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A Weak Grothendieck Compactness Principle for Banach spaces with a Symmetric Basis

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Abstract The Grothendieck compactness principle states that every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence. In [1], an analogue of the Grothendieck compactness principle for the weak topology was used to characterize Banach spaces with the Schur property. Using a different analogue of the Grothendieck compactness principle for the weak topology, a characterization of the Banach spaces with a symmetric basis that are not isomorphic to ℓ^1 and do not contain a subspace isomorphic to c_0 is given. As a corollary, it is shown that, in the Lorentz space $d(w, 1)$, every weakly compact set is contained in the closed convex hull of the rearrangement invariant hull of a norm null sequence.

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1 Introduction

The Grothendieck compactness principle states that every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence [3, p. 112]. In [1], an analogue of the Grothendieck compactness principle for the weak topology was used to characterize Banach spaces with the Schur property. To be specific, it was shown that a Banach space X has the Schur property if and only if every weakly compact subset of X is contained in the closed convex hull of a weakly null sequence.

In this article, another variant of the Grothendieck compactness principle for the weak topology is considered in Banach spaces with a symmetric basis. The weak Grothendieck compactness principle considered here involves rearrangement invariant hulls as well as convex hulls and is used to characterize the Banach spaces with a symmetric basis that are not isomorphic to ℓ^1 and do not contain a subspace isomorphic to c_0. Subsets of such Banach spaces X are relatively weakly compact if and only they are contained in the closed convex hull of the rearrangement invariant hull of a weakly null sequence in X. Moreover, in the Lorentz sequence space $d(w, 1)$, every weakly compact set is contained in the closed convex hull of the rearrangement invariant hull of a norm null sequence.

2 The main results

Before proving the main result, let us recall a few definitions and fix some notation. If $x = (a_n)$ is an element of c_0, let $x^* = (a_n^*)$ denote its nonincreasing rearrangement; i.e., (a_n^*) is an enumeration of the nonzero terms of $(|a_n|)$ such that $a_1^* \geq a_2^* \geq \cdots$ followed by infinitely many zeros if necessary. If S is a subset of c_0, the *rearrangement invariant hull* of S is

$$\text{ri}(S) = \{y \in c_0 : y^* = x^* \text{ for some } x \in S\}.$$

A set S is *rearrangement invariant* if $\text{ri}(S) = S$. A Banach space X with a 1-symmetric basis is rearrangement invariant and $\|x^*\| = \|x\|$, for all $x \in X$. If S is a weakly closed subset of a Banach space with a 1-symmetric basis, it does not necessarily follow that $\text{ri}(S)$ is weakly closed. For example, if $\{e_n\}_{n\in\mathbb{N}}$ is the canonical unit vector basis in ℓ^2 and $S = \{e_1\}$, then $\text{ri}(S) = \{e_n : n \in \mathbb{N}\}$, a set which is not weakly closed in ℓ^2. With this motivation, if S is a subset of a Banach space with a 1-symmetric basis, $\overline{\text{ri}}(S)$ will denote the weak closure of $\text{ri}(S)$. Finally, the notation will follow the notation in [1]. In particular, if $\{e_i\}_{i=1}^\infty$ is a Schauder basis and $n < m$, $P_{[n,m)}$ denotes the projection defined by $P_{[n,m)}\left(\sum_{i=1}^\infty a_i e_i\right) = \sum_{i=n}^{m-1} a_i e_i$, and, for a subset J of $\mathbb{N}$, P_J is defined similarly. As usual, P_n will denote the projection $P_{[1,n]}$.

The proof of the main result uses the following fact:

Lemma 1 *If S is a subset of a Banach space with a symmetric basis, then* $\overline{\mathrm{ri}}\,\overline{\mathrm{co}}(S) \subseteq \overline{\mathrm{co}}\,\overline{\mathrm{ri}}(S)$.

Since the proof of Lemma 1 is somewhat technical and the results may be of independent interest, the proof of Lemma 1 is postponed until after Theorem 2.

Theorem 1 *Let X be a Banach space with a symmetric basis. The space X is not isomorphic to ℓ^1 and does not contain a subspace isomorphic to c_0 if and only if the following are equivalent for all subsets $K \subseteq X$:*

1. *K is relatively weakly compact.*
2. *There exists a weakly null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in X such that*
$$K \subseteq \overline{\mathrm{ri}}\{\mathbf{x}_n : n \in \mathbb{N}\} \ .$$
3. *There exists a weakly null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in X such that*
$$K \subseteq \overline{\mathrm{ri}}\,\overline{\mathrm{co}}\{\mathbf{x}_n : n \in \mathbb{N}\} \ .$$
4. *There exists a weakly null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in X such that*
$$K \subseteq \overline{\mathrm{co}}\,\overline{\mathrm{ri}}\{\mathbf{x}_n : n \in \mathbb{N}\} \ .$$

Proof Let $\{e_i\}_{i=1}^{\infty}$ denote the symmetric basis of X. Since a Banach space with a symmetric basis $\{e_i\}$ can be endowed with an equivalent norm such that $\{e_i\}$ is a 1-symmetric basis with respect to the new norm, there is no loss of generality in assuming that $(e_i)_{i=1}^{\infty}$ is normalized and 1-symmetric. If X is isomorphic to ℓ^1 then the set $K = \mathrm{ri}\{\mathrm{e}_1\} = \{\pm \mathrm{e}_\mathrm{n} : \mathrm{n} \in \mathbb{N}\}$ is not relatively weakly compact and yet it is contained in the weak closure of the rearrangement invariant hull of the weakly null sequence $(\mathbf{x}_n)_n = (e_1, 0, 0, 0, ...)$. Thus, statement (2) is true but statement (1) is false. Consequently, the four properties are not equivalent for all $K \subseteq X$ if X is isomorphic to ℓ^1. We now show that the condition that X not contain a subspace isomorphic to c_0 is necessary for all four properties to be equivalent for all $K \subset X$. In particular, we will show that if X contains a subspace isomorphic to c_0, then there exists a $K \subset X$ such that (2) is true, but (1) is false.

We assume that X contains a subspace isomorphic to c_0, which implies, due to c_0 not being distortable [5], that there exists a normalized block basis $(u_i)_{i=1}^{\infty}$ of $(e_i)_{i=1}^{\infty}$ which satisfies,

$$\max_{1\le i<\infty} |a_i| \le \|\sum_{i=1}^{\infty} a_i u_i\| \le 2 \max_{1\le i<\infty} |a_i| \qquad \text{for all } (a_i) \in c_0. \qquad (*)$$

For all $n \in \mathbb{N}$, we define the right shift operator $R_n : X \to X$ by $R_n(x) = \sum_{i=1}^{\infty} e_i^*(x) e_{i+n}$. In other words, if $x = (a_1,\, a_2, \cdots)$ then $R_n(x) = (\underbrace{0, \ldots, 0}_{n}, a_1,\, a_2, \cdots)$.

As $(u_i)_{i=1}^{\infty}$ is a block basis of $(e_i)_{i=1}^{\infty}$, we may choose a subsequence $(k_i)_{i=1}^{\infty}$ such that $(R_{k_n}(\sum_{i=1}^n u_i))_{n=1}^\infty$ is a block basis of $(e_i)_{i=1}^\infty$. We let $x_n = R_{k_n}(\sum_{i=1}^n u_i)$ for all $n \in \mathbb{N}$. By $(*)$, we have that $\|x_n\| \le 2$ for all $n \in \mathbb{N}$. We define $K = \bigcup_{1\le n<\infty}\{\sum_{i=1}^n u_i\}$. Since K consists of a sequence which is equivalent to the summing basis for c_0, K is not relatively weakly compact. However, $K \subseteq \overline{\mathrm{ri}}\{x_n : n \in \mathbb{N}\}$. We thus just need to show that $(x_n)_{n=1}^\infty$ is weakly null. We assume that $(x_n)_{n=1}^\infty$ is not weakly null, and hence has a subsequence $(x_{p_n})_{n=1}^\infty$ which is equivalent to the unit vector basis for ℓ^1.

Let $\varepsilon > 0$ and let $x \in X$ be such that $(R_{k_{p_n}}x)_{n=1}^\infty$ is a block basis of $(e_i)_{i=1}^\infty$. As X is not isomorphic to ℓ^1 and $(e_i)_{i=1}^\infty$ is symmetric, $(R_{k_{p_n}}x)_{n=1}^\infty$ is not equivalent to the unit vector basis for ℓ^1, and is hence weakly null. Thus there exists $(\lambda_i)_{i=1}^M \subset (0,1]$ such that $\sum_{i=1}^M \lambda_i = 1$ and $\|\sum_{i=1}^M \lambda_i R_{k_{p_i}}x\| < \varepsilon$. Let C_M be the group of M cyclic permutations of the set $\{1, 2, \cdots, M\}$. As $(e_i)_{i=1}^\infty$ is 1-symmetric and $(R_{k_{p_i}}x)_{i=1}^\infty$ is a block basis of elements with the same distribution, we have that $\|\sum_{i=1}^M \lambda_{\pi(i)} R_{k_{p_i}}x\| = \|\sum_{i=1}^M \lambda_i R_{k_{p_i}}x\| < \varepsilon$ for all $\pi \in C_M$. By averaging over all cyclic permutations, we obtain that $\|\sum_{i=1}^M \frac{1}{M} R_{k_{p_i}}x\| = \|\frac{1}{M}\sum_{\pi\in C_M}\sum_{i=1}^M \lambda_{\pi(i)} R_{k_{p_i}}x\| \le \frac{1}{M}\sum_{\pi\in C_M}\|\sum_{i=1}^M \lambda_{\pi(i)} R_{k_{p_i}}x\| < \varepsilon$. We may thus choose a strictly increasing sequence $(q_i)_{i=1}^\infty$ of positive integers with $q_1 = 1$ such that $\|\frac{1}{q_{n+1}-q_n}\sum_{i=q_n+1}^{q_{n+1}} R_{k_{p_i}}(\sum_{j=1}^{p_{q_n}} u_j)\| < 2^{-n}\varepsilon$ for all $n \ge 2$. We set $y_1 = x_{p_1}$ and $y_n = \frac{1}{q_{n+1}-q_n}\sum_{i=q_n+1}^{q_{n+1}} x_{p_i}$ for all $n \ge 2$, then set $z_1 = 0$ and $z_n = \frac{1}{q_{n+1}-q_n}\sum_{i=q_n+1}^{q_{n+1}} R_{p_{k_i}}(\sum_{j=1}^{p_{q_n}} u_j)$ for all $n \ge 2$. We have that $(y_i)_{i=1}^\infty$ is a block sequence of convex combinations of $(x_{p_i})_{i=1}^\infty$ and is hence equivalent to the unit vector basis of ℓ^1. Thus if $\varepsilon > 0$ is sufficiently small, then $(y_i - z_i)_{i=1}^\infty$ is equivalent to the unit vector basis for ℓ^1 as $\|z_i\| < 2^{-n}\varepsilon$ for all $n \in \mathbb{N}$. However, this is a contradiction as $(y_i - z_i)_{i=1}^\infty$ is 1-equivalent to a seminormalized block basis of $(u_i)_{i=1}^\infty$ and is hence equivalent to the unit vector basis for c_0.

We now show that the conditions that X is not isomorphic to ℓ^1 and X contains no subspace isomorphic to c_0 are sufficient for all four properties to be equivalent for all $K \subseteq X$. Assume (1) holds and, without loss of generality, assume that K is nonempty and weakly compact. Let (e_n) be a 1-symmetric basis of X. Since X is separable, we can choose a countable dense subset $\{\mathbf{z}_n = (z_{n,1}, z_{n,2}, \cdots)\}_{n\in\mathbb{N}}$ of K. For $n \in \mathbb{N}$, define $\mathbf{x}_n \in X$ by $\mathbf{x}_n = (\underbrace{0,\ldots,0}_{n}, z_{n,1}, z_{n,2}, \cdots)$. Then $\mathbf{x}_n^* = \mathbf{z}_n^*$; the sequence $(\mathbf{x}_n)$ converges to $\mathbf{0}$ coordinatewise; and $\overline{\mathrm{ri}}(K) = \overline{\mathrm{ri}}\{\mathbf{x}_n : n \in \mathbb{N}\}$.

For the sake of contradiction, assume that $(\mathbf{x}_n)$ does not converge weakly to $\mathbf{0}$. Then, as in the proof of Lemma 1.c.11 in [8], there exists a subsequence $(\mathbf{x}_{n_j})$ of $(\mathbf{x}_n)$ such that $(\mathbf{x}_{n_j})$ is equivalent to a block basis of (e_n) and is equivalent to the unit vector basis of ℓ^1. Since K is weakly compact, by taking a subsequence if necessary, there is no loss in generality in assuming that the sequence $(\mathbf{z}_{n_j})$ in K corresponding to $(\mathbf{x}_{n_j})$ converges weakly to $\mathbf{z} = (z_1, z_2, \cdots)$ in K. Let $\mathbf{h}_j = \mathbf{z}_{n_j} - \mathbf{z} = (h_{j,1}, h_{j,2}, \cdots)$, and define

$$\zeta_j = (\underbrace{0,\ldots,0}_{n_j}, z_1, z_2, \cdots) \quad \text{and} \quad \eta_j = (\underbrace{0,\ldots,0}_{n_j}, h_{j,1}, h_{j,2}, \cdots)\,.$$

Then $\mathbf{x}_{n_j} = \zeta_j + \eta_j$ and, since $(\mathbf{x}_{n_j})$ is equivalent to the usual vector basis of ℓ^1, Rosenthal's ℓ^1 Theorem implies that either (ζ_j) or (η_j) is equivalent to the usual vector basis of ℓ^1.

Assume that (η_j) is equivalent to the unit vector basis of ℓ^1. Since $(\mathbf{h}_j)$ converges weakly to $\mathbf{0}$, there exists a perturbation $(\widetilde{\mathbf{h}}_j)$ of $(\mathbf{h}_j)$ which is equivalent to $(\mathbf{h}_j)$, converges weakly to $\mathbf{0}$, and, by taking a subsequence if necessary, has disjoint supports. By taking yet another subsequence if necessary, we can assume that the corresponding translations $(\widetilde{\eta}_j)$ of $(\widetilde{\mathbf{h}}_j)$ also have disjoint supports and $(\widetilde{\eta}_j)$ is equivalent to (η_j). Therefore $(\widetilde{\eta}_j)$ is equivalent to the unit vector basis of ℓ^1. Since the basis (e_n) is 1-symmetric, there exists $C > 0$ such that

$$\left\| \sum_{j=1}^{\infty} \alpha_j \widetilde{\mathbf{h}}_j \right\| = \left\| \sum_{j=1}^{\infty} \alpha_j \widetilde{\eta}_j \right\| \geq C \sum_{j=1}^{\infty} |\alpha_j| \, .$$

Therefore $(\widetilde{\mathbf{h}}_j)$ is equivalent to the unit vector basis of ℓ^1, contradicting that $(\widetilde{\mathbf{h}}_j)$ converges weakly to $\mathbf{0}$. Consequently, (ζ_j) must be equivalent to the unit vector basis of ℓ^1.

Choose $0 < K < 1$ so that $\| \sum_{j=1}^{\infty} \alpha_j \zeta_j \| \geq K \sum_{=1}^{\infty} |\alpha_j|$ for $(\alpha_j) \in \ell^1$. Since $\mathbf{z} \in X$, there exists $N \in \mathbb{N}$ such that $\|P_{[N+1,\infty)}(\mathbf{z})\| < \frac{K}{2}$. Define

$$\zeta'_j = (\underbrace{0, \ldots, 0}_{n_j}, z_1, \cdots, z_N, 0, 0, \cdots) \quad \text{and} \quad \zeta''_j = (\underbrace{0, \ldots, 0}_{n_j+N}, z_{N+1}, \cdots, z_{N+2}, \cdots) \, .$$

By the triangle inequality, it is easy to check that $\| \sum_{j=1}^{\infty} \alpha_j \zeta'_j \| \geq \frac{K}{2} \sum_{j=1}^{\infty} |\alpha_j|$. Therefore (ζ'_j) is equivalent to the usual vector basis for ℓ^1. By taking a subsequence if necessary, we can assume that the ζ'_j are disjointly supported. Then, using the 1-symmetry of (e_n),

$$\begin{aligned} \frac{K}{2} \sum_{j=1}^{\infty} |\alpha_j| &\leq \left\| \sum_{j=1}^{\infty} \alpha_j \zeta'_j \right\| \\ &\leq \left\| \sum_{j=1}^{\infty} \alpha_j \left(\sum_{i=1}^{N} z_i \, e_{n_j+i} \right) \right\| \\ &= \left\| \sum_{i=1}^{N} z_i \left(\sum_{j=1}^{\infty} \alpha_j e_{n_j+i} \right) \right\| \\ &\leq N \|\mathbf{z}\|_\infty \left\| \sum_{j=1}^{\infty} \alpha_j e_j \right\| \, . \end{aligned}$$

Therefore $\| \sum_{j=1}^{\infty} \alpha_j e_j \| \geq \frac{K}{2N\|z\|_\infty} \sum_{j=1}^{\infty} |\alpha_j|$. But this implies that X is isomorphic to ℓ^1, a contradiction that finishes the proof of (1) implies (2).

The implication (2) implies (3) is clear, and the implication (3) implies (4) follows immediately from Lemma 1.

Assume that (4) holds and, without loss of generality, that (e_n) is a normalized 1-symmetric basis for X. To see that (4) implies (1), it suffices to prove that $\mathrm{ri}(\mathbf{x}_n)$ is relatively weakly compact. If not, there exists a subsequence $(\mathbf{x}_{n_i})$ of $(\mathbf{x}_n)$ and a sequence $(\widetilde{\mathbf{x}}_{n_i})$ such that $\widetilde{\mathbf{x}}_{n_i}^* = \mathbf{x}_{n_i}^*$ and $(\widetilde{\mathbf{x}}_{n_i})$ has no weakly convergent subsequence. Without loss of generality, assume that $(\widetilde{\mathbf{x}}_{n_i})$ converges coordinatewise to $\mathbf{x}$. Suppose that $\widetilde{\mathbf{x}}_{n_i} = \sum_{j=1}^\infty \widetilde{\xi}_j^i e_j$ and $\mathbf{x}$ is the formal sum $\sum_{j=1}^\infty \xi_j e_j$. Fix $k \in \mathbb{N}$ and choose i_0 large enough so that $\left\|\sum_{j=1}^k (\xi_j - \widetilde{\xi}_j^{i_0}) e_j\right\| < 1$. With $M = \sup_n \|\mathbf{x}_n\|$, it follows that

$$\left\|\sum_{j=1}^k \xi_j e_j\right\| \le \left\|\sum_{j=1}^k (\xi_j - \widetilde{\xi}_j^{i_0}) e_j\right\| + \left\|\sum_{j=1}^k \widetilde{\xi}_j^{i_0} e_j\right\| < 1 + \|P_k(\widetilde{\mathbf{x}}_{n_{i_0}})\| \le 1 + \|\mathbf{x}_{n_{i_0}}\| \le 1 + M.$$

Since X contains no subspace isomorphic to c_0, the basis (e_n) is boundedly complete. Therefore $\mathbf{x} \in X$.

For $i \in \mathbb{N}$, define $\widetilde{\mathbf{h}}_i = \widetilde{\mathbf{x}}_{n_i} - \mathbf{x}$. Since no subsequence of $(\widetilde{\mathbf{h}}_i)$ converges weakly to $\mathbf{0}$, there exists $\delta > 0$ such that $\|\widetilde{\mathbf{h}}_i\| > \delta$ for each $i \in \mathbb{N}$. A small perturbation of $(\widetilde{\mathbf{h}}_i)$ yields a sequence $(\mathbf{h}_i')$ in X such that $(\mathbf{h}_i')$ is a block basic sequence of (e_n), $(\mathbf{h}_i')$ has no subsequence converging weakly to $\mathbf{0}$, and $\|\mathbf{h}_i'\| > \delta$ for each $i \in \mathbb{N}$. As before, by passing to a subsequence if necessary, there is no loss of generality in assuming that $(\mathbf{h}_i')$ and $(\widetilde{\mathbf{h}}_i)$ are equivalent to the usual vector basis for ℓ^1.

Since $\mathbf{x} \in X$, there exists $k_1 \in \mathbb{N}$ such that $\left\|P_{[k_1+1,\infty)}(\mathbf{x})\right\| < \frac{\delta}{3\cdot 2^2}$. By tossing out finitely many terms of $(\widetilde{\mathbf{x}}_{n_i})$ if necessary, there is no loss of generality in assuming that $\left\|P_{[1,k_1]}(\widetilde{\mathbf{h}}_1)\right\| < \frac{\delta}{3\cdot 2^2}$. Then choose $\ell_1 > k_1$ such that $\left\|P_{[\ell_1+1,\infty)}(\widetilde{\mathbf{h}}_1)\right\| < \frac{\delta}{3\cdot 2^2}$ and define $\mathbf{h}_1 = P_{[k_1+1,\ell_1]}(\widetilde{\mathbf{h}}_1) = \sum_{j=k_1+1}^{\ell_1} \left(\widetilde{\xi}_j^1 - \xi_j\right) e_j$. Let $A_1 = \{\, i \in [1,k_1] : \widetilde{\xi}_i^1 \ne 0\}$; $B_1 = \{\, i \in [k_1+1,\ell_1] : \widetilde{\xi}_i^1 \ne 0\}$; and, for $i \in \mathbb{N}$, denote $\mathbf{x}_{n_i} = \sum_{j=1}^\infty \xi_j^i e_j$. There exists a one-to-one map $\pi_1 : A_1 \cup B_1 \to \mathbb{N}$ such that $\widetilde{\xi}_i^1 = \xi_{\pi(i)}^1$. Let $\mathbf{y}_1 = \sum_{j=1}^\infty \upsilon_j^1 e_j$ where

$$\upsilon_j^1 = \begin{cases} \widetilde{\xi}_i^1 & \text{if } j = \pi_1(i),\ i \in A_1 \\ \widetilde{\xi}_i^1 - \xi_i & \text{if } j = \pi_1(i),\ i \in B_1 \\ 0 & \text{otherwise.} \end{cases}$$

The element $\mathbf{y}_1$ is defined using some of the coordinates of $\widetilde{\mathbf{x}}_{n_1}$ (and $\mathbf{x}_{n_1}$) and the coordinates of $P_{B_1}(\widetilde{\mathbf{h}}_1)$. If $i \in [k_1+1,\ell_1] \setminus B_1$, $\widetilde{\xi}_i^1 - \xi_i = -\xi_i$ and, since $\left\|P_{[k_1+1,\ell_1]}(\mathbf{x})\right\| < \frac{\delta}{3\cdot 2^2}$,

$$\left\|\widetilde{\mathbf{h}}_1 - P_{B_1}(\mathbf{h}_1)\right\| \le \left\|\widetilde{\mathbf{h}}_1 - \mathbf{h}_1\right\| + \|\mathbf{h}_1 - P_{B_1}(\mathbf{h}_1)\| < \frac{2\delta}{3\cdot 2^2} + \frac{\delta}{3\cdot 2^2} = \frac{\delta}{2^2}.$$

Note also that

$$\begin{aligned}\|\mathbf{x}_{n_1} - \mathbf{y_1}\| &\le \|P_{[\ell_1+1,\infty)}(\widetilde{\mathbf{x}}_{n_1})\| + \|P_{[k_1+1,\ell_1]}(\mathbf{x})\| \\ &\le \|P_{[\ell_1+1,\infty)}(\mathbf{x})\| + \|P_{[\ell_1+1,\infty)}(\widetilde{\mathbf{h}}_1)\| + \|P_{[k_1+1,\ell_1]}(\mathbf{x})\| \\ &< \frac{\delta}{2^2}\,.\end{aligned}$$

Let $\kappa_1 = \max \pi_1(A_1 \cup B_1)$ and define $\mu_1 = \min\{|\xi^1_j| : \xi^1_j \neq 0 \text{ and } 1 \le j \le \kappa_1\}$. Then $\kappa_1 > 0$. Similar to the above, there exists $k_2 > \ell_1$ such that $\left\|P_{[k_2+1,\infty)}(\mathbf{x})\right\| < \frac{\delta}{3\cdot 2^3}$. Again, by tossing out finitely many terms of $(\widetilde{\mathbf{x}}_{n_i})$ if necessary, since $(\mathbf{x}_n)$ converges weakly to $\mathbf{0}$, there exists $n_2 > n_1$ such that $\|P_{[1,\kappa_1]}(\mathbf{x}_{n_2})\| < \frac{\delta}{3\cdot 2^3}$, $\|P_{[1,\kappa_1]}(\mathbf{x}_{n_2})\|_\infty < \frac{\mu_1}{2}$, and $\left\|P_{[1,k_2]}(\widetilde{\mathbf{h}}_2)\right\| < \frac{\delta}{3\cdot 2^3}$. Then choose $\ell_2 > k_2$ such that $\left\|P_{[\ell_2+1,\infty)}(\widetilde{\mathbf{h}}_2)\right\| < \frac{\delta}{3\cdot 2^3}$. Define $\mathbf{h}_2 = P_{[k_2+1,\ell_2]}(\widetilde{\mathbf{h}}_2) = \sum_{j=k_2+1}^{\ell_2}\left(\widetilde{\xi}^2_j - \xi_j\right)e_j$. Let $A_2 = \{\, i \in [1,k_2] : \widetilde{\xi}^2_i \neq 0\}$ and $B_1 = \{\, i \in [k_2 + 1, \ell_2] : \widetilde{\xi}^2_i \neq 0\}$. There exists a one-to-one map $\pi_2 : A_2 \cup B_2 \to \mathbb{N}$ such that $\widetilde{\xi}^2_i = \xi^2_{\pi_2(i)}$. Let $\mathbf{y}_2 = \sum_{j=1}^\infty \upsilon^2_j e_j$ where

$$\upsilon^2_j = \begin{cases} \widetilde{\xi}^2_i & \text{if } j = \pi_2(i),\ i \in A_2 \\ \widetilde{\xi}^2_i - \xi_i & \text{if } j = \pi_2(i),\ i \in B_2 \\ 0 & \text{otherwise.}\end{cases}$$

Similar to before,

$$\left\|\widetilde{\mathbf{h}}_2 - P_{B_2}(\mathbf{h}_2)\right\| < \frac{\delta}{2^3} \quad\text{and}\quad \|\mathbf{x}_{n_2} - \mathbf{y_2}\| < \frac{\delta}{2^3}\,.$$

Note that, since $\left\|P_{[1,\kappa_1]}(\mathbf{x}_{n_2})\right\|_\infty < \frac{\mu_1}{2}$, the support of $\mathbf{y}_2$ is disjoint from the support of $\mathbf{y}_1$.

Continuing the above argument, we get a subsequence $(\mathbf{x}_{n_i})$ (of the originally-denoted $(\mathbf{x}_{n_i})$) and sequences $(\mathbf{h}_i)$ and $(\mathbf{y}_i)$ such that

$$\left\|\widetilde{\mathbf{h}}_i - P_{B_i}(\mathbf{h}_i)\right\| < \frac{\delta}{2^{i+1}} \quad\text{and}\quad \|\mathbf{x}_{n_i} - \mathbf{y_i}\| < \frac{\delta}{2^{i+1}}$$

and, if $i \neq j$, the supports of $\mathbf{y}_i$ and $\mathbf{y}_j$ are disjoint.

Since $(\widetilde{\mathbf{h}}_i)$ is equivalent to the usual vector basis of ℓ^1, this implies, by taking a subsequence if necessary, that $(P_{B_i}(\mathbf{h}_i))$ is an ℓ^1-basis. Note also that $(\mathbf{y}_i)$ is a bounded sequence in X and, since the basis (e_n) is 1-symmetric, there exists $C > 0$ such that

$$\left\|\sum_{i=1}^\infty \alpha_i \mathbf{y}_i\right\| \ge \left\|\sum_{i=1}^\infty \alpha_i P_{\pi_i(B_i)}(\mathbf{y}_i)\right\| = \left\|\sum_{i=1}^\infty \alpha_i P_{B_i}(\mathbf{h}_i)\right\| \ge C\sum_{i=1}^\infty |\alpha_i|\,.$$

Thus $(\mathbf{y}_i)$ is equivalent to the usual vector basis of ℓ^1 and, by perturbation, we can assume, by taking a subsequence if necessary, that $(\mathbf{x}_{n_i})$ is equivalent to the usual vector basis of ℓ^1. But this contradicts that (x_{n_i}) converges weakly to $\mathbf{0}$ and completes the proof.

In Theorem 1, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold even if X contains a subspace isomorphic to c_0. However, if X has a subspace isomorphic to c_0, condition (4) does not imply condition (1). To see this, choose a countable dense subset $\{\mathbf{z}_n : n \in \mathbb{N}\}$ of B_{c_0} and define $\mathbf{x}_n$ as in the proof of Theorem 1. Since $\overline{\mathrm{ri}}\{\mathbf{x}_n : n \in \mathbb{N}\} = B_{c_0}$, the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a weakly null sequence in c_0 with $B_{c_0} = \overline{\mathrm{co}}\,\overline{\mathrm{ri}}\{\mathbf{x}_n : n \in \mathbb{N}\}$, but B_{c_0} is not relatively weakly compact.

Examples of spaces with 1-symmetric bases for which the four conditions of Theorem 4 are equivalent are all the rearrangement invariant Köthe sequence spaces that are perfect, minimal and not equal to ℓ^1. (For further information about Köthe sequence spaces, see [9, §1.b and §2.a], [6, §30], and [2]. Note that, in [2], Köthe sequence spaces are assumed to be perfect.)

In the setting of Lorentz sequence spaces, Theorem 1 can be expanded. Recall that, if (w_n) is a nonincreasing sequence of positive numbers with $(w_n) \in c_0 \setminus \ell^1$, the Lorentz sequence space $d(w,1)$ (sometimes denoted $\ell_{w,1}$) consists of all sequences $\mathbf{x} = (a_n)$ of scalars for which $\|\mathbf{x}\| = \sum_{n=1}^{\infty} a_n^* w_n < \infty$. The set $d(w,1)$ endowed with the norm $\|\cdot\|$ is a Banach space. (For more information about Lorentz spaces, see [8,9].)

Theorem 2 *Let K be a subset of $d(w,1)$, where $w \in c_0\setminus\ell^1$. The following are equivalent:*

1. *K is relatively weakly compact.*
2. *There exists a norm null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $d(w,1)$ such that*
$$K \subseteq \overline{\mathrm{ri}}\,\overline{\mathrm{co}}\{\mathbf{x}_n : n \in \mathbb{N}\}\,.$$
3. *There exists a norm null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $d(w,1)$ such that*
$$K \subseteq \overline{\mathrm{co}}\,\overline{\mathrm{ri}}\{\mathbf{x}_n : n \in \mathbb{N}\}\,.$$
4. *There exists a weakly null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $d(w,1)$ such that*
$$K \subseteq \overline{\mathrm{co}}\,\overline{\mathrm{ri}}\{\mathbf{x}_n : n \in \mathbb{N}\}\,.$$
5. *There exists a weakly null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $d(w,1)$ such that*
$$K \subseteq \overline{\mathrm{ri}}\,\overline{\mathrm{co}}\{\mathbf{x}_n : n \in \mathbb{N}\}\,.$$

Proof Let K be a relatively weakly compact set in $d(w,1)$. Assume, for the sake of contradiction, that the set $K^* \stackrel{\mathrm{def}}{=} \{\mathbf{x}^* : \mathbf{x} \in K\}$ is not relatively compact. Then there exists a sequence $(\mathbf{x}_n)$ in K such that $(\mathbf{x}_n^*)$ has no convergent subsequence. By passing to a subsequence if necessary, assume that $(\mathbf{x}_n^*)$ converges coordinatewise to $\mathbf{x}$. Since the unit vector basis (e_n) of $d(w,1)$ is boundedly complete, $\mathbf{x} \in X$. Define $\widetilde{\mathbf{h}}_n = \mathbf{x}_n^* - \mathbf{x}$ for $n \in \mathbb{N}$. By perturbation of $(\widetilde{\mathbf{h}}_n)$, there exists a seminormalized block basis $(\mathbf{h}_n)$ of (e_n) equivalent to $(\widetilde{\mathbf{h}}_n)$ and, since $\mathbf{x}_n^*$ and $\mathbf{x}$ are nonincreasing sequences, it is easy to ensure that $\lim_n \|\mathbf{h}_n\|_\infty = 0$. Proposition 4.e.3 in [8, p. 177] gives a subsequence $(\mathbf{h}_{n_i})$ of $(\mathbf{h}_n)$ that is equivalent to the usual vector basis of ℓ^1. Consequently, the sequence $(\widetilde{\mathbf{h}}_{n_i})$ can be assumed to be equivalent to the usual vector basis of

ℓ^1. The weak compactness of K and the proof of (4) implies (1) in Theorem 1 combine to give a subsequence of $(\mathbf{x}_{n_i})$ that is both equivalent to the usual vector basis of ℓ^1 and is weakly convergent, a contradiction proving that K^* is relatively compact. Then, by Grothendieck's compactness principle, there exists a norm null sequence $(\mathbf{y}_n)$ in $d(w,1)$ such that $K^* \subseteq \overline{\mathrm{co}}\{\mathbf{y}_n : n \in \mathbb{N}\}$. Since $K \subseteq \overline{\mathrm{ri}}(K^*) \subseteq \overline{\mathrm{ri}}\,\overline{\mathrm{co}}\{\mathbf{y}_n : n \in \mathbb{N}\}$, the implication (1) implies (2) is proven.

The remaining implications are either immediate or follow from Lemma 1 and Theorem 1.

It should be noted that the proofs of Theorem 1 and Theorem 2 could be shortened using Theorem 14 and Theorem 16 in [2]. However, the proofs provided above are more self-contained.

3 The proof of Lemma 1

We now return to the proof of Lemma 1. Recall that a subset T of c_0 is called *rearrangement invariant* if $z \in T$ whenever $z \in c_0$ and $z^* = x^*$ for some $x \in T$. For sequences $x, y \in c_0$, *x is weakly majorized by y* (or *x is dominated by y*) if, for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} x_j^* \leq \sum_{j=1}^{n} y_j^* \,. \tag{$\star$}$$

In this case, we write $x \prec\prec y$. (See, for example, [4, page 166], [9, page 123], or [11, page 185].)

Considering a Banach space with a Schauder basis as a sequence space, a Banach space with a 1-symmetric basis has the *domination property* [11]: If X is a Banach space with a 1-symmetric basis, for all $y \in X$ and all $x \in c_0$ with $x \prec\prec y$, it follows that $x \in X$ and $\|x\| \leq \|y\|$. Indeed, if $x \prec\prec y$ for $y \in X$ and $x \in c_0$, assume that $\|x\|_1 = \infty$. (If $\|x\|_1 < \infty$, the result is easy.) Let $\varepsilon > 0$ be given and choose $m \in \mathbb{N}$ so that $\|(I - P_m)(y^*)\| < \varepsilon$. Then choose a natural number $k > m$ such that $\|P_k(x^*)\| > \|P_m(y^*)\|$. By ($\star$) and the choice of k, it follows that, if $k < \ell$,

$$\sum_{j=k+1}^{\ell} x_j^* = \sum_{j=1}^{\ell} x_j^* - \sum_{j=1}^{k} x_j^* \leq \sum_{j=1}^{\ell} y_j^* - \sum_{j=1}^{m} y_j^* = \sum_{j=m+1}^{\ell} y_j^* \,.$$

This shows that $P_{(k,\ell]}(x^*) \prec\prec P_{(m,\ell]}(y^*)$ and, by Remark 1 in [9, page 124], it follows that

$$\|P_{(k,\ell]}(x^*)\| \leq \|P_{(m,\ell]}(y^*)\| \leq \|(I - P_m)(y^*)\| < \varepsilon \,.$$

Therefore the sequence $(P_n(x^*))$ converges to x^* in X and, since the basis for X is 1-symmetric, $x \in X$. It is then easy to check that $\|x\| \leq \|y\|$. Therefore a Banach space with a 1-symmetric basis has the domination property. (This result also follows from Theorem 3.2 in [10].)

In what follows we will sometimes identify vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with
$(x_1, \dots, x_n, 0, \dots, 0, \dots) \in c_{00}$. If $n \in \mathbb{N}$, $\mathbb{N}_n := \{1, \dots, n\}$, and $y \in \mathbb{R}^n$, define the set

$$\mathrm{Sym}_n(y) = \{(\varepsilon_j\, y_{\pi(j)})_{j=1}^n \in \mathbb{R}^n |\, \pi : \mathbb{N}_n \longrightarrow \mathbb{N}_n \text{ is a bijection and } \varepsilon \in \{-1,1\}^{\mathbb{N}_n}\}\,.$$

Note that $\mathrm{Sym}_n(y) = \mathrm{ri}\{y\}$ for all $y \in \mathbb{R}^n$. A connection between weak majorization of elements in $\mathbb{R}^n$ and rearrangement invariant hulls is given by the Theorem 1.2 in [10].

Theorem 3 (Markus) *Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$. The following are equivalent:*

1. $x \prec\prec y$.
2. $x \in \mathrm{co}\,\mathrm{Sym}_n(y)$.

Define the *weakly majorized rearrangement invariant hull* of a set S in a Banach space X with a symmetric basis to be

$$\mathrm{ri}_{\mathrm{wm}}(S) = \{y \in c_0 : y \prec\prec x \text{ for some } x \in \mathrm{co}(S^*)\}$$

where $S^* = \{x^* : x \in S\}$.

Lemma 2 *Let S be a subset of a Banach space X with a symmetric basis. Then $\mathrm{ri}_{\mathrm{wm}}(S)$ is a convex, rearrangement invariant subset of X containing S.*

Proof Without loss of generality, assume that X is a Banach space with a 1-symmetric basis. Since Banach spaces with a 1-symmetric basis have the domination property, $\mathrm{ri}_{\mathrm{wm}}(S)$ is contained in X. It is easy to check that $\mathrm{ri}_{\mathrm{wm}}(S)$ is a rearrangement invariant set containing S. To show that the set $\mathrm{ri}_{\mathrm{wm}}(S)$ is convex, let $y, z \in \mathrm{ri}_{\mathrm{wm}}(S)$; let $0 \le t \le 1$; and let $w = (1-t)y + tz$. By hypothesis, there exists $x, q \in \mathrm{co}(S^*)$ such that for all $n \in \mathbb{N}$,

$$\sum_{j=1}^n y_j^* \le \sum_{j=1}^n x_j^* \quad \text{and} \quad \sum_{j=1}^n z_j^* \le \sum_{j=1}^n q_j^*\ .$$

Set $p = (1-t)x + tq \in \mathrm{co}(S^*)$ and note that $x^* = x$, $q^* = q$, and $p^* = p$. It is straightforward to check that, for all $u \in c_0$ and all $n \in \mathbb{N}$,

$$\sum_{j=1}^n u_j^* = \max_{i_1 < i_2 < \cdots < i_n} \sum_{j=1}^n |u_{i_j}|\ .$$

Consequently, for all $u, v \in c_0$, $(u+v)^* \prec\prec u^* + v^*$. Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{j=1}^{n} w_j^* &= \sum_{j=1}^{n} \big((1-t)y + tz\big)_j^* \\ &\le (1-t)\sum_{j=1}^{n} y_j^* + t\sum_{j=1}^{n} z_j^* \\ &\le (1-t)\sum_{j=1}^{n} x_j^* + t\sum_{j=1}^{n} q_j^* \\ &= \sum_{j=1}^{n} \big((1-t)\,x_j + t\,q_j\big) \\ &= \sum_{j=1}^{n} p_j \\ &= \sum_{j=1}^{n} p_j^* \ .\end{aligned}$$

Therefore $w \prec\prec p$. This proves that $w \in \mathrm{ri}_{\mathrm{wm}}(S)$ and $\mathrm{ri}_{\mathrm{wm}}(S)$ is convex.

The next result establishes the relationship between the various rearrangement invariant hulls.

Lemma 3 *Let X be a Banach space with a symmetric basis. If S is a subset of X, then*

$$\overline{\mathrm{co}}\,\overline{\mathrm{ri}}(S) = \overline{\mathrm{co}}\ \mathrm{ri}(S) = \overline{\mathrm{ri}}_{\mathrm{wm}}(S)\,.$$

Proof Without loss of generality, assume that X has a 1-symmetric basis. The first equality is easy: $\overline{\mathrm{co}}\ \mathrm{ri}(S) \subseteq \overline{\mathrm{co}}\,\overline{\mathrm{ri}}\,S \subseteq \overline{\mathrm{co}}\left(\overline{\mathrm{co}}\,(\mathrm{ri}\ S)\right) = \overline{\mathrm{co}}\ \mathrm{ri}\ S$.

Since, by Lemma 2, $\mathrm{ri}_{\mathrm{wm}}(S)$ is a convex, rearrangement invariant set containing S, it follows that $\mathrm{co}\,\mathrm{ri}(S) \subseteq \mathrm{ri}_{\mathrm{wm}}(S)$. Therefore $\overline{\mathrm{co}}\ \mathrm{ri}(S) \subseteq \overline{\mathrm{ri}}_{\mathrm{wm}}(S)$.

It remains to show that $\overline{\mathrm{ri}}_{\mathrm{wm}}(S) \subseteq \overline{\mathrm{co}}\ \mathrm{ri}(S)$. Fix $y \in \mathrm{ri}_{\mathrm{wm}}(S)$.

Case 1: Assume that $y^* = y$. By hypothesis $y \prec\prec x$ for some $x \in \mathrm{co}(S^*)$ and note that $x^* = x$. Then, for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} y_j \le \sum_{j=1}^{n} x_j \ .$$

If $n \in \mathbb{N}$ is fixed, the above shows that $P_n y \prec\prec P_n x$. By Markus' theorem, $P_n y \in \mathrm{co}\ \mathrm{Sym}_n(P_n x)$. Thus, there exist finite sequences $\big(g_\beta^{(n)}\big)_{\beta\in F_n}$ in $\mathrm{Sym}_n(P_n x)$ and $\big(t_\beta^{(n)}\big)_{\beta\in F_n}$ in $[0,1]$ such that $\sum_{\beta\in F_n} t_\beta^{(n)} = 1$ and

$$P_n y = \sum_{\beta\in F_n} t_\beta^{(n)}\, g_\beta^{(n)}\ .$$

We now define the finite sequence $\big(h_\beta^{(n)}\big)_{\beta\in F_n}$ in $\mathrm{ri}\{x\}$ by

$$h_\beta^{(n)} = (g_{\beta,1}^{(n)}, \dots, g_{\beta,n}^{(n)}, x_{n+1}, x_{n+2}, x_{n+3}, \dots) \quad \text{for all } \beta \in F_n \ .$$

We claim that each $h_\beta^{(n)} \in \mathrm{co}\ \mathrm{ri}(S)$. Indeed, since $x \in \mathrm{co}(S^*)$, there exist $z_1, \dots, z_\nu \in S^* \subseteq \mathrm{ri}(S)$, and $s_1, \dots, s_\nu \in [0,1]$ summing to 1, such that $x = \sum_{k=1}^{\nu} s_k\, z_k$. Clearly, by the construction, for each $n \in \mathbb{N}$ and for every $\beta \in F_n$, $h_\beta^{(n)} = \big(\eta_{n,\beta,j}\, x_{\rho_{n,\beta}(j)}\big)_{j\in\mathbb{N}}$, for some one-to-one and onto function $\rho_{n,\beta} : \mathbb{N} \longrightarrow \mathbb{N}$, and some sequence $(\eta_{n,\beta,j})_{j\in\mathbb{N}}$ with values in $\{-1,1\}$. From above,

$$x_i = \sum_{k=1}^{\nu} s_k\, z_{k,i} \quad \text{for } i \in \mathbb{N}\,.$$

Fix $n \in \mathbb{N}$ and then fix $\beta \in F_n$. Clearly,

$$h_{\beta,j}^{(n)} = \eta_{n,\beta,j}\, x_{\rho_{n,\beta}(j)} = \sum_{k=1}^{\nu} s_k\, \eta_{n,\beta,j}\, z_{k,\rho_{n,\beta}(j)} \quad \text{for } j \in \mathbb{N}\,.$$

If $k \in \{1, \dots, \nu\}$, define $\psi_\beta^{(n,k)} = (\psi_{\beta,j}^{(n,k)})_{j\in\mathbb{N}} \in c_0$ by

$$\psi_{\beta,j}^{(n,k)} = \eta_{n,\beta,j}\, z_{k,\rho_{n,\beta}(j)} \quad \text{for } j \in \mathbb{N}\,.$$

Then

$$h_\beta^{(n)} = \sum_{k=1}^{\nu} s_k\, \psi_\beta^{(n,k)}$$

and, since $\big(\psi_\beta^{(n,k)}\big)^* = z_k^* = z_k$, it follows that $\psi_\beta^{(n,k)} \in \mathrm{ri}(S)$ and $h_\beta^{(n)} \in \mathrm{co}\ \mathrm{ri}(S)$ for every $n \in \mathbb{N}$ and $\beta \in F_n$.

Therefore we may define $H^{(n)} \in \mathrm{co}\ \mathrm{ri}(S)$ by

$$H^{(n)} = \sum_{\beta\in F_n} t_\beta^{(n)}\, h_\beta^{(n)}\,.$$

If $n \in \mathbb{N}$ and $Q_n = I - P_n$, then

$$H^{(n)} - P_n y = Q_n x$$

and

$$\|y - H^{(n)}\| \le \|y - P_n y\| + \|P_n y - H^{(n)}\| = \|Q_n y\| + \|Q_n x\|\,.$$

Since the sequences $(\|Q_n x\|)$ and $(\|Q_n y\|)$ converge to 0, the sequence $(H^{(n)})$ converges to y and $y \in \overline{\mathrm{co}}\ \mathrm{ri}(S)$.

Case 2: Assume that $y \in \mathrm{ri}_{\mathrm{wm}}(S)$ is arbitrary. Then $y \prec\!\prec x$ for some $x \in \mathrm{co}\, S^*$. So $x^* = x$ and, for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} y_j^* \le \sum_{j=1}^{n} x_j\,.$$

As in Case 1, for each $n \in \mathbb{N}$, there exists a finite sequence $\left(h^{(n)}_{\beta}\right)_{\beta \in F_n}$ in co ri(S) and $H^{(n)} \in$ co ri(S) of the form

$$H^{(n)} = \sum_{\beta \in F_n} t^{(n)}_{\beta}\, h^{(n)}_{\beta}$$

such that the sequence $(H^{(n)})$ converges to y^*

There exists a one-to-one mapping $\pi : \mathbb{N} \to \mathbb{N}$ such that $y^*_j = |y_{\pi(j)}|$ for all $j \in \mathbb{N}$. Thus, for each coordinate $j \in \mathbb{N}$, there exists $\varepsilon_{\pi(j)} \in \{-1, 1\}$ such that $y^*_j = \varepsilon_{\pi(j)}\, y_{\pi(j)}$. For each $n \in \mathbb{N}$ and $j \in \mathbb{N}$, define

$$J^{(n)}_j = \varepsilon_{\pi(j)}\, H^{(n)}_j = \sum_{\beta \in F_n} t^{(n)}_{\beta}\, \varepsilon_{\pi(j)}\, h^{(n)}_{\beta,j}\,.$$

Then, for every $n \in \mathbb{N}$, $J^{(n)} \in$ co ri(S) and, by the rearrangement invariance of the norm, the sequence $(J^{(n)})$ converges to y_π. Let $A := \mathrm{Range}(\pi)$, which is an infinite subset of $\mathbb{N}$. Then, for all $k \in \mathbb{N} \setminus A$,

$$|y_k| \le |y_{\pi(j)}| = y^*_j \quad \text{for all } j \in \mathbb{N}\,.$$

Since $y^* \in c_0$, it follows that $y_k = 0$ for all $k \in \mathbb{N} \setminus A$. Consider the inverse mapping $\sigma = \pi^{-1} : A \to \mathbb{N}$. Fix $n \in \mathbb{N}$ and then fix $\beta \in F_n$. Define $\varphi^{(n)}_{\beta} = \left(\varphi^{(n)}_{\beta,k}\right) \in c_0$ by

$$\varphi^{(n)}_{\beta,k} = \begin{cases} \varepsilon_k\, h^{(n)}_{\beta,\sigma(k)} & \text{for all } k \in A \\ 0 & \text{for all } k \in \mathbb{N} \setminus A\,. \end{cases}$$

Note that, for all $\beta \in F_n$, $\varphi^{(n)}_{\beta} \in$ co ri(S). Then, if $\Phi^{(n)} \in$ co ri(S) is defined by

$$\Phi^{(n)} = \sum_{\beta \in F_n} t^{(n)}_{\beta}\, \varphi^{(n)}_{\beta}\,,$$

the rearrangement invariance of the norm implies that $\|y - \Phi^{(n)}\| = \|y_\pi - J^{(n)}\|$. Since the sequence $(J^{(n)})$ converges to y_π, $(\Phi^{(n)})$ converges to y. Thus $y \in \overline{\text{co}}$ ri(S), completing the proof of the Lemma 3.

Lemma 4 *Let U be a rearrangement invariant subset of a Banach space X with a symmetric basis. Then $\overline{U}^{\text{norm}}$ is rearrangement invariant.*

Proof Without loss of generality, let X have a 1-symmetric basis. Let $z \in c_0$ be such that $z^* = w^*$ for some $w \in \overline{U}^{\text{norm}}$. We will show that $z \in \overline{U}^{\text{norm}}$. There exists a sequence $(x_n)_{n \in \mathbb{N}}$ in U converging to w. Markus' Inequality [10, Theorem 5.4] states that for all $u, v \in c_0$, $u^* - v^* \prec\prec u - v$. So, by the domination property of X, each $w^* - x^*_n \in X$ and

$$\|w^* - x^*_n\| \le \|w - x_n\|\,.$$

Therefore (x^*_n) converges to w^* and, since U is rearrangement invariant, $x^*_n \in U$ for each $n \in \mathbb{N}$. Therefore, $z^* = w^* \in \overline{U}^{\text{norm}}$. (Note that, in [10], Markus'

Inequality is stated for compact operators on ℓ^2. That it applies in our setting is seen by considering diagonal operators.)

There exists a one-to-one mapping $\pi : \mathbb{N} \to \mathbb{N}$ such that $z_j^* = |z_{\pi(j)}|$, for all $j \in \mathbb{N}$. Thus, for each coordinate $j \in \mathbb{N}$, there exists $\varepsilon_{\pi(j)} \in \{-1, 1\}$ such that $z_j^* = \varepsilon_{\pi(j)}\, z_{\pi(j)}$. An argument similar to the argument in Case 2 of Lemma 3, using the rearrangement invariance of the set U and the norm $\|\cdot\|$, implies that $z \in \overline{U}^{\text{norm}}$.

Proof (Proof of Lemma 1.) Let $S \subseteq X$ and assume that X is a Banach space with a 1-symmetric basis. Clearly $S \subseteq \text{ri}(S)$ and $\overline{\text{co}}(S) \subseteq \overline{\text{co}}\ \text{ri}(S)$. By Lemma 3, $\overline{\text{co}}(S) \subseteq \overline{\text{ri}}_{\text{wm}}(S)$. Since $\text{ri}_{\text{wm}}(S)$ is convex and rearrangement invariant, Lemma 4 implies that $\overline{\text{ri}}_{\text{wm}}(S)$ is rearrangement invariant. Consequently, $\text{ri}\ \overline{\text{co}}(S) \subseteq \overline{\text{ri}}_{\text{wm}}(S)$ and $\overline{\text{ri}}\,\overline{\text{co}}(S) \subseteq \overline{\text{ri}}_{\text{wm}}(S)$. A final application of Lemma 3 yields that $\overline{\text{ri}}\,\overline{\text{co}}(S) \subseteq \overline{\text{co}}\,\overline{\text{ri}}(S)$, completing the proof of the lemma.

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