $|J| \geqslant 19m^2$, and after replacing J with a smaller set, we may in fact assume that $|J| = 19m^2$. It then follows by (4) that

$$\Big\|\sum_{i\in J} g_i^{(m)}\Big\|^2 \leqslant 2|J|\ .$$

On the other hand, the choice of J yields

$$\Big\|\sum_{i\in J} B^{(1)} g_i^{(m)}\Big\| \geqslant |J| \cdot \frac{2}{3m} - |J| \cdot \frac{1}{3m} = \frac{|J|}{3m}\ .$$

The last two inequalities and the fact that $\|B^{(1)}\| \leqslant \|B\| \leqslant 1$ imply that $|J| \leqslant 18m^2$ which is a contradiction. This shows that $|H| \leqslant \frac{v_m}{m}$, and hence

$$\tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\|D^{(1)} B^{(1)} g_i^{(m)}\big\| \leqslant \frac{|H|}{v_m} + \frac{1}{m} \leqslant \frac{2}{m}\ . \tag{9}$$

To estimate the second term on the right-hand side of (8), we shall first apply Lemma 3 with B, D, N replaced by $B^{(2)}, D^{(2)}$ and $\{n \in N : n \geqslant n_0\}$, respectively, to obtain $J_n \subset \{1, \dots, v_n\}$, $n \in N, n \geqslant n_0$, with $\sum |J_n| \leqslant s_m$, and an almost commuting diagram

$$\begin{array}{ccccc} \ell_2^{u_m} & \xrightarrow{B^{(2)}} & \Big(\bigoplus_{n\in N, n\geqslant n_0} \ell_2^{u_n}\Big)_{\ell_p} & \xrightarrow{D^{(2)}} & \Big(\bigoplus_{n\in N, n\geqslant n_0} \ell_\infty^{v_n}\Big)_{c_0} \\ \Big\downarrow P & & & & \Big\uparrow R \\ \Big(\bigoplus_{n\in N, n\geqslant n_0} \ell_2^{J_n}\Big)_{\ell_p} & & \xrightarrow{\qquad I \qquad} & & \Big(\bigoplus_{n\in N, n\geqslant n_0} \ell_\infty^{J_n}\Big)_{c_0} \end{array}$$

with $\|D^{(2)}B^{(2)} - RIP\| \leqslant \frac{1}{m}$, $\|P\| \leqslant 2$ and $\|R\| \leqslant 1$. From this we obtain

$$\frac{1}{v_m}\sum_{i=1}^{v_m} \big\|D^{(2)}B^{(2)}g_i^{(m)}\big\| \leqslant \frac{1}{v_m}\sum_{i=1}^{v_m} \big\|IPg_i^{(m)}\big\| + \frac{1}{m}\ . \tag{10}$$

Let us set $H = \big\{i : \big\|IPg_i^{(m)}\big\| > \frac{1}{m}\big\}$, and assume for a contradiction that $|H| > \frac{v_m}{m}$. For each $i \in H$, there exist $n \in N, n \geqslant n_0$, and $j \in J_n$ with $\big|\big[Pg_i^{(m)}\big]_{n,j}\big| \geqslant \frac{1}{m}$, where $[y]_{n,j}$ denotes the (n, j)-coordinate of an element y of $\big(\bigoplus_{n\in N, n\geqslant n_0} \ell_2^{J_n}\big)_{\ell_p}$. Hence by pigeon-hole principle, there exist $n \in N, n \geqslant n_0$, $j \in J_n$ and $J \subset H$ with $|J| \geqslant \frac{|H|}{\sum |J_n|}$ such that $\big|\big[Pg_i^{(m)}\big]_{n,j}\big| \geqslant \frac{1}{m}$ for all $i \in J$. It follows from (3) that $|J| > \frac{v_m}{m\cdot s_m} \geqslant 9m^2$. After replacing J by a smaller set, we may in fact assume that $|J| = 9m^2$. Hence by (4) we have

$$\Big\|\sum_{i\in J} \varepsilon_i g_i^{(m)}\Big\|^2 \leqslant 2|J|$$

where ε_i is the sign of $\big[Pg_i^{(m)}\big]_{n,j}$ for each $i \in J$. On the other hand, by the choice of J and since $\|P\| \leqslant 2$, we get

$$2\Big\|\sum_{i\in J} \varepsilon_i g_i^{(m)}\Big\| \geqslant \Big\|\sum_{i\in J} \varepsilon_i P g_i^{(m)}\Big\| \geqslant \sum_{i\in J} \varepsilon_i \big[Pg_i^{(m)}\big]_{n,j} \geqslant |J| \cdot \frac{1}{m}\ .$$

The last two inequalities yield $|J| \leqslant 8m^2$ which is a contradiction. Hence $|H| \leqslant \frac{v_m}{m}$ and

$$\frac{1}{v_m}\sum_{i=1}^{v_m} \big\|IPg_i^{(m)}\big\| \leqslant \frac{2|H|}{v_m} + \frac{1}{m} \leqslant \frac{3}{m}\ .$$

This inequality together with (10) yields the upper bound

$$\frac{1}{v_m}\sum_{i=1}^{v_m} \big\|D^{(2)}B^{(2)}g_i^{(m)}\big\| \leqslant \frac{4}{m}\ .$$

Combining this with (9) and (8), we finally obtain

$$\big|\Phi_m(AT_N B)\big| \leqslant \frac{6}{m}$$

for all $m \in M \setminus N$. This completes the proof of (6). $\square$

We conclude this section with a proof of Theorem 2 which begins with a lemma.

Lemma 5. *For each $m \in \mathbb{N}$, the formal identity $I\colon \ell_2^m \to \ell_\infty^m$ factors through T_n for all sufficiently large $n \in \mathbb{N}$ via operators that are uniformly bounded.*

Proof. Let us fix $m \in \mathbb{N}$. Assume that $n \in \mathbb{N}$ satisfies

$$m \cdot \sqrt{\frac{m(m-1)}{M-1}} < \frac{1}{2}\ , \quad \text{where } M = (s_{n-1}+1) \vee 19n^2\ .$$

Then by (5) we have

$$\sum_{i,j=1}^{M} \big|\langle g_i^{(n)}, g_j^{(n)}\rangle\big|^2 \leqslant 2M\ ,$$

and hence

$$\sum_{\substack{i,j=1\\ i\neq j}}^{M} \big|\langle g_i^{(n)}, g_j^{(n)}\rangle\big|^2 \leqslant M\ .$$

Next, let S be a random (with respect to the uniform distribution) subset of $\{1,2,\dots,M\}$ of size m. Then for $i \neq j$ we have $\mathbb{P}(i,j\in S) = \binom{M-2}{m-2}/\binom{M}{m}$, and thus

$$\mathbb{E}\sum_{\substack{i,j\in S\\ i\neq j}} \big|\langle g_i^{(n)}, g_j^{(n)}\rangle\big|^2 \leqslant M \cdot \binom{M-2}{m-2}\cdot\binom{M}{m}^{-1} = \frac{m(m-1)}{M-1}\ .$$

Thus for some set S the above inequality holds. After relabelling, we may assume that $S = \{1,2,\dots,m\}$. Let us now define $B\colon \ell_2^m \to \ell_2^{u_n}$ by $Be_i = g_i^{(n)}$ for $1 \leqslant i \leqslant m$, where (e_i) is the standard basis of $\mathbb{R}^m$. By (4) we have $\|B\| \leqslant 2$. Let $P\colon \ell_\infty^{v_n} \to \ell_\infty^m$ be the projection onto the first m co-ordinates. Finally, for each $1 \leqslant i \leqslant m$ set $u_i = \big(\langle g_i^{(n)}, g_j^{(n)}\rangle\big)_{j=1}^m$. Observe that if $\big\|\sum_{i=1}^m \lambda_i e_i\big\|_{\ell_\infty^m} = 1$, then

$$\Big\|\sum_{i=1}^m \lambda_i(e_i-u_i)\Big\|_{\ell_\infty^m} \leqslant \sum_{i=1}^m \|e_i-u_i\|_{\ell_\infty^m} = \sum_{i=1}^m \max_{\substack{1\leqslant j\leqslant m\\ j\neq i}} \big|\langle g_i^{(n)}, g_j^{(n)}\rangle\big|$$

$$\leqslant m\cdot\sqrt{\frac{m(m-1)}{M-1}} < \frac{1}{2}\ .$$

It follows that there is a well-defined linear map $A\colon \ell_\infty^m \to \ell_\infty^m$ with $Au_i = e_i$ for all i. Moreover, the above calculation shows that $\|A\| \leqslant 2$. We complete the proof by observing the factorization $I = APT_nB$. $\square$

Proof of Theorem 2. It follows from Theorem 1 that if $[M] < [N]$ for infinite subsets M and N of $\mathbb{N}$, then $\mathcal{J}^{T_M} \subsetneq \mathcal{J}^{T_N}$. Next, recall that $I_{U,V}$ can be viewed as the diagonal operator whose m^{th} diagonal entry is the formal identity $I\colon \ell_2^{u_m} \to \ell_\infty^{u_m}$. It then follows from Lemma 5 that $I_{U,V}$ factors through T_M for every infinite subset M of $\mathbb{N}$. Thus, $\varphi([M]) < \varphi([N])$ whenever M is finite and N is infinite. This completes the proof that φ is strictly monotonic.

To show that φ preserves the join operation, it is of course sufficient to consider incomparable elements of $\mathfrak{B}$. In particular, it is enough to show that $\mathcal{J}^{T_M} \vee \mathcal{J}^{T_N} = \mathcal{J}^{T_{M\cup N}}$ for infinite sets M and N. The left-to-right inclusion is clear, since both T_M and T_N trivially factor through $T_{M\cup N}$. Conversely, $T_{M\setminus N}$ factors through T_M, and so $T_{M\cup N} = T_{M\setminus N} + T_N \in \mathcal{J}^{T_M} + \mathcal{J}^{T_N} \subset \mathcal{J}^{T_M} \vee \mathcal{J}^{T_N}$. This shows the reverse inclusion, as required.

It follows from the properties established so far that φ is injective. Finally, Corollary 4 completes the proof of the theorem. $\square$

Remark. Some comments about the choice of $\varphi([M])$, M finite, are in order. We have $V = \mathrm{c}_0$ and, when $1 < p < \infty$, we have $U \sim \ell_p$. However, the formal inclusion I_{ℓ_p,c_0} is different from $I_{U,V}$. Since I_{ℓ_p,c_0} factors through every non-compact operator in $\mathcal{L}(\ell_p,\mathrm{c}_0)$, we have $\mathcal{J}^{I_{\ell_p,\mathrm{c}_0}} \subset \mathcal{J}^{I_{U,V}}$. When $2 \leqslant p < \infty$ then $I_{U,V}$

factors through I_{ℓ_p,c_0} via the formal inclusion from $U = \big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{\ell_p}$ to $\ell_p = \big(\bigoplus_{n=1}^\infty \ell_p^{u_n}\big)_{\ell_p}$. It follows that in this case the ideals $\mathcal{J}^{I_{\ell_p,c_0}}$ and $\mathcal{J}^{I_{U,V}}$ are identical. However, for $1 < p < 2$, we have $\mathcal{J}^{I_{\ell_p,c_0}} \subsetneq \mathcal{J}^{I_{U,V}}$, and hence our choice of $\varphi([\emptyset])$ yields a stronger statement. We do not know whether $\mathcal{J}^{I_{U,V}} = \bigcap_{M\subset\mathbb{N},\ |M|=\infty} \mathcal{J}^{T_M}$, or indeed whether φ preserves the meet operation, *i.e.*, whether φ is a lattice isomorphism. To see that $I_{U,V} \notin \mathcal{J}^{I_{\ell_p,c_0}}$ for $1 < p < 2$, define for each $m \in \mathbb{N}$ a functional Ψ_m on $\mathcal{L}(U,V)$ by setting

$$\Psi_m(S) = \tfrac{1}{u_m}\sum_{i=1}^{u_m} \big\langle Se_i^{(m)}, e_i^{(m)*}\big\rangle \qquad (S \in \mathcal{L}(U,V))$$

where $(e_i^{(m)})_{i=1}^{u_m}$ is the unit vector basis of $\ell_2^{u_m}$ in $U = \big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{\ell_p}$, and $(e_i^{(m)*})_{i=1}^{u_m}$ is the unit vector basis of $\ell_1^{u_m}$ in $V^* = \big(\bigoplus_{n=1}^\infty \ell_1^{u_n}\big)_{\ell_1}$. It is easy to verify that $\Psi_m \in \mathcal{L}(U,V)^*$ with $\|\Psi_m\| \leqslant 1$ and that $\Psi_m(I_{U,V}) = 1$ for all $m \in \mathbb{N}$. On the other hand, given $A \in \mathcal{L}(c_0,V)$ and $B \in \mathcal{L}(U,\ell_p)$ with $\|A\| \leqslant 1$ and $\|B\| \leqslant 1$, we have

$$\big|\Psi_m(AI_{\ell_p,c_0}B)\big| \leqslant \tfrac{1}{u_m}\sum_{i=1}^{u_m} \big\|I_{\ell_p,c_0}B_m e_i^{(m)}\big\|$$

where $B_m\colon \ell_2^{u_m} \to \ell_p$ is the restriction of B to $\ell_2^{u_m}$. An application of [20, Lemma 4] now gives $\Psi_m(AI_{\ell_p,c_0}B) \to 0$ as $m \to \infty$. Thus, letting Ψ be a weak*-accumulation point of (Ψ_m) in $\mathcal{L}(U,V)^*$, we obtain $\Psi(I_{U,V}) = 1$ and $\mathcal{J}^{I_{\ell_p,c_0}} \subset \ker\Psi$.

3. Closed ideals in $\mathcal{L}(\ell_p,\ell_\infty)$ and $\mathcal{L}(\ell_1,\ell_p)$

In this section we prove results analogous to Theorems 1 and 2 for $\mathcal{L}(\ell_p,\ell_\infty)$ and $\mathcal{L}(\ell_1,\ell_p)$ in the range $1 < p < \infty$. In both cases a fairly straightforward modification of the proof of Theorem 1 will suffice. We shall also present a duality argument that establishes some, but not all, of the results for $\mathcal{L}(\ell_1,\ell_p)$ directly from the results for $\mathcal{L}(\ell_p,\ell_\infty)$. Such a duality argument was used by Sirotkin and Wallis in [19], and we shall say more about that in the next section.

Fix $1 \leqslant p < \infty$ and set $U = \big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{\ell_p}$, $V = \big(\bigoplus_{n=1}^\infty \ell_\infty^{v_n}\big)_{c_0}$ and $W = \big(\bigoplus_{n=1}^\infty \ell_\infty^{v_n}\big)_{\ell_\infty}$, where (u_n) and (v_n) are sequences satisfying (1)–(5) on page 4. Similar to the previous section, we will prove our first result for $\mathcal{L}(U,W)$, which of course is the same as working with $\mathcal{L}(\ell_p,\ell_\infty)$ when $1 < p < \infty$. We shall also use the operators $T_M\colon U \to V$ defined in the previous section. Finally, we will denote by J the formal inclusion $I_{V,W}$ of V into W.

Theorem 6. *Let M and N be infinite subsets of $\mathbb{N}$.*

(i) If $M \setminus N$ is infinite, then $J \circ T_M \notin \mathcal{J}^{T_N}(U,W)$.

(ii) If $N \setminus M$ is finite, then $\mathcal{J}^{T_N}(U,W) \subset \mathcal{J}^{T_M}(U,W)$.

It follows that the map $\varphi\colon \mathfrak{B} \to \mathfrak{H}(U,W)$ given by

$$\varphi([M]) = \begin{cases} \mathcal{J}^{T_M}(U,W) & \text{if } M \text{ is infinite, and}\\ \mathcal{J}^{I_{U,V}}(U,W) & \text{if } M \text{ is finite,}\end{cases}$$

is well defined, injective, monotone and preserves the join operation. In particular, $\mathcal{L}(U,W)$ has continuum many closed ideals between $\mathcal{J}^{I_{U,V}}$ and $\mathcal{FS}$.

Proof. We follow the proof of Theorem 1. For $m \in \mathbb{N}$ and $S \in \mathcal{L}(U,W)$ we define

$$\Psi_m(S) = \tfrac{1}{v_m}\sum_{i=1}^{v_m} \big\langle Sg_i^{(m)}, f_i^{(m)}\big\rangle$$

where $f_i^{(m)} \in W^*$ are chosen so that $\|f_i^{(m)}\| = 1$ and $J^* f_i^{(m)} = e_i^{(m)}$ for each i. (In other words, $f_i^{(m)}$ is a Hahn–Banach extension of $e_i^{(m)}$ to W.) We then have $\Psi_m \in \mathcal{L}(U,W)^*$ with $\|\Psi_m\| = 1$ for all $m \in \mathbb{N}$, and $\Psi_m(J \circ T_M) = \Phi_m(T_M) = 1$ for all $m \in M$. The proof of (i) will then be complete if we show that

$$\lim_{m\to\infty,\ m\in M\setminus N} \Psi_m(AT_N B) = 0 \qquad \text{for all } A \in \mathcal{L}(V,W),\ B \in \mathcal{L}(U)\ . \tag{11}$$

To see this, fix operators $A \in \mathcal{L}(V,W)$ and $B \in \mathcal{L}(U)$, and assume without loss of generality that $\|A\| \leqslant 1$ and $\|B\| \leqslant 1$. We then have

$$\big|\Psi_m(AT_N B)\big| \leqslant \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\|T_N B g_i^{(m)}\big\|\ .$$

This is the analogue of inequality (7) from Theorem 1. Since the right-hand sides of these inequalities are identical, from here onwards we can simply follow the proof of Theorem 1. This then completes the proof of part (i).

Part (ii) is again trivial, whereas the rest of the theorem is proved exactly as Theorem 2. □

Remark. The same argument works for any Banach space W that contains a copy of c_0 with J being an isomorphic embedding of V into W. We could also replace the ideals $\mathcal{J}^{T_M}$ with $\mathcal{J}^{J\circ T_M}$, and $\mathcal{J}^{I_{U,V}}$ with $\mathcal{J}^{I_{U,W}}$.

We now turn to the duality argument. Given Banach spaces X and Y, for a subset $\mathcal{J}$ of $\mathcal{L}(Y^*,X^*)$ we let $\mathcal{J}_* = \{T \in \mathcal{L}(X,Y) : T^* \in \mathcal{J}\}$. The following result is straightforward to verify.

Proposition 7. *Let X and Y be Banach spaces. The map $\mathcal{J} \mapsto \mathcal{J}_*$ is a monotone map from $\mathfrak{H}(Y^*,X^*)$ to $\mathfrak{H}(X,Y)$ that preserves the meet operation. Moreover, if $\mathcal{I} \setminus \mathcal{J}$ contains a dual operator for some closed ideals $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{L}(Y^*,X^*)$, then $\mathcal{I}_* \not\subset \mathcal{J}_*$.*

Note that U is the dual space of U_*, where $U_* = \big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{c_0}$ when $p=1$, and $U_* = \big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{\ell_q}$ when $1 < p < \infty$ and q is the conjugate index to p. In the next result we shall write W_* for $\big(\bigoplus_{n=1}^\infty \ell_1^{v_n}\big)_{\ell_1} \cong \ell_1$, being the predual of W. Of course, we have $W_* \cong V^*$, and the formal inclusion $J = I_{V,W}$ is nothing else but the canonical embedding of V into its bidual $V^{**} \cong W$.

Theorem 8. *Let φ be the map defined in Theorem 6. Define $\psi\colon \mathfrak{B} \to \mathfrak{H}(W_*,U_*)$ by $\psi([M]) = \varphi([M])_*$ for every $M \subset \mathbb{N}$. Then ψ is injective and monotone. In particular, $\mathcal{L}(W_*,U_*)$ contains continuum many closed ideals.*

Proof. Fix an infinite set $M \subset \mathbb{N}$. Note that $J \circ T_M$ is the diagonal operator $\operatorname{diag}(S_n)_{n\in\mathbb{N}}\colon U \to W$ where $S_n = T_n$ when $n \in M$, and $S_n = 0$ otherwise. Define S_M to be the diagonal operator $\operatorname{diag}(S_n^*)_{n\in\mathbb{N}}\colon W_* \to U_*$. It is then clear that $S_M^* = J \circ T_M$. In particular, $J \circ T_M$ is a dual operator. The theorem now follows from Theorem 6 and Proposition 7. □

We conclude this section by proving a slightly stronger result for $\mathcal{L}(W_*,U_*)$ (and hence for $\mathcal{L}(\ell_1,\ell_q)$ for $1 < q < \infty$) by following the proof of Theorem 1. In this context it will be more appropriate to write V^* in place of W_*. We shall use the notation established in the proof of Theorem 8 and denote by $S_M\colon V^* \to U_*$ the diagonal operator with $S_M^* = J \circ T_M$ for an infinite subset M of $\mathbb{N}$.

Theorem 9. *Let M and N be infinite subsets of* $\mathbb{N}$.
(i) If $M \setminus N$ is infinite, then $S_M \notin \mathcal{J}^{S_N}(V^*,U_*)$.
(ii) If $N \setminus M$ is finite, then $\mathcal{J}^{S_N}(V^*,U_*) \subset \mathcal{J}^{S_M}(V^*,U_*)$.

It follows that the map $\psi\colon \mathfrak{B} \to \mathfrak{H}(V^*, U_*)$ *given by*

$$\psi([M]) = \begin{cases} \mathcal{J}^{S_M}(V^*, U_*) & \text{if } M \text{ is infinite, and} \\ \mathcal{J}^{I_{V^*,U_*}}(V^*, U_*) & \text{if } M \text{ is finite,} \end{cases}$$

is well defined, injective, monotone and preserves the join operation. In particular, $\mathcal{L}(V^*, U_*)$ *has continuum many closed ideals between* $\mathcal{J}^{I_{V^*,U_*}}$ *and* $\mathcal{FS}$.

Proof. For $m \in \mathbb{N}$ consider $\Phi_m \in \mathcal{L}(U, V)^*$ as defined at the start of the proof of Theorem 1. We also let

$$\Psi_m(S) = \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\langle S e_i^{(m)}, g_i^{(m)} \big\rangle \qquad \text{for } S \in \mathcal{L}(V^*, U_*)\ ,$$

where $(e_i^{(m)})_{i=1}^{v_m}$ is the unit vector basis of $\ell_1^{v_m}$ as in the proof of Theorem 1. It is clear that $\Psi_m \in \mathcal{L}(V^*, U_*)^*$ with $\|\Psi_m\| = 1$. Since

$$\big\langle S_M e_i^{(m)}, g_i^{(m)} \big\rangle = \big\langle e_i^{(m)}, S_M^* g_i^{(m)} \big\rangle = \big\langle e_i^{(m)}, J \circ T_M g_i^{(m)} \big\rangle = \big\langle T_M g_i^{(m)}, e_i^{(m)} \big\rangle\ ,$$

it follows that $\Psi_m(S_M) = \Phi_m(T_M) = 1$ for all $m \in M$. As in the proof of Theorem 1 we will now show that

$$\lim_{m\to\infty,\ m\in M\setminus N} \Psi_m(A S_N B) = 0 \qquad \text{for all } A \in \mathcal{L}(U_*),\ B \in \mathcal{L}(V^*)\ .$$

This will then complete the proof of part (i) of Theorem 9. To see the above, we follow closely the proof of Theorem 1. We fix $m \in M \setminus N$ and assume that $\|A\| \leqslant 1$ and $\|B\| \leqslant 1$. We then observe the following analogue of (7).

$$\big|\Psi_m(A S_N B)\big| \leqslant \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\| S_N^* A^* g_i^{(m)} \big\| = \tfrac{1}{v_m} \sum_{i=1}^{v_m} \big\| T_N A^* g_i^{(m)} \big\|\ .$$

Here A^* is an element of $\mathcal{L}(U)$, and hence this upper bound on $|\Psi_m(A S_N B)|$ is of the same form as the right-hand side of (7). Thus, the rest of the proof of Theorem 1 can now be used to complete the proof of (i).

The rest of Theorem 9 is proved exactly as Theorem 2 using the following observations. Firstly, dualizing Lemma 5 shows that I_{V^*,U_*} factors through S_N for every infinite set $N \subset \mathbb{N}$. Secondly, if $1 < q < \infty$, then $U_* \sim \ell_q$, and if $q = \infty$ (or indeed, if $2 \leqslant q \leqslant \infty$), then S_N factors through $\ell_2 \cong \big(\bigoplus_{n=1}^\infty \ell_2^{u_n}\big)_{\ell_2}$ via S_N viewed as a map to $V^* \to \ell_2$ followed by the formal inclusion of ℓ_2 into U_*. In either case, we deduce that S_N is finitely strictly singular from Theorem 10 below. ☐

Theorem 10. *Let* $1 < q < \infty$. *Then every operator* $\ell_1 \to \ell_q$ *is finitely strictly singular. More generally, given Banach spaces* X *and* Y, *if* X *does not contain uniformly complemented and uniformly isomorphic copies of* ℓ_2^n, $n \in \mathbb{N}$, *and* Y *has non-trivial type, then* $\mathcal{L}(X, Y) = \mathcal{FS}(X, Y)$.

Proof. We begin by recalling some results from the local theory of Banach spaces. A Banach space Z is *locally* π*-euclidean* if there is a constant λ and a function $k \mapsto N(k)$ such that every subspace of Z of dimension $N(k)$ contains a further subspace 2-isomorphic to ℓ_2^k and λ-complemented in Z. This notion was introduced by Pełczyński and Rosenthal in [14].

Having non-trival type is equivalent to not containing ℓ_1^n uniformly (a result of Maurey and Pisier [12]), which in turn is equivalent to K-convexity (due to Pisier [16, Theorem 2.1]). Figiel and Tomczak-Jaegermann proved that a K-convex Banach space is locally π-euclidean [7]. Thus in particular the space Y in our theorem is locally π-euclidean.

Let us now assume that $T \in \mathcal{L}(X, Y)$ is not finitely strictly singular. Then there is an $\varepsilon > 0$ and a sequence of finite-dimensional subspaces E_n of X such that $\|Tx\| \geqslant \varepsilon\|x\|$ for all $n \in \mathbb{N}$ and for all $x \in E_n$, and such that $\dim E_n \to \infty$. Since Y is locally π-euclidean, we may assume after passing to a subsequence and replacing the E_n by suitable subspaces that $T(E_n)$ is 2-isomorphic to ℓ_2^n and λ-complemented in Y for some constant λ. It follows that E_n is C-isomorphic to ℓ_2^n and C-complemented in X, where $C = \max\{2, \lambda\} \cdot \|T\|/\varepsilon$. This contradicts the assumption on X. □

4. Further results, remarks and open problems

The main result of this section, Theorem 12 below, is concerned with the structure of "large" ideals in $\mathcal{L}(\ell_p, \mathrm{c}_0)$ for $1 < p < \infty$. By "large" ideal we mean ideals containing operators that are not finitely strictly singular. To place Theorem 12 into context, we shall begin with a brief sketch of the proof of Sirotkin and Wallis [19] that $\mathcal{L}(\ell_1, \ell_q)$, $1 < q \leqslant \infty$, $\mathcal{L}(\ell_1, \mathrm{c}_0)$, and $\mathcal{L}(\ell_p, \ell_\infty)$, $1 \leqslant p < \infty$, contain uncountable many closed ideals. We shall not define all the terms used and refer the reader to [19] and to the work of Beanland and Freeman [2] whose results play an important rôle.

Fix $1 \leqslant p < \infty$ and let q be the conjugate index of p. For an ordinal α, $0 < \alpha < \omega_1$, let T_α be the p-convexified Tsirelson space of order α. Recall that T_α is a reflexive sequence space not containing any copy of ℓ_r, $1 \leqslant r < \infty$, or c_0. Its unit vector basis (t_i) is dominated by the unit vector basis (e_i) of ℓ_p, and moreover, $(t_i)_{i\in F}$ uniformly dominates $(e_i)_{i\in F}$ for every Schreier-α set F.

Let us denote by J_α the formal embedding of ℓ_p into T_α, which is the dual operator of the formal embedding I_α of T_α^* into ℓ_q when $1 < p < \infty$, and into c_0 when $p = 1$. Fix a quotient map $Q_\alpha\colon \ell_1 \to T_\alpha^*$ and set $S_\alpha = I_\alpha \circ Q_\alpha$. Note that $S_\alpha^* = Q_\alpha^* \circ J_\alpha$. Since Q_α^* is an isomorphic embedding, it follows that S_α^* is an isomorphism on $(e_i)_{i\in F}$ for every Schreier-α set F. We shall now consider the closed ideals $\mathcal{J}_\alpha = \mathcal{J}^{S_\alpha^*}(\ell_p, \ell_\infty)$.

Beanland and Freeman [2] introduced an ordinal-index which for a strictly singular operator T from ℓ_p into an arbitrary Banach space quantifies the failure of T to preserve a copy of ℓ_p. Using methods of Descriptive Set Theory, they proved that for such an operator T, there is a countable ordinal α such that T fails to preserve ℓ_p specifically on Schreier-α sets. This means that for every bounded sequence (x_n) in ℓ_p and for any $\varepsilon > 0$ there is a Schreier-α set F and scalars $(a_i)_{i\in F}$ such that

$$\Big\| \sum_{i\in F} a_i T x_i \Big\| < \varepsilon \Big\| \sum_{i\in F} a_i x_i \Big\| \ .$$

Let us temporarily say that such an operator T is *α-singular*. Although operators with this property do not form an ideal, every operator in the closed ideal generated by them is $\alpha\omega$-singular.

Let us now return to the closed ideals $\mathcal{J}_\alpha$ of $\mathcal{L}(\ell_p, \ell_\infty)$. Since T_α contains no copy of ℓ_p, it follows from the result of Beanland–Freeman that S_α^* is β-singular for some countable ordinal β. Set $\gamma = \beta\omega$. Since S_γ^* is an isomorphism on $(e_i)_{i\in F}$ for every Schreier-γ set F, it follows that S_γ^* is not γ-singular, and hence $S_\gamma^* \notin \mathcal{J}_\alpha$. This shows that there is an ω_1-sequence of countable ordinals α for which the closed ideals $\mathcal{J}_\alpha$ are distinct, and moreover they are distinguished by dual operators. Hence by Proposition 7, the corresponding closed ideals $\big(\mathcal{J}_\alpha\big)_*$ of $\mathcal{L}(\ell_1, \ell_q)$ when $1 < p < \infty$, and of $\mathcal{L}(\ell_1, \mathrm{c}_0)$ when $p = 1$, are also distinct. We remark that the closed ideals constructed by Sirotkin and Wallis are slightly different, but the proof here is essentially a streamlined version of their proof.

Let us emphasize the key aspects of the above argument. Firstly, all operators in $\mathcal{J}_\alpha$ are γ-singular for a sufficiently large γ. On the other hand, J_γ is not γ-singular, as it is isomorphic on $(e_i)_{i\in F}$ for every Schreier-γ set F. Since Q_γ^* is an isomorphic embedding, this property is inherited by S_γ^* which therefore does not belong to $\mathcal{J}_\alpha$. Since no operator in $\mathcal{L}(\ell_p, \mathrm{c}_0)$ can be uniformly isomorphic on $(e_i)_{i\in F}$ even for the class of Schreier-1 sets F, this argument cannot be used to construct infinitely many closed ideals in $\mathcal{L}(\ell_p, \mathrm{c}_0)$ for $1 < p < \infty$. Moreover, one cannot use duality via Proposition 7 either, as the S_α are not dual operators. Recall that Wallis [22] observed that the method of [20] extends to yield continuum many closed ideals in $\mathcal{L}(\ell_p, \mathrm{c}_0)$ for $1 < p < 2$, but this approach does not seem to go beyond this range of values for p either. The main result of this section shows that there is another obstruction. Theorem 12 below shows that at least for $p = 2$ there are no proper closed ideals in $\mathcal{L}(\ell_p, \mathrm{c}_0)$ containing an operator that is not finitely strictly singular. In other words, there are no proper large ideals. We begin with a simple lemma.

Lemma 11. *Let E be a finite-dimensional Banach space and let $J\colon E \to \ell_\infty^m$ be an isomorphic embedding with $\|x\| \leqslant \|Jx\|$ for all $x \in E$. Then every operator $T\colon E \to \ell_\infty^n$ factors through J: more precisely, there is an operator $A\colon \ell_\infty^m \to \ell_\infty^n$ such that $T = A \circ J$ and $\|A\| \leqslant \|T\|$.*

Proof. We may assume that $\|T\| \leqslant 1$. Then there are functionals $f_i, g_j \in E^*$ with $\|f_i\| \leqslant \|J\|$ and $\|g_j\| \leqslant 1$ such that

$$Jx = \big(\langle x, f_i\rangle\big)_{i=1}^m \quad \text{and} \quad Tx = \big(\langle x, g_j\rangle\big)_{j=1}^n \quad \text{for all } x \in E\ .$$

Since $\|x\| \leqslant \|Jx\|$ for all $x \in E$, it follows from the geometric Hahn–Banach theorem that the unit ball of E^* is contained in $\operatorname{conv}\{f_i : 1 \leqslant i \leqslant m\}$. In particular, each g_j can be expressed as a convex combination $g_j = \sum_{i=1}^m t_{i,j} f_i$. Define $A\colon \ell_\infty^m \to \ell_\infty^n$ by

$$A\big((y_i)_{i=1}^m\big) = \Big(\sum_{i=1}^m t_{i,j} y_i\Big)_{j=1}^n\ .$$

It is straightforward to verify that $\|A\| \leqslant 1$ and $T = A \circ J$. □

From now on we fix $1 < p < \infty$ and two diagonal operators K and L defined as follows. For each $n \in \mathbb{N}$ fix embeddings $K_n\colon \ell_2^n \to \ell_\infty^{k_n}$ and $L_n\colon \ell_p^n \to \ell_\infty^{m_n}$ satisfying $\|x\| \leqslant \|K_n x\| \leqslant 2\|x\|$ for all $x \in \ell_2^n$, and $\|x\| \leqslant \|L_n x\| \leqslant 2\|x\|$ for all $x \in \ell_p^n$, and then set $K = \operatorname{diag}(K_n)_{n\in\mathbb{N}}\colon \big(\bigoplus_{n=1}^\infty \ell_2^n\big)_{\ell_p} \to \big(\bigoplus_{n=1}^\infty \ell_\infty^{k_n}\big)_{\mathrm{c}_0}$ and $L = \operatorname{diag}(L_n)_{n\in\mathbb{N}}\colon \big(\bigoplus_{n=1}^\infty \ell_p^n\big)_{\ell_p} \to \big(\bigoplus_{n=1}^\infty \ell_\infty^{m_n}\big)_{\mathrm{c}_0}$. Observe that by Lemma 11, the closed ideals $\mathcal{J}^K(X,Y)$ and $\mathcal{J}^L(X,Y)$ are independent of the particular choice of embeddings K_n and L_n for any pair (X,Y) of Banach spaces.

Theorem 12. *Let $1 < p < \infty$ and let K, L be defined as above. Then*
(i) $\mathcal{J}^L(\ell_p, \mathrm{c}_0) = \mathcal{L}(\ell_p, \mathrm{c}_0)$, and
(ii) if $T \in \mathcal{L}(\ell_p, \mathrm{c}_0) \setminus \mathcal{FS}(\ell_p, \mathrm{c}_0)$, then $\mathcal{J}^K \subset \mathcal{J}^T$.
It follows that if $\mathcal{J}$ is a non-trivial, proper closed ideal of $\mathcal{L}(\ell_p, \mathrm{c}_0)$, then either $\mathcal{J} = \mathcal{K}$ or $\mathcal{J}^{I_{\ell_p,\mathrm{c}_0}} \subset \mathcal{J} \subset \mathcal{FS}$ or $\mathcal{J}^K \subset \mathcal{J}$. In particular, for $p = 2$ we have the following: if $\mathcal{J}$ is a non-trivial, proper closed ideal of $\mathcal{L}(\ell_2, \mathrm{c}_0)$, then either $\mathcal{J} = \mathcal{K}$ or $\mathcal{J}^{I_{\ell_2,\mathrm{c}_0}} \subset \mathcal{J} \subset \mathcal{FS}$.

Proof. Let $T \in \mathcal{L}(\ell_p, \mathrm{c}_0)$. It is well known that T can be arbitrarily well approximated by the sum of two block-diagonal operators. So to show (i), we may as well assume that $T = \operatorname{diag}(T_n)\colon \big(\bigoplus_{n=1}^\infty \ell_p^n\big)_{\ell_p} \to \big(\bigoplus_{n=1}^\infty \ell_\infty^{r_n}\big)_{\mathrm{c}_0}$ for some integers r_n and maps $T_n\colon \ell_p^n \to \ell_\infty^{r_n}$. By Lemma 11, there are operators $A_n\colon \ell_\infty^{m_n} \to \ell_\infty^{r_n}$ such that $\|A_n\| \leqslant \|T_n\| \leqslant \|T\|$ and $T_n = A_n L_n$. Setting $A = \operatorname{diag}(A_n)_{n\in\mathbb{N}}$, we obtain $T = A \circ L \in \mathcal{J}^L$, as required.

Let us next consider $T \in \mathcal{L}(\ell_p, \mathrm{c}_0)$ that is not finitely strictly singular. By standard basis arguments, after perturbing T by a compact (or even a nuclear) operator, we may assume that the columns of T with respect to the canonical bases of ℓ_p and c_0 are finite, and hence T maps finitely supported vectors to finitely supported vectors. Since $T \notin \mathcal{FS}(\ell_p, c_0)$, it follows that there exists $\varepsilon > 0$ and finite-dimensional subspaces E_n of ℓ_p with $\dim E_n \to \infty$ such that $\|Tx\| \geqslant \varepsilon\|x\|$ for all $x \in E_n$ and for all $n \in \mathbb{N}$. After perturbing the E_n and replacing them by suitable subspaces, we may assume that for some $p_1 < q_1 < p_2 < q_2 < \dots$, we have E_n and $T(E_n)$ are both contained in $\operatorname{span}\{e_i : p_n \leqslant i \leqslant q_n\}$ for all $n \in \mathbb{N}$. After passing to further subspaces of E_n, we may also assume that each E_n is 2-isomorphic to ℓ_2^n by Dvoretzky's theorem. Let us now fix for each $n \in \mathbb{N}$ an isomorphism $J_n\colon \ell_2^n \to E_n$ satisfying $\frac{1}{\varepsilon}\|x\| \leqslant \|J_n x\| \leqslant \frac{2}{\varepsilon}\|x\|$ for all $x \in \ell_2^n$, and define T_n to be the restriction of T to E_n viewed as a map $T_n\colon E_n \to \ell_\infty^{r_n} \cong \operatorname{span}\{e_i : p_n \leqslant i \leqslant q_n\}$. Since $\|x\| \leqslant \|T_n J_n x\|$ for all $n \in \mathbb{N}$ and $x \in \ell_2^n$, it follows from Lemma 11 that the K_n factor uniformly through $T_n J_n$. Hence K factors through $\operatorname{diag}(T_n J_n)\colon \big(\bigoplus_{n=1}^\infty \ell_2^n\big)_{\ell_p} \to \big(\bigoplus_{n=1}^\infty \ell_\infty^{r_n}\big)_{c_0}$, and thus through T, as required.

The rest of the theorem now follows. □

We conclude with some open problems. As already mentioned in the Introduction, our method does not work for operators from ℓ_1 to c_0. Sirotkin and Wallis [19] have shown that $\mathcal{L}(\ell_1, c_0)$ contains uncountably many closed ideals, but the following remains open.

Problem 13. *Does $\mathcal{L}(\ell_1, c_0)$ contain continuum many closed ideals?*

There are various other questions that arise naturally in this paper but remain unanswered. We summarize these in the next two problems.

Problem 14. *Does the map φ in Theorem 2, 6 or 9 preserve the meet operation?*

A positive answer would in particular imply that for disjoint infinite subsets M and N of $\mathbb{N}$ we have $\mathcal{J}^{T_M} \cap \mathcal{J}^{T_N} = \mathcal{J}^{I_{U,V}}$, and thus the definition of $\varphi([\emptyset])$ is optimal.

Problem 15. *Does the inclusion $\mathcal{FS} \subset \mathcal{J}^K$ hold in Theorem 12?*

This would show that $\mathcal{J} \subset \mathcal{FS} \subset \mathcal{J}^K \subset \mathcal{J}'$ for all "small" ideals $\mathcal{J}$ and for all large ideals $\mathcal{J}'$, and would thus shed further light on the lattice $\mathfrak{H}(\ell_p, c_0)$.

Let us now turn attention to the remaining pairs of classical sequence spaces that we have hitherto not mentioned.

Problem 16. *What can be said about the lattice of closed ideals in $\mathcal{L}(\ell_\infty, \ell_p)$, $1 \leqslant p \leqslant \infty$, and in $\mathcal{L}(\ell_\infty, c_0)$? In particular, are they infinite?*

We note that the structure of $\mathfrak{H}(c_0, \ell_\infty)$ on the other hand is well understood and is as simple as possible.

Theorem 17. *The compact operators are the only non-trivial proper closed ideal in $\mathcal{L}(c_0, \ell_\infty)$.*

Proof. We need to show that if $T\colon c_0 \to \ell_\infty$ is a non-compact operator, then it generates $\mathcal{L}(c_0, \ell_\infty)$. By standard basis arguments, we find a block subspace Y in c_0 spanned by a block sequence (x_n) of the unit vector basis (e_n) such that T is an isomorphism on Y. Let $Z = T(Y)$, and let S be an arbitrary operator in $\mathcal{L}(c_0, \ell_\infty)$. Define $B\colon c_0 \to c_0$ by $Be_n = x_n$. Since (Tx_n) is equivalent to (e_n), we can define $A_1\colon Z \to \ell_\infty$ by $A_1(Tx_n) = Se_n$. Finally, since ℓ_∞ is injective, A_1 extends to an operator $A\colon \ell_\infty \to \ell_\infty$. Note that $S = ATB$, and hence the proof is complete. □

Finally, it follows from the work of Bourgain, Rosenthal and Schechtman that $L_p[0,1]$ for $1 < p < \infty$, $p \neq 2$, has up to isomorphism uncountably many complemented subspaces [3]. It follows easily that $\mathcal{L}(L_p[0,1])$ has uncountably many closed ideals. (In fact, although it is not known whether $L_p[0,1]$ has continuum many complemented subspaces, it follows from [20] that $\mathcal{L}(L_p[0,1])$ has continuum many closed ideals. This is because $L_p[0,1]$ contains a complemented copy of $\ell_p \oplus \ell_2$). It is a well known unsolved conjecture that every complemented subspace of $L_1[0,1]$ is isomorphic either to ℓ_1 or to $L_1[0,1]$. The following related question is therefore of interest.

Problem 18. *Does $\mathcal{L}(L_1[0,1])$ contain infinitely many closed ideals?*

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