A WEAK GROTHENDIECK COMPACTNESS PRINCIPLE

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Abstract. The Grothendieck compactness principle states that every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence. In this article, an analogue of the Grothendieck compactness principle is considered when the norm topology of a Banach space is replaced by its weak topology. It is shown that every weakly compact subset of a Banach space is contained in the closed convex hull of a weakly null sequence if and only if the Banach space has the Schur property.

In [3, p. 112], Alexander Grothendieck proved that every norm compact subset of a Banach space X is contained in the closed convex hull of a norm null sequence. Grothendieck remarks that this result is implicitly contained in an article of J. Dieudonné and L. Schwartz [1, proof of Th. 5]. Despite Grothendieck's remark, we refer to this result as the *Grothendieck compactness principle*.

It is known that an analogue of Grothendieck's compactness principle does not hold for all Banach spaces if the norm topology is replaced by the weak topology. Lindenstrauss and Phelps [5, Corollary 1.2] proved that the closed unit ball in an infinite-dimensional reflexive Banach space cannot be contained in the closed convex hull of a weakly null sequence. Also, in [7], it is noted that the closed unit ball of ℓ^2, considered as a subset of c_0, is not contained in the closed convex hull of a weakly null sequence in c_0.

The question that provides the impetus for the current article is: For which Banach spaces do analogues of Grothendieck's compactness principle hold if the norm topology of a Banach space is replaced by its weak topology? That is, for which Banach spaces is it true that every weakly compact set in the Banach space is contained in the closed convex hull of a weakly null sequence?

If X is a Banach space with the Schur property, that is, a space in which weak convergence and norm convergence of sequences coincide, then every weakly compact set is norm compact; and therefore, by Grothendieck's result, in spaces with the Schur property, every weakly compact set is contained the closed convex hull of a weakly (in fact, norm) null sequence. It turns out that no other spaces have this property.

Theorem 1. *Every weakly compact subset of a Banach space X is contained in the closed convex hull of a weakly null sequence if and only if X has the Schur property.*

Before proving the theorem, let us recall a definition and a few facts. A Schauder basis $\{e_i\}_{i=1}^{\infty}$ for a Banach space is *bimonotone* if, for every $n, m \in \mathbb{N}$ with $n < m$, the projection $P_{[n,m)}$ defined by $P_{[n,m)}\left(\sum_{i=1}^\infty a_i e_i\right) = \sum_{i=n}^{m-1} a_i e_i$ satisfies $\|P_{[n,m)}\| = 1$. It is worth noting that, if $\{e_i\}$ is a basis for a Banach space X, then X can be renormed to have an equivalent bimonotone norm: $|\!|\!| \sum_{i=1}^\infty a_i e_i |\!|\!| = \sup_{n<m} \|P_{[n,m)}\left(\sum_{i=1}^\infty a_i e_i\right)\|$. Consequently, since every separable Banach space is isometrically isomorphic to a closed subspace of $C[0,1]$, every separable Banach space can be considered to be a subspace of a space with a normalized bimonotone basis.

As a last bit of notation before proving the theorem, for a subset J of $\mathbb{N}$, P_J is defined similar to the above and, as usual, P_n will denote the projection $P_{[1,n]}$.

Proof of Theorem 1. Let X be a Banach space such that every weakly compact subset of X is contained in the closed convex hull of a weakly null sequence in X. In order to reach a contradiction, assume that X does not have the Schur property. Then there exists a normalized sequence $(x_i)_{i=1}^\infty$ in X that converges weakly to 0. Define

$$K_n = (n \cdot \overline{\text{co}}(x_i)_{i=1}^\infty) \cap \frac{1}{n} B_X\,, \qquad \text{for all } n \in \mathbb{N}\,,$$

and

$$K = \cup_{n=1}^\infty K_n\,,$$

where B_X is the closed unit ball of X. Since the sequence $(x_i)_{i=1}^\infty$ converges weakly to 0, the sets $\overline{\text{co}}(x_i)_{i=1}^\infty$ and K_n for $n \in \mathbb{N}$ are weakly compact. Note that, since the diameters of the sets K_n converge to 0, any weakly open set containing 0 will contain all but finitely many of the K_n's. Thus, if $\{U_\alpha\}_{\alpha\in A}$ is a weakly open cover of K, there exists $\beta \in A$ such that all but finitely many of the K_n are subsets of U_β. The collection of K_n which are not a subset of U_β is a finite collection of weakly compact sets, and are thus covered by a finite subcollection of sets in $\{U_\alpha\}_{\alpha\in A}$. Therefore K is weakly compact.

Since K is a weakly compact subset of X, there exists a weakly null sequence $(y_i)_{i=1}^\infty$ such that $K \subseteq \overline{\text{co}}(y_i)$. If Y denotes the closed span of $\{y_i\}_{i=1}^\infty$, then Y is a separable Banach space containing K and $\overline{\text{co}}(y_i)$. By the remarks preceding the proof, we may consider Y to be a subspace of a Banach space with a normalized bimonotone basis $\{e_i\}_{i=1}^\infty$.

Fix $N \in \mathbb{N}$, $0 < \varepsilon < 1$, and a sequence $(\varepsilon_i)_{i=1}^\infty$ decreasing to 0 with $\sum_{i=1}^\infty \varepsilon_i < \varepsilon$. Since the sequences $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ are weakly null, there exist increasing sequences $(m_i)_{i=1}^\infty$, $(p_i)_{i=1}^\infty$, and $(q_i)_{i=1}^\infty$ in $\mathbb{N}$ satisfying the following properties:

(1) $\|x_{m_j} - P_{[q_{2j-1},q_{2j})}(x_{m_j})\| < \varepsilon_j^2\,, \qquad$ for all $j \in \mathbb{N}$.

(2) $\|P_{[q_{2j-1},q_{2j})}(x_{m_i})\| < \varepsilon_i\varepsilon_j\,, \qquad$ for all $i \neq j$.

(3) $\|P_{[q_{2j-1},q_{2j})}(y_i)\| < \varepsilon_j\,, \qquad$ for all $i \notin [p_j, p_{j+1})$.

Since $(x_{m_i})_{i=1}^\infty$ is weakly null, $0 \in \overline{\text{co}}(x_{m_i})$. After scaling by N, this implies that there exists $M \in \mathbb{N}$ and $(\lambda_i)_{i=1}^M \subseteq [0,1]$ such that $\|\sum_{i=1}^M \lambda_i x_{m_i}\| \leq \frac{1}{N}$ and $\sum_{i=1}^M \lambda_i = N$. (Indeed, choose N convex combinations of the x_{m_i}'s, in blocks with disjoint support, such that the norms of the convex combinations are each less than $\frac{1}{N^2}$ and sum them.) This is equivalent to $\sum_{i=1}^M \lambda_i x_{m_i} \in K_N$. If we now fix $\delta = \frac{\varepsilon}{M}$, there exists a sequence $(\alpha_i)_{i=1}^\infty$ in $[0,1]$ such that $\sum_{i=1}^\infty \alpha_i = 1$ and $\|\sum_{i=1}^M \lambda_i x_{m_i} - \sum_{i=1}^\infty \alpha_i y_i\| < \delta$.

For each $1 \le j \le M$, since (e_i) is bimonotone, we obtain the following estimate:

$$\begin{aligned}
\delta &> \left\| P_{[q_{2j-1},q_{2j})}\left(\sum_{i=1}^M \lambda_i x_{m_i} - \sum_{i=1}^\infty \alpha_i y_i\right)\right\| \\
&\ge \left\|\sum_{i=1}^M \lambda_i P_{[q_{2j-1},q_{2j})}(x_{m_i})\right\| - \sum_{i=1}^\infty \alpha_i \|P_{[q_{2j-1},q_{2j})}(y_i)\| \\
&\ge \left\|\sum_{i=1}^M \lambda_i P_{[q_{2j-1},q_{2j})}(x_{m_i})\right\| - \sum_{i\in[p_j,p_{j+1})} \alpha_i\|y_i\| - \sum_{i\notin[p_j,p_{j+1})} \alpha_i \|P_{[q_{2j-1},q_{2j})}(y_i)\| \\
&> \left\|\sum_{i=1}^M \lambda_i P_{[q_{2j-1},q_{2j})}(x_{m_i})\right\| - \sum_{i\in[p_j,p_{j+1})} \alpha_i\|y_i\| - \varepsilon_j\,, \qquad \text{by (3)} \\
&\ge \lambda_j\|x_{m_j}\| - \lambda_j\|x_{m_j} - P_{[q_{2j-1},q_{2j})}(x_{m_j})\| - \sum_{\substack{i\le M\\ i\ne j}} \lambda_i \|P_{[q_{2j-1},q_{2j})}(x_{m_i})\| - \sum_{i\in[p_j,p_{j+1})} \alpha_i\|y_i\| - \varepsilon_j \\
&> \lambda_j - \varepsilon_j^2 - \sum_{\substack{i\le M\\ i\ne j}} \varepsilon_i\varepsilon_j - \sum_{i\in[p_j,p_{j+1})} \alpha_i\|y_i\| - \varepsilon_j \qquad \text{by (1) and (2)} \\
&> \lambda_j - \sum_{i\in[p_j,p_{j+1})} \alpha_i\|y_i\| - 2\varepsilon_j
\end{aligned}$$

Summing these inequalities over $1 \le j \le M$ yields

$$M\delta \;>\; \sum_{j=1}^M \lambda_j - \sum_{i=1}^\infty \alpha_i\|y_i\| - 2\varepsilon \;\ge\; N - \sup_{i\in\mathbb{N}} \|y_i\| - 2\varepsilon$$

so that

$$\sup_{i\in\mathbb{N}} \|y_i\| > N - M\delta - 2\varepsilon = N - 3\varepsilon\,.$$

However, since $\varepsilon > 0$ can chosen to be arbitrarily small, $\sup_{i\in\mathbb{N}} \|y_i\| \ge N$. Then, since N is an arbitrary positive integer, $\sup_{i\in\mathbb{N}} \|y_i\| = \infty$, contradicting that (y_i) is a weakly null sequence. Therefore X must have the Schur property. □

It is known [4] that a Banach space X with a 1-symmetric basis has the Schur property if and only if $X = \ell^1$. Since a Banach space with a symmetric basis $\{e_n\}$ can be endowed with an equivalent norm such that $\{e_n\}$ is a 1-symmetric basis with respect to the new norm, Theorem 1 provides a characterization of the Banach spaces with a symmetric basis that have the analogue of the Grothendieck compactness principle for the weak topology. (For information about Banach spaces with symmetric bases, see [6, Chapter 3].)

Corollary 2. *Let X be a Banach space with a symmetric basis. Every weakly compact subset of X is contained in the closed convex hull of a weakly null sequence if and only if X is isomorphic to ℓ^1.*

Other variants of the Grothendieck compactness principle can be considered, and results on the weak Grothendieck compactness principle in the setting of Banach spaces with a symmetric basis will appear elsewhere [2].

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