Spin wavelets on the sphere and their discretizations into frames

Azita Mayeli
SUNY Stony Brook
CUNY City Tech College

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Outline

Let $f \in \mathcal{S}(\mathbb{R}^+)$, $f \not\equiv 0$, but $f(0) = 0$.

Calderón reproducing formula: if $c \in (0, \infty)$ is defined by

$$c = \int_0^\infty |f(t)|^2 \frac{dt}{t},$$

then for all $s > 0$,

$$\int_0^\infty |f(ts)|^2 \frac{dt}{t} = c < \infty.$$

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Discretizing formula: for $a > 1$ sufficiently close to 1

$$0 < A_a \leq \sum_{j=-\infty}^{\infty} |f(a^{2j}s)|^2 \leq B_a < \infty \quad \forall \ s > 0.$$

Moreover

$$B_a/A_a \to 1 \ \text{ as } \ a \to 1^+$$

Example: (Daubechies) $f(s) = se^{-s}, a = 2^{1/3} \Rightarrow B_a/A_a = 1.0000$

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Associated to f, one can find $\psi := \psi_f \in \mathcal{S}(\mathbb{R})$ such that

$$A\|F\|_2^2 \leq \sum_j \|F * \psi_{a^j}\|_2^2 \leq B\|F\|_2^2,$$

And for $b > 0$ sufficiently small,

$$A\|F\|_2^2 \leq \sum_{j,k} |\langle F, w_{jk}\rangle|_2^2 \leq B\|F\|_2^2,$$

$$w_{jk}(x) = \sqrt{ba^j}\,\overline{\psi}_{a^j}(x - bka^j)$$

Thus, for some $C > 0$

$$F \sim C \sum \langle F, w_{jk}\rangle w_{jk}$$

Spectral Theory

Generalization of this procedure to *many* different situations:

Let T be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$.

If

$$s \longrightarrow T$$

in

$$\int_0^\infty |f(ts)|^2 \frac{dt}{t} = c < \infty \qquad (s > 0)$$

$$0 < A_a \leq \sum_{j=-\infty}^{\infty} |f(a^{2j}s)|^2 \leq B_a < \infty \qquad (s > 0)$$

Spectral Theory

then by spectral theory we obtain

Theorem (Geller-M. '07)

If P be the projection onto the null space of T,

$$\int_0^\infty |f|^2(tT)\frac{dt}{t} = c(I-P)$$

$$A_a(I-P) \leq \sum_{j=-\infty}^{\infty} |f|^2(a^{2j}T) \leq B_a(I-P).$$

Example

- $\mathcal{H} = L^2(\mathbb{R})$: $T = -d^2/dx^2$. We could write

$$[f(t^2T)F](x) = \int_0^\infty K_t(x,y)F(y)dy$$

with

$$K_t(x,y) = \psi_t(x-y).$$

Therefore

$$f(t^2T)F = \psi_t * F$$

Example

- $\mathcal{H} = L^2(\mathbf{M})$: $\mathbf{M}$ is a smooth and compact Riemannian manifold. Now instead take $T = \Delta$, the Laplace-Beltrami operator on $\mathbf{M}$. Then

$$[f(t^2T)F](x) = \int_{\mathbf{M}} K_t(x,y)F(y)d\mu(y)$$

for a suitable K_t.

Spectral Theory - discretization

Theorem (Geller-M. '07)

Let $a > 1$*. Pick* $b > 0$*, and for each* j*, write* $\mathbf{M} = \cup_k E_{j,k}$ *where* $\mathrm{diam}(E_{j,k}) \leq ba^j$*.*
Assume also for some $c_0 > 0$ *and some* J*,* $\mu(E_{j,k}) \geq c_0(ba^j)^n$ *whenever* $j < J$*. Take* $x_{j,k} \in E_{j,k}$*. Let*

$$w_{j,k}(x) = \sqrt{\mu(E_{j,k})}\,\overline{K_{a^j}}(x_{j,k}, x).$$

Then, if b *is sufficiently small,*

$\{w_{j,k}\}$ *is a nearly tight frame for* $(I - P)L^2(\mathbf{M})$*.*

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Compare with $w_{j,k}$ on $\mathbb{R}$:

$$w_{jk}(x) = \sqrt{ba^j}\,\overline{\psi}_{a^j}(x - bka^j)$$

Example: On the circle, where $\Delta = -d^2/d\theta^2$, if $f(u) = ue^{-u}$,

$$K_t(\theta, \phi) = \sum_{m=-\infty}^{\infty} (t^2 m^2 e^{-t^2 m^2}) e^{im\theta} e^{-im\phi}$$

so that, if $F(\phi) = \sum_m \hat{F}(m) e^{im\phi}$,

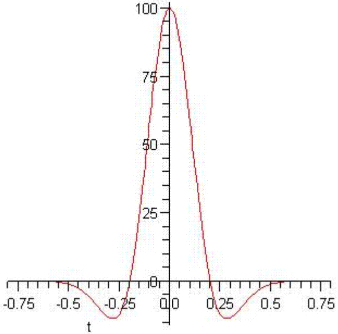
$$f(t^2\Delta)F = t^2 \Delta e^{-t^2\Delta} F = \sum_m (t^2 m^2 e^{-t^2 m^2}) \hat{F}(m) e^{im\theta}.$$

Mexcian Needlet: On the sphere, where Δ is sphercial Laplacian, if $f(u) = ue^{-u}$,

$$K_t(x,y) = \sum_{l=0}^{\infty} \sum_{|m|\leq l} (t^2\lambda_l e^{-t^2\lambda_l}) Y_{l,m}(x)\overline{Y_{l,m}(y)}$$

where $\lambda_l = l(l+1)$ for $l \in \mathbb{N}_0$.

So, if $H_t(x \cdot y) := 4\pi K_t(x,y)$, then for $y = (0,0,1)$, if $t = 0.1$



Applications

- Spherical wavelet frames are often used for CMB data analysis.
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- CMB models are best analyzed in the frequency domain, so having wavelets that are explicit on the frequency side is very useful.
- Partial sky coverage in the presence of the Mask and other missing observations make the evaluation of exact spherical harmonic transforms troublesome.
- The combination of above facts makes the time-frequency localization of wavelets most valuable.

Theorem (space-frequency localizations, Geller-M. '07)

For every pair of C^∞ differential operators X (in x) and Y (in y) on $\mathbf{M}$, and for every integer $N \geq 0$,

$$|XYK_t(x,y)| \leq C_{N,X,Y} \frac{t^{-(n+\deg X+\deg Y)}}{(1+\frac{d(x,y)}{t})^N}.$$

New Application

- Geller, D., Marinucci, D., *Spin Wavelets on the Sphere*, to appear in JFAA.
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- The CMB radiation has both a temperature, as a scalar field, and a polarization, as a tensor field.

$$\text{Photon} \rightarrow \text{temperature} + \text{polarization}$$

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Photon $\rightarrow$ temperature + polarization

- Polarization caries the directional information.
- Polarization data will help physicists to determine for instance when the first stars formed.
- Taking into account the huge amount of polarization data which will be available in the next 1-2 decades, as well as the important cosmological information contained in these data, it is clear that efficient mathematical tools for data analysis are in high demand.
- Polarization is viewed as a spin 2 quantity.

Integer Spin s

Definition (Physicist's Definition by Newman - Penrose '66)

A quantity η defined on the sphere has spin weight $s \in \mathbb{Z}$, provided that, whenever a tangent vector m at a point transforms under coordinate change by

$$m' = e^{i\psi} m,$$

the quantity η, at that point, transforms by

$$\eta' = e^{is\psi} \eta.$$

Integer Spin s functions

For Mathematics Definition:

Let $\mathbf{N}$ be the north pole $(0,0,1)$, let $\mathbf{S}$ be the south pole, $(0,0,-1)$. Let

$$U_I = S^2 \setminus \{\mathbf{N}, \mathbf{S}\}.$$

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On U_I we use standard spherical coordinates (θ, ϕ) ($0 < \theta < \pi$, $-\pi \leq \phi < \pi$), and analogously, on any U_R we use coordinates (θ_R, ϕ_R) obtained by rotation of the coordinate system on U_I.

Integer Spin s functions - definition continued

At each point p of U_R we let $\rho_R(p)$ be the unit tangent vector at p which is tangent to the circle $\theta_R =$ constant, pointing in the direction of increasing ϕ_R. (This is well-defined since $R\mathbf{N}, R\mathbf{S} \notin U_R$.)

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If $p \in U_{R_1} \cap U_{R_2}$, we let

$\psi_{pR_2R_1}$ be the angle from $\rho_{R_1}(p)$ to $\rho_{R_2}(p)$.

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Definition

Let $\Omega \subseteq S^2$ be open. $F = (F_R)_{R \in SO(3)}$ is called a spin s function if F_R's are defined on $U_R \cap \Omega$ and for all $p \in U_{R_1} \cap U_{R_2} \cap \Omega$,

$$F_{R_2}(p) = e^{is\psi} F_{R_1}(p),$$

$\psi = \psi_{pR_2R_1}$.

Definition (smooth integer spins)

$$C_s^{\infty}(\Omega) := \{F = (F_R)_{R \in SO(3)};\ \text{all } F_R \in C^{\infty}(U_R \cap \Omega)\}.$$

Definition (L_s^2 space)

$$L_s^2(S^2) := \{F = (F_R)_{R \in SO(3)};\ \text{all } F_R \in L^2(U_R)\}$$

with an inner product given by

$$\langle F, G \rangle = \langle F_R, G_R \rangle$$

Spin-Raising operator

$$\eth : C_s^\infty(\Omega) \to C_{s+1}^\infty(\Omega)$$

given by

$$(\eth F)_R = \eth_{sR} F_R$$

where

$$\eth_{sR} F_R = -(\sin\theta_R)^s \left(\frac{\partial}{\partial\theta_R} + \frac{i}{\sin\theta_R} \frac{\partial}{\partial\phi_R} \right) (\sin\theta_R)^{-s} F_R .$$

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Spin-Lowering operator

$$\overline{\eth} : C_s^\infty(\Omega) \to C_{s-1}^\infty(\Omega)$$

given by

$$(\overline{\eth} F)_R = \overline{\eth_{-s,R} \overline{F_R}}$$

Spin s Spherical Laplacian

Define Δ_s acting on $C_s^\infty(\Omega)$

$$\Delta_s = \begin{cases} -\overline{\partial}\partial & \text{if } s \geq 0 \\ -\partial\overline{\partial} & \text{if } s < 0 \end{cases}$$

Then Δ_0 is the usual spherical Laplacian.

Spherical Harmonics

Recall: Δ_0 has an orthonormal basis of spherical harmonics (eigenfunctions)

$$\{Y_{lm} : \quad l \in \mathbb{N}_0, -l \leq m \leq l\}$$

with eigenvalues

$$\lambda_l = l(l+1)$$

Call the eigenspace $\mathcal{H}_l$. Then

$$L^2(S^2) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l.$$

Spin s Spherical Harmonics

For general s:

$$L_s^2(S^2) = \bigoplus_{l=|s|}^{\infty} \mathcal{H}_{ls}.$$

Each $\mathcal{H}_{ls}$ is an eigenspace of Δ_s, with an explicit orthonormal basis

$$\{{}_sY_{lm} : -l \leq m \leq l\}$$

obtained by applying the spin-raising and spin-lowering operators to the usual spherical harmonics Y_{lm}.

Spin s Spherical Harmonics

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Example: $s = 2$;

$$Y_{lm} \to \partial Y_{lm} \to \partial(\partial Y_{lm})$$

or up to constant factor

$$Y_{lm} \to {}_1Y_{lm} \to {}_2Y_{lm}$$

Spin s Wavelets

Set

$$K_{t,R',R}(x,y) = \sum_{l\geq|s|} \sum_{m=-l}^{l} f(t^2\lambda_{ls})\ {}_sY_{lmR'}(x)\ \overline{{}_sY_{lmR}(y)}$$

where λ_{ls} are eigenvalues for Δ_s. Then

$$(f(t^2\Delta_s)F)_{R'}(x) = \int_{S^2} K_{t,R',R}(x,y)F_R(y)d\mu(y).$$

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$$(f(t^2\Delta_s)F)_{R'}(x) = \int_{S^2} K_{t,R',R}(x,y)F_R(y)d\mu(y).$$

Localization Theorem: $K_{t,R',R}$ satisfies the usual estimates

$$|XYK_{t,R',R}(x,y)| \leq C_{N,X,Y}\frac{t^{-(n+\deg X+\deg Y)}}{(1+\frac{d(x,y)}{t})^N},$$

as long as x stays in a compact subset of $U_{R'}$ and y stays in a compact subset of U_R.

Spin s Frames

Recall $K_{t,R',R}(x,y) = \sum_{l \geq |s|} \sum_{m=-l}^{l} f(t^2 \lambda_{ls}) \, {}_sY_{lmR'}(x) \, \overline{{}_sY_{lmR}(y)}$.

Spin s Frames

Recall $K_{t,R',R}(x,y) = \sum_{l\geq|s|} \sum_{m=-l}^{l} f(t^2\lambda_{ls})\ {}_sY_{lmR'}(x)\ \overline{{}_sY_{lmR}(y)}$.

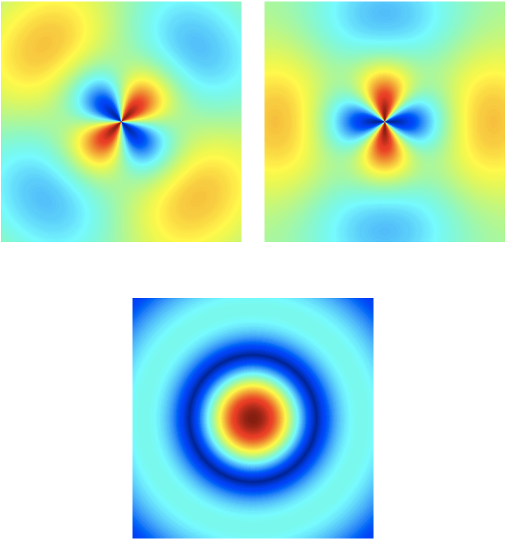
Theorem (discretization into frames)

Choose the sets E_{jk} as in the scalar theorem, pick $x_{jk} \in E_{jk}$, pick R_{jk} with $x_{jk} \in U_{R_{jk}}$, and set

$$w_{j,k}(x) = \sqrt{\mu(E_{j,k})} \sum_{l\geq|s|} \sum_{m=-l}^{l} f(a^{2j}\lambda_{ls})\ \overline{{}_sY_{lmR_{jk}}(x_{jk})}\ {}_sY_{lm}(x).$$

so that $w_{jk} \in C_s^\infty(S^2)$. Then, for a close enough to 1 and b small enough, the w_{jk} are a nearly tight frame for $(I-P)L_s^2(S^2)$, where $P = \mathcal{H}_{|s|,s}$ is the null space of Δ_s.

Spin 2 wavelet ${}_2w_{j,k}$ when $j = 10$



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